CHOOL MATHEMATICS 2

TEACHERS' EDITION

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Teachers' Edition to accompany

School Mathematics 2

1974 Edition

CHARLES R. FLEENOR

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Mathematics Manipulative Aids

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Illustrations by Betty Delphey

FOREWORD

The *School Mathematics* series integrates a sound mathematics program with a teaching strategy that encourages the student to participate actively in the learning process. The essence of the *School Mathematics* series is reflected in some basic beliefs about the learning process to which we are committed:

1. Effective learning requires active participation on the part of the learner.
2. Understanding of mathematical concepts is a necessary prerequisite to adequate development and acquisition of computational skills.
3. Learning proceeds from concrete experiences to abstractions.
4. Learning is enhanced by motivation.
5. Each learner must experience earned success.

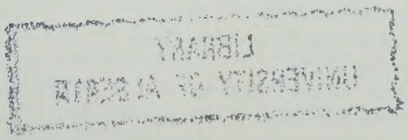
School Mathematics provides for the necessary mastery of basic mathematical concepts and computational skills while allowing for the wide range of abilities that are possessed by students at this level. Students are encouraged to discover patterns and relationships, reason logically, and develop their own generalizations.

Problem solving skills are carefully developed by means of diagrams, flow charts, formulas, and equations. Numerous sets of word problems give the student experience with applications of mathematics to the fields of physics, chemistry, geography, history, and economics as well as everyday applications of mathematics in simple business transactions such as sales tax, discounts, and interest.

Many other special features of this series such as the Think problems, Research Projects, and Mathematical Recreations are provided in the belief that the study of mathematics can be stimulating, enjoyable, and rewarding.

The authors wish to express their appreciation to numerous institutions and individuals in developing this series. Our teaching experiences at Ball State University, the Educational Research Council of Greater Cleveland, and Illinois State University have given us the opportunity to experiment, and to develop and test our ideas. We are indebted to many outstanding mathematicians and mathematics educators for their continual efforts to upgrade the teaching of mathematics in our schools. Special thanks go to Ms Jeanette Powelson for her contributions to the program, and to the editorial and production staffs of our publisher. Finally, we are deeply appreciative of the many teachers and students who have proved conclusively that the study of mathematics can be an exciting and gratifying experience.

Charles Fleenor
Robert Eicholz
Phares O'Daffer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THE BOOK 2 PROGRAM

Mathematics of the Book 2 Program

The program for *School Mathematics,* Book 2, like that for the previous book in the series, is organized into four learning units. Each learning unit contains three or four subunits, called *modules*. Each module is devoted to a specific mathematical topic.

Unit E: Integers
The first module of this unit reviews the basic ideas of numeration, numbers, and fundamental operations. Special attention is given to rounding, exponents, and estimation. The binary numeration system is introduced to provide a contrast to the decimal numeration system and to provide motivating material for more able students.

Module 2 provides some excellent background for pre-algebra students. Algebraic expressions and equations whose replacement sets are whole numbers are introduced. Flow charts and functions are used to reinforce algebraic concepts. Only the minimum amount of set theory and symbolism necessary for the algebraic work is introduced.

Some basic ideas of number theory are reviewed and extended in Module 3. A special feature in this module is the work with sequences of whole numbers.

Module 4 is devoted to Integers. Addition, subtraction, multiplication, and division of integers is covered in depth in this module. Since many students will have been introduced to integers in the previous book of this series, students may be expected to attain some proficiency in the basic operations with integers. Additional applications of integers are provided by work with inequalities, absolute value, graphs of functions, and equations.

Unit F: Rational Numbers
Module 1 reviews the basic idea of fractional numbers, but the principal emphasis is on the introduction of negative rational numbers. Since efficient computation with negative rational numbers depends strongly on the ability to compute with integers, the preceding module of Unit E is a prerequisite. Throughout the unit, computation with fractions having unwieldy denominators is kept to a minimum. Emphasis is placed on fractions which are powers of 10, in order to relate such fractions to the decimal system and the work in the metric system.

Module 2, Decimals, reviews all of the basic operations with decimals. This module also presents many relevant problem situations involving applications of decimals. Students who already have excellent paper and pencil skills in work with decimals might be additionally motivated if they are permitted to use minicomputers or desk calculators at appropriate times.

Module 3 provides important work with ratio, proportion, and percents, and should further strengthen students' ability to work with rational numbers in both fractional and decimal form.

Unit G: Geometry
Module 1 of this unit reviews the three rigid motions introduced in Book 2 of this series. Congruent figures are introduced in terms of combinations of rigid motions. Three triangle congruence theorems (Side-Side-Side, Side-Angle-Side, and Angle-Side-Angle) are highlighted. The geometry of the module is almost exclusively nonmetric.

Metric geometry is the topic of Module 2. Standard metric units of measure for length, area, and volume are emphasized throughout the module. Included in this module on measurement is angle measure and the Pythagorean Theorem. (The latter topic involves triangles with sides whose lengths are rational numbers.) The ideas of ratio and proportion that were presented in Module 3 of Unit F are prerequisite for the work in this module. The work with similar triangles culminates with a brief introduction to three of the trigonometric functions —the sine, cosine, and tangent functions. However, mastery in work with such functions is not an objective for the module.

The final module of the Geometry unit covers the standard topics of circles, cylinders, cones, and spheres. Work with area and volume of these figures comprises the main emphasis in the module.

Unit H: Some Special Topics in Mathematics
The first module, on Probability, offers an excellent opportunity for active student involvement and is a highly motivating topic for most students. Two of the topics, permutations and combinations, will be new to most students.

Three measures of central tendency—mean, median, and mode—are reviewed in Module 2. Interpretation of graphical data is an important aspect of this module. The module concludes with some basic ideas related to sampling.

Module 3, Real Numbers, could be taught following Module 2 of Unit F (Decimals). Knowledge of the Pythagorean Theorem is essential for the introduction to square roots of rational numbers. This module is more important conceptually than it is computationally. While the student is expected to realize the role of the real numbers in completing the number line and something of the nature of irrational numbers, the student is not expected to become proficient in computing with nonrational real numbers.

The final module of the book, Other Topics in Mathematics, should be regarded as optional material. The main emphasis is on the mathematical structure found in some diverse systems such as modular arithmetic, rotations and flips of a square around its lines of symmetry, the set of vectors and the operation of addition, and the mathematical model of a switching circuit with parallel and series switches.

Features of Book 2

School Mathematics, Book 2, consists of four *learning units*. Each learning unit is comprised of three or four *modules*. The four learning units are published as a single case-bound book or in four soft-bound booklets of single learning units.

Each module opening presents a list of the major objectives for the module. These objectives serve as general guidelines for the student's work in the module. At the end of the module, a self-evaluation test, called *Test Yourself,* is presented so that the students themselves may determine their degree of mastery of the objectives. Answers to the self-evaluation test are included in the student text so that immediate reinforcement is available.

Each lesson, presented on two facing pages, normally constitutes enough material for a class session of 45 to 60 minutes.

The core lessons are designed in one of two ways: those lessons which include an Investigation activity begin on the left-hand page with the Investigation, followed by a set of Discussion Exercises and then by a set of exercises on the right-hand page for the student to work independently. (The titles of these sections are *Investigating the Ideas, Discussing the Ideas,* and *Using the Ideas,* respectively.) Other lessons are designed around a set of discussion exercises *(Discussing the Ideas)* and a set of independent exercises *(Using the Ideas).*

Many lessons throughout the book contain starred exercises for enrichment and a *Think* problem for the more able or interested student. At the bottom of many right-hand pages, the student is referred to the special sets of Supplementary Exercises found in the back of the student text. These exercises provide additional practice and reinforcement of basic mathematical skills.

At the end of each module, a set of review exercises called *Reviewing the Ideas* is presented. This is followed by the self-evaluation test *(Test Yourself)* previously discussed. At the end of each learning unit, there are two pages of *Cumulative Review* exercises which cover the significant ideas of all the modules of the learning unit. Separate module achievement tests and unit achievement tests are also available as part of the evaluation program.

At the end of each module is a list of suggested Research Projects. These projects may encourage individual students to study some mathematical topics independently and in a greater depth than is possible in group work. Many of the suggested explorations could be developed into projects suitable for mathematics fairs and exhibits.

Each module concludes with a Mathematical Recreations page which will attract the attention of many students. Most of these recreations require little mathematical background. The puzzles, games, and activities that are suggested will serve as excellent motivation for many students.

The Teaching Strategy

The *School Mathematics* series is adaptable to a wide variety of classroom organizations and teaching strategies. Although teachers are encouraged to use their own teaching techniques and methods, we offer a broad general teaching strategy for this book. Each day's lesson in which the student is presented with a new concept can be divided into these stages:

1. *Preparation*
2. *Investigation*
3. *Discussion*
4. *Utilization*
5. *Extension*

Preparation: The Preparation stage is usually a "prebook" stage. Sometimes this stage consists in a review of concepts studied previously, in order to provide readiness for new ideas. At other times, it may consist in collecting and assembling material necessary for the lesson. In any event, the time devoted to the Preparation should be very short, and care should be taken that any activity undertaken at this stage does not preempt the Investigation which follows.

Investigation: Generally, the Investigation is a student-centered activity which presents the rudiment of the concept treated in the lesson. Each Investigation presents the students with a challenge question. To answer the challenge question, the students must speculate, experiment, probe, estimate, and explore. The Investigation stage enables students to develop nonroutine methods of problem solving. They are often asked to find as many ways as they can to do a certain thing. This allows for individual differences in students, since the open-ended questions permit less able students to experience success even when they have found only a few ways to accomplish a certain task.

In general, the Investigation should be carried out by students working independently or in small groups. The length of time needed to carry out an Investigation will vary, but should not be unduly prolonged. It is not necessary that every student complete all of the Investigation. During the Investigation stage, the teacher should remain somewhat in the background, acting as a moderator and advisor to those students needing help. Encourage students to do their own thinking and exploring. Do not give too much help at this stage.

Discussion: Following the Investigation, students are given an opportunity in the Discussion section to talk about what they did and to summarize the mathematical ideas of the lesson in preparation for working independently in the Using the Ideas section. Generally, the beginning discussion exercises are designed to stimulate students to talk about what they did in the Investigation. You should encourage them to discuss the various methods that they used to investigate and explore the concepts. Also, you should follow your teachers' guide carefully to be sure that whatever mathematical ideas are to be developed in the section are actually summarized and understood. You should provide additional examples as needed in order to strengthen understanding before proceeding.

Utilization: The material presented on the right-hand page of the student book, entitled *Using the Ideas,* gives students the opportunity to apply or utilize what they have learned in the preceding phases of the lesson.

You should provide for individual differences in assigning problems in this section. The teachers' commentary for each lesson provides suggestions for assignments on three levels: Minimum, Average, and Maximum. These suggested assignments should be regarded as guidelines, not prescriptions; you should feel free to vary assignments to meet the needs of your students.

Extension: The *Investigating School Mathematics* series provides opportunity for a wide variety of extension experiences. The Extension material can be thought of as including three principal types of activities: *enrichment*, *reinforcement*, and *remedial*.

The enrichment material is primarily, but not exclusively, intended for use by more able students. The reinforcement material should be used for all students who would benefit from further experiences in both the reteaching and practice areas. The remedial material, though primarily drill and practice, can be effectively used by any students who, for one reason or another, need to improve basic skills. Note that there is considerable overlapping in the categories of Extension materials, and no student should necessarily be excluded from participation in any one of the forms of Extension activities. Your choice of Extension suggestions for individual students constitutes one of your best opportunities for providing for individual differences and needs.

Materials that can be utilized in the Extension stage are offered in the student text, in the Teachers' Edition commentary, and in the supplements *(Workbooks, Duplicator Masters)*.

Within the student text, the starred exercises represent either reinforcement exercises or enrichment. The Think problems are most often enrichment problems. The Research Projects and Mathematical Recreations at the end of each module also should be considered enrichment material. For remedial work as well as for reinforcement, the Supplementary Exercises at the back of the text constitute a source of more than 1400 problems.

In the Teachers' Edition, in the *Extension* section, you will often find additional suggestions either for enrichment or for remedial work. You should also check through the list of Resources for Active Learning in the introductory notes for each module for outside sources of material that might be used for Extension activities.

Other materials that may be used in the Extension stage are provided in the supplements designed for use with the student text. The *Duplicator Masters* provide additional practice for remediation and reinforcement. The *Workbooks* also can serve as a reteaching tool for remedial or reinforcement work.

The Teachers' Edition

General Format

The Teachers' Edition provides a wealth of suggestions and aids designed to assist you in providing an effective mathematics program for your students. On each Teachers' Edition page, the student page is reproduced in full size and includes annotated answers. The teaching commentary pertaining to each page is found in the margins beside the appropriate student page.

Preceding each module are two pages of introductory notes for teachers which contain general objectives for the module, performance objectives, comments concerning the mathematical content of the module, suggestions for teaching the module, a suggested lesson schedule, a discussion of evaluation procedures, and recommended resources for active learning. Teachers should find it very worthwhile to read these opening pages before teaching the material in the module.

At the top of each left-hand page of the Teachers' Edition, under the heading *Objective(s)*, the goal of the lesson is stated in terms of what the student should be able to do as a result of the lesson. This objective summarizes the key idea of the lesson in terms of the student's performance.

Suggestions for the Preparation, Investigation, and Discussion stages also follow in the left-hand margin of the Teachers' Edition.

In the right-hand margin of the Teachers' Edition appear suggestions for the Utilization stage, along with certain selected answers where space was not available for annotations on the student page. Also in this column will be found follow-up suggestions for the Extension stage, which may be either remedial or enrichment material. Included here also are suggestions for *Supplementary Exercises* in the back of the text, *Duplicator Master* sheets, and *Workbook* assignments. Complete solutions to most Think problems occur under the Extension section.

General Suggestions

We offer two general suggestions regarding use of the module and individual lesson notes:

(1) Read and consider each point as it applies to the immediate objectives for the lesson and the overall objectives of the module.
(2) Do not allow the Teachers' Edition notes to deter you from using your own effective teaching methods or to stifle creative efforts.

The Teachers' Edition does not attempt to dictate all of the activities in the day-to-day handling of your class and the individuals in it. You should use your Teachers' Edition as a guide to be coordinated with those methods which you have found to be most effective in teaching mathematics in the past.

Provision for Individual Differences

Each module introduction begins with a listing of both general objectives and performance objectives. The performance objectives are related in chart form to materials that would be helpful for background, reteaching, reinforcement, and related activities. These materials are given in the chart in abbreviated form. The Abbreviation Key follows (with Addison-Wesley's Order Code Numbers given in parentheses).

Abbreviation Key

WB	*Workbook* (1272)
DM	*Duplicator Masters* (1273)
ASC	*Arithmetic Skill Cards* for Development, Practice, and Applications: W—Whole Numbers; F—Fractions; D—Decimals and Percent; AP—Applications (1599)
SWM 1	*Success with Mathematics 1* (1580)
SWM 2	*Success with Mathematics 2* (1590)
SWM 3	*Success with Mathematics: A Course for Secondary School* (2411)
SP	*Skills and Patterns* (2156)
PD	*Patterns and Discovery* (2296)
DS	*Discovery and Structure* (2321)

Each performance objective listed in the chart is classified as red, yellow, or green.

LONG-RANGE PLANNING CHART							
Learning Module	**Basic Course** (Red)		**Average Course** (Yellow)		**Maximum Course** (Green)		**Review and Evaluation** pages
	Lesson pages	Time (days)	Lesson pages	Time (days)	(Green) Lesson pages	Time (days)	
E–1 Numerals, Numbers, and Operations	E 4–17	10	E 2–21	10	E 2–25	8	E 26, 27
E–2 Sets, Equations, and Inequalities	E 30–45	15	E 30–53	15	E 30–57	12	E 58, 59
E–3 Number Theory	E 62–65, 68–71, 74–75	10	E 62–71, 74–79	10	E 62–85	10	E 86, 87
E–4 Integers	E 90–109	15	E 90–119	18	E 90–123	18	E 124–127
F–1 Rational Numbers	F 2–25	15	F 2–29	12	F 2–33	12	F 34, 35
F–2 Decimals	F 38–47, 52–57	15	F 38–49, 52–63	13	F 38–63	12	F 64, 65
F–3 Ratio and Percent	F 68–73, 76–78, 80–85	15	F 68–89	12	F 68–91	10	F 92–95
G–1 Rigid Motions and Congruence	G 2–9, 12–15, 22–25	12	G 2–9, 12–25	10	G 2–25	10	G 26, 27
G–2 Measurement	G 30–39, 42–43, 46–47	12	G 30–47	12	G 30–51	12	G 52, 53
G–3 Similar Triangles and Trigonometry	G 56–61	8	G 56–65	8	G 56–69	10	G 70, 71
G–4 Circles, Cylinders, Cones, and Spheres	G 74–86	10	G 74–89	10	G 74–91	10	G 92–95
H–1 Probability	H 2–9, 12–17	12	H 2–19	12	H 2–23	14	H 24, 25
H–2 Statistics	H 28–41	10	H 28–43, 46	10	H 28–47	12	H 48, 49
H–3 Real Numbers	H 52–63	10	H 52–67	10	H 52–69	12	H 70, 71
H–4 Other Topics in Mathematics			H 74–87	8	H 74–91	8	H 92–95

The color assignments are made on the following basis:

1. The red objectives present minimal performance standards. When only these are studied, the resulting course is the Basic Course, as outlined in the Long-Range Planning Chart (see page x).

2. The yellow objectives are within the capabilities of average students. When both the red and the yellow objectives are studied, the resulting course is the Average Course, as indicated in the Long-Range Planning Chart.

3. The green objectives represent a maximal level of achievement for the module. These green objectives fall into two categories:

(a) material that is considered to be more difficult or challenging, and

(b) material that is considered to be extension or supplementary to a basic program.

As an additional aid to individualization, minimum, average, and maximum assignments are suggested in the commentary accompanying each lesson other than review lessons.

Long-Range Planning Chart

The long-range planning chart on the next page provides the teacher with some general guidelines in planning *basic*, *average*, and *maximum* coverage of *School Mathematics*.

The *basic course* covers all the essential parts of the program necessary for adequate student preparation at the next level in mathematics. The *average course* covers all the material of the basic course but offers more extensive coverage of in-depth and extension material. The *maximum course* provides for nearly total coverage of the text material. The suggested time schedule based on 170 teaching days should be viewed only as an *aid* in helping you plan a time schedule that allows for individual differences within your class. You should not view it as a rigid schedule. Adjust the schedule according to the needs and abilities of your students.

Classroom Organization

The Book 2 program can be used in situations where the entire class works together on the same lesson, where small groups work together on the same lesson, or where individual students are allowed to proceed at their own rate of speed. You will want to employ the type of classroom organization that best suits the physical facilities of your particular situation. Teachers often find it stimulating to vary group sizes for different lessons and units of work. Smaller-size groups can often work more effectively together and allow greater opportunity to participate in the Investigation and the Discussion. Whatever class organization you choose, keep in mind that the key to the *School Mathematics* program is active student participation.

Evaluation of Progress

Students' attitudes toward mathematics are often influenced by the methods used for evaluating their progress. All too often, evaluation procedures focus attention on what the student *did not* understand or master, rather than on what the student *did* accomplish. In evaluating progress, try to maintain a positive view, one which capitalizes on successes and develops confidence.

Pretests and achievement tests for Book 8 may be obtained from the publisher. Module reviews as well as cumulative reviews are provided in the text, to aid in evaluation of students' progress. However, you may find that a day-to-day evaluation of students, often involving personal interviews, will be a significant factor in determining how well they grasp the concepts and how well they are able to apply them.

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Books cited in the "Research Projects" at the conclusion of each module provide useful reference material. A complete list of books is included here to provide historical background and reference material for additional research projects. They are also valuable for their contribution to recreational reading.

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BIBLIOGRAPHY OF RESOURCES FOR ACTIVE LEARNING

Since implementing the investigative approach means more student involvement within the classroom or in a mathematics laboratory, it requires that the teacher be familiar with commonly available, free, and inexpensive material and supplies. It also requires knowledge of commercial aids and teacher resource material.

Included in each module introduction is a section titled "Resources for Active Learning." This list correlates appropriate "Activities" from teacher sourcebooks to the objectives of that module. The "Manipulative Devices" and "Commerical Games" can be used to support the mathematical objectives. They also provide for a means of reinforcement, extension, application, and fun.

At the time of this writing, the resources included were those which directly complement the active-learning approach and which are of good quality and present useful ideas.

Familiarize yourself with these materials and others which have been marketed subsequently. Check with your principal, supervisor, or mathematics coordinator for more recent materials. Only a few suppliers are listed for each device or game. Often they can be obtained from your local supplier. A complete bibliography and list of Suppliers of Resources for Active Learning is included immediately following this list.

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Suppliers of Resources for Active Learning

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Childcraft Education Corp., New York.
Concept Co., Belmont, Massachusetts.
Creative Playthings, Princeton, New Jersey.
Creative Publications, Palo Alto, California.
Edmund Scientific Co., Barrington, New Jersey.
Educational Teaching Aids, Chicago.
Encyclopaedia Britannica Publications Ltd., Toronto, Ontario.
Gamco Industries Inc., Big Spring, Texas.
Geyer Instructional Aids, Fort Wayne, Indiana.
J. L. Hammett Co., Braintree, Massachusetts.
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School Mathematics 2

1974 Edition

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UNIT E: INTEGERS

Module 1: Numbers, Numerals, and Operations

General Objectives

To review some ancient numeration systems and the decimal numeration system.

To review the basic operations of addition, subtraction, multiplication, and division of whole numbers.

To provide experience in rounding, estimation, and exponential notation.

To compare the decimal numeration system with the binary numeration system.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
1	**RED** Given a decimal numeral of six or fewer places, the student can give the place value of each digit.	E-4, 5	*WB 1	SP a-1	
2	**RED** Given appropriate examples, the student can use exponents to write powers of numbers.	E-10, 11	WB 3 DM 2	DS f-1	
3	**RED** Given a whole number of six digits or less, the student can round the number to a specified place.	E-12, 13	WB 4		
4	**RED** Given problems involving whole numbers and the basic operations, the student can estimate sums, differences, products, or quotients.	E-14, 15	WB 5 DM 3		SWM 1 178, 302
5	**RED** Given problems involving the basic operations, the student can add, subtract, multiply, and divide whole numbers.	E-6, 7, 8, 9 E-16, 17	WB 2, 6 DM 1, 4-7	ASC W-1 through 10 SP a-1, 2, 3, 4 PD a-1, 2, 3, 4	SWM 2 36, 195 SWM 3 14, 28, 57, 162, 230-243, 262, 263
6	**YELLOW** Given a numeral in an ancient numeration system (Roman, Babylonian, Egyptian, or Greek), the student can interpret and write it as a decimal numeral, and conversely.	E-2, 3			
7	**YELLOW** Given a base-two numeral, the student can give an equivalent base-ten numeral, and conversely.	E-18, 19 E-20, 21	WB 7	DS a-3	SWM 1 253 SWM 2 97
8	***GREEN*** *Given two base-two numerals, the student can add, subtract, multiply, or divide using these numerals.*	E-24, 25	DM 8		
	Reviewing the Ideas	E-26	WB 8		

*See Abbreviation Key on page ix.

MATHEMATICS

This module plays an important role in the overall development of the coming year's work. Although most of the material will be review for many students, the emphasis upon the decimal numeration system, place value, basic operations, basic principles for the operations, and exponents form a solid foundation upon which succeeding modules of the text are based.

The decimal numeration system is used almost universally today. The word decimal comes from the Latin word *decem*, meaning ten. Since we group by tens, the base of the system is ten. Using ten as a base, the Hindu-Arabic digits 0–9, and the idea of *place value*, we can write numerals for every whole number. The choice of base ten is quite useful but other bases can be used. A numeration system employing place value can be developed as follows:

1. Choose a base b, where b is a positive integer greater than one.
2. Select the set of digits $\{0, 1, 2, 3, 4, \ldots, (b-2), (b-1)\}$. If b is greater than ten, new symbols for the digits representing numbers greater than nine must be created.

Using the idea of place value, every nonnegative whole number can be represented in the form

$$a_n\, a_{n-1}\, a_{n-2} \cdots a_1\, a_0$$

where a_i is any of the digits of the system. Using expanded notation, this symbol means

$$a_n \cdot b^n + a_{n-1}\, b^{n-1} + a_{n-2}\, b^{n-2} + a_1\, b^1 + a_0\, b^0$$

Thus, $23415_{(ten)}$ in *expanded notation* is

$$2 \cdot 10^4 + 3 \cdot 10^3 + 4 \cdot 10^2 + 1 \cdot 10^1 + 5 \cdot 10^0.$$

However, $23415_{(8)}$ means

$$2 \cdot 8^4 + 3 \cdot 8^3 + 4 \cdot 8^2 + 1 \cdot 8^1 + 5 \cdot 8^0.$$

One of the more important reasons for studying bases other than ten is the insight that can be gained into the decimal system by contrasting it with a nondecimal system. An important by-product is the interest and motivation it may provide for students.

TEACHING THE MODULE

Vocabulary

addend
addition
associative principle
base
base two
binary system
bit
commutative principle
decimal system
digit
distributive principle
division
estimation
expanded notation
exponent
factor
Hindu-Arabic numeral
identity element
multiplication
numeral
one principle
operation
palindrome
period
place value
power
product
rounding
subtraction
zero principle

Since much of the material in this module will be review for most students, careful attention should be given to pacing. In particular, the first lessons involving numeration, place value, and basic operations should be covered quickly. This will give additional time for such developmental topics as exponents, rounding, and estimation.

The material on the binary system should be regarded as optional material. If you prefer, it could be omitted or it could be taught as a "mini-course" at any convenient time during the year.

The Research Projects on page E–27 and the Mathematical Recreation on page E–28 can be utilized to provide interesting extensions of the ideas presented in the regular lessons of the module. Encourage as many of your students as possible to participate in these projects.

Lesson Schedule

A daily lesson is presented on each pair of facing pages of the module. For fast-paced classes, some lessons may be combined so that two lessons can be covered in one day, thus shortening the time requirement. Students working individually may proceed quite rapidly through the module. If optional material is not taught, 8 to 10 days should be sufficient time to cover the salient portions of the module.

Evaluation

As you evaluate students' understanding of the ideas presented in this module, keep in mind that the material covered is predominantly concept-oriented rather than computational. Students should be able to add, subtract, multiply, and divide whole numbers, but the other objectives listed at the beginning of the module should be given equal emphasis.

The review exercises (Reviewing the Ideas) on page E–26 are designed to prepare the student for the self-evaluation test (Test Yourself) on page E–27. Module Achievement Tests designed for use with *School Mathematics* are available from the publisher, or you could use the self-evaluation test as a model for your own test covering the module.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 3, pp. 67, 89, Creative Publications.
Cloudburst, Vol. 5, Nos, 1348, 8318, 8328, Midwest Publications
Computers, Cuisenaire Co.
ESS: Peas and Particles, Webster, McGraw-Hill
Experiments in Mathematics, Stage 1 (patterns), pp. 56–58; Stage 2 (codes and devices), pp. 22–23, 48–51; Stage 3, pp. 54–57, Houghton Mifflin
From Fingers to Computers; Mathematics: Man's Key to Progress, Book B, pp. 21–31, Rand McNally
Let's Play Games in Metrics (games and activities involving exponents), pp. 90–91, Addison-Wesley
Nuffield Project: *Computation and Structure 4,* "Extension of place value," pp. 1–12; Large Numbers and Indices," pp. 50–63; *Computers and Young Children,* Wiley
Powers, Roots and Logarithms, Set 5M, Set 5E, Cards 1–17, Herder and Herder
Sourcebook for Substitutes, pp. 90–91, Addison-Wesley

Manipulative Devices

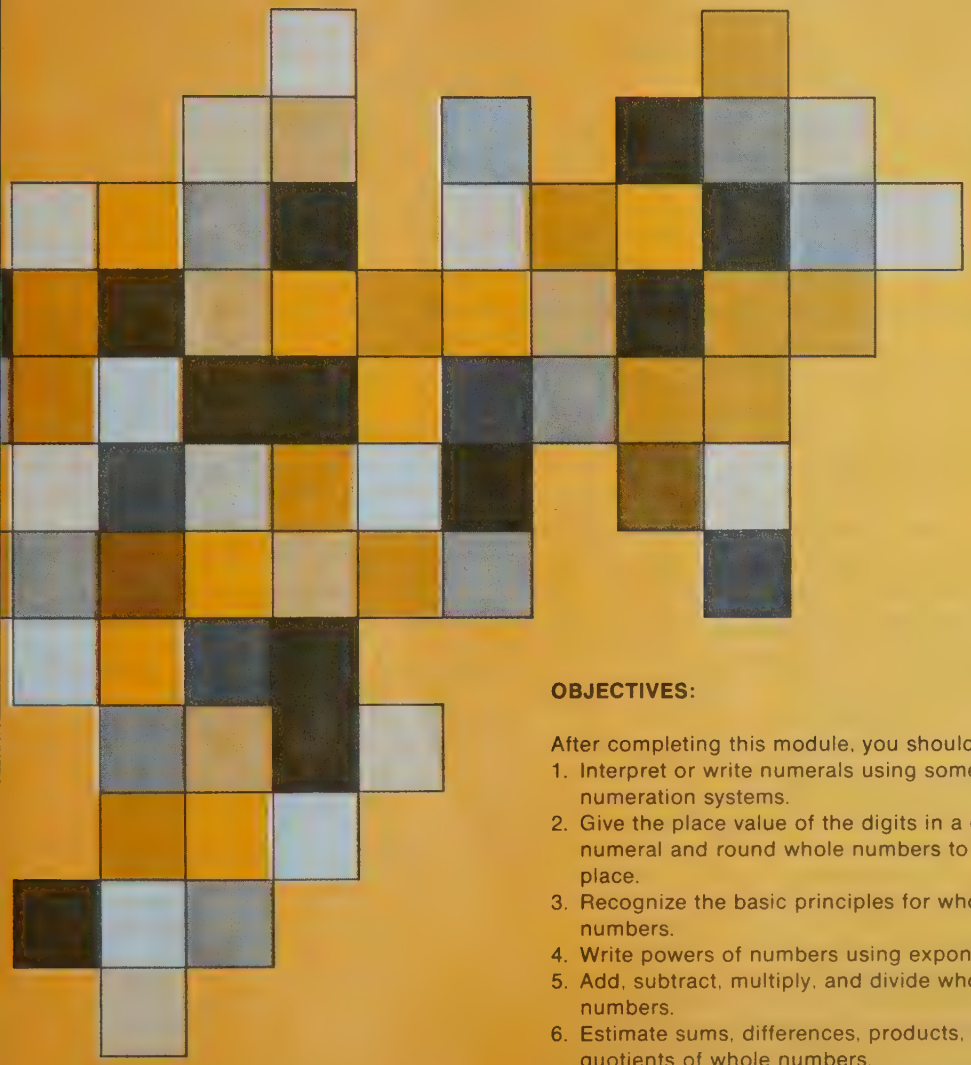
Abacuses—a variety (Creative Publications; Selective Educational Equipment)
Binary Counter, (Creative Publications; Sargent-Welch)
Binary and multi-base materials (Educational Teaching Aids)
Calculators (Cuisenaire Co.; Selective Educational Equipment)
Multi-base Converter (Educational Teaching Aids)
Napiers Bones (Rods) (Creative Publications; Midwest Publications).
Slide rules (Creative Publications; Geyer; Pickett)

Games and Puzzles

In Order Game Kit (Midwest Publications)
Numo (Gamco; Midwest Publications)
Radix (Lang)

UNIT E: *Integers*

MODULE 1: *Numerals, Numbers, and Operations*



OBJECTIVES:

After completing this module, you should be able to:

1. Interpret or write numerals using some ancient numeration systems.
2. Give the place value of the digits in a decimal numeral and round whole numbers to a given place.
3. Recognize the basic principles for whole numbers.
4. Write powers of numbers using exponents.
5. Add, subtract, multiply, and divide whole numbers.
6. Estimate sums, differences, products, and quotients of whole numbers.
7. Write a base-two numeral for a base-ten numeral or convert a base-two numeral to a base-ten numeral.
8. Perform calculations using base-two numerals.

Objective

The student can interpret Egyptian, Babylonian, Greek, and Roman numerals, given a table of symbols for the numeration systems.

PREPARATION

Discuss with the class:

1. Man's need for numbers.
2. The need for symbols (numerals) to record numbers.
3. The fact that different cultures developed different numeration symbols.

If you have posters, pictures, or other materials concerning ancient numeration systems, display these in your classroom.

INVESTIGATION

The Investigation involves the students in some simple activities that utilize the information in the table. Most students will have had some experience with Roman numerals. You should not expect everyone to complete all parts of the Investigation correctly. Corrections can be made as the material is discussed, following the Investigation.

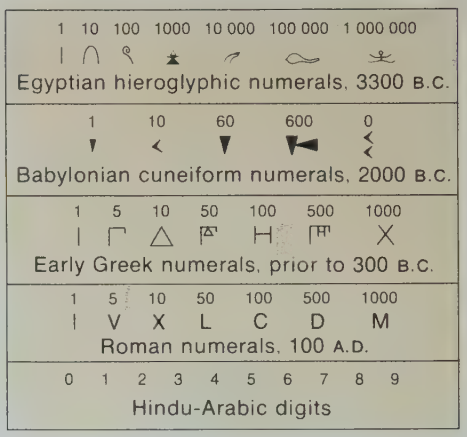
DISCUSSION

The emphasis of the Discussion should be on the fact that the Egyptian and Greek numeration systems were *additive* systems. They can be contrasted to the Roman numeration system, which has *subtractive* as well as additive features. The Babylonians utilized a place-value system based on 60. This base is supposed to have been chosen because the Babylonian year contained 360 days; they divided a circle in 360 parts. However, the numerals they used are relatively awkward, and we would find computation with them difficult.

Ancient Numeration Systems

Investigating the Ideas

Symbols for numbers are called **numerals**. Some ancient numerals as well as the present-day Hindu-Arabic numerals are given in the table at the right.


1 10 100 1000 10 000 100 000 1 000 000
Egyptian hieroglyphic numerals, 3300 B.C.
1 10 60 600 0
Babylonian cuneiform numerals, 2000 B.C.
1 5 10 50 100 500 1000
Early Greek numerals, prior to 300 B.C.
1 5 10 50 100 500 1000
I V X L C D M
Roman numerals, 100 A.D.
0 1 2 3 4 5 6 7 8 9
Hindu-Arabic digits

How many of these questions can you answer by writing the numeral indicated?

A What is your age in years? (Egyptian)

B What is your height in centimetres? (Babylonian)

C What is the number of people in your classroom? (Greek)

D What is the year of your birth? (Roman)

Discussing the Ideas

1. The Egyptian numeration system was an **additive system**. The numbers for the symbols were simply added together. For 23, they might have written 𓎆𓍢𓍢 or 𓍢𓍢𓎆 or 𓍢𓍢 . What are some other numerals you might have written for your age for question **A** above?

2. The Babylonians wrote in clay with a **stylus**. The figure shows how they might have represented 172. (For the numbers less than 60, the values of the numerals were added.) What Babylonian symbol would represent 215?

3. The Greek system did not use place value either. Like the Egyptian system, it was an additive system. What number would 𐅄𐅄𐅃𐅃𐅃𐅂𐅁𐅁 represent?

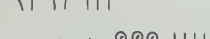
4. The early Roman system was also an additive system. Later it also had some **subtractive** phases. Thus nine, once written as VIIII, became IX. What is the Roman numeral for the present year?

Using the Ideas

1. Write the Hindu-Arabic numeral for each Egyptian numeral.

A

B

C 

D

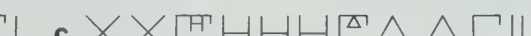
E

F

2. Write the Hindu-Arabic numeral for each Babylonian numeral.

A **B** **C** **D**

3. Write the Hindu-Arabic numeral for each Greek numeral.

A **B** **C** 

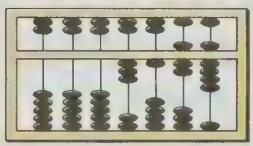
4. Write the Hindu-Arabic numeral for each Roman numeral.

A V	**D** X	**G** LX	**J** CD	**M** MCLIX
B VI	**E** XI	**H** XL	**K** MC	**N** CLXXVIII
C IV	**F** IX	**I** DC	**L** CM	**O** MCMLXX

5. Write the numeral 247 using

A Egyptian hieroglyphics. **B** Greek numerals. **C** Roman numerals.

6. The Romans often used an **abacus** to calculate and then recorded their answers using numerals. The answer to a problem is shown on the abacus. Record the answer using Roman numerals.

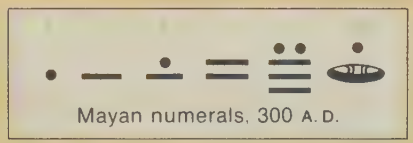


think

The Mayan Indians of Yucatan used a system based on 20 and wrote their numerals vertically.

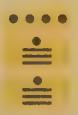
The diagram at the right shows how the Mayans utilized place value. Because they were interested in astronomy and used 360 for the number of days in a year, they used 18 · 20 in the third place instead of 20 · 20. The symbol was used as a zero to show an empty place.

This Mayan numeral represents a famous historical date. What is it?



Mayan numerals, 300 A.D.

· 18 · 20 · 20 = 7200
· 18 · 20 = 2160
· 20 = 60
= 12



UTILIZATION

Since the purpose of the lesson is basically to stimulate interest and provide some historical background in numeration systems, it is not necessary to emphasize skill in working with any of the ancient numeration systems. In fact, you might wish to treat several of these exercises as an extension of the Discussion Exercises on page E-2. Such a discussion may heighten student interest considerably.

EXTENSION

This lesson provides good opportunities for student activities and projects. Students interested in art could be encouraged to create an "Egyptian papyrus manuscript" inscribed with hieroglyphics. Other students might use modelling clay and a wedge-shaped stylus to make Babylonian "clay tablets." Bulletin board displays showing early schemes for writing numerals would add to the interest and help begin the year in a stimulating surrounding.

The Thirty-first Yearbook of the National Council of Teachers of Mathematics, *Historical Topics for the Mathematics Classroom* (NCTM, Washington, D.C., 1969), is an excellent source of reference in mathematical history. Standard histories of mathematics could also be utilized.

Think Solution

← 4 · 18 · 20 = 1440

← 16 · 20 = 320

← 16 = $\frac{16}{1776}$

Assignments

Minimum 1–4. Average 1–5. Maximum 1–6.

Objective

Given a decimal numeral, the student can give the meaning of the digits in each place and name the periods in a numeral.

PREPARATION

Provide a small piece of tracing paper for students to use in the Investigation.

This lesson reviews the basic ideas of the decimal system of numeration. Because of student familiarity with this topic you will want to progress quickly through the material. With good students you may want to cover this and the next lesson on the same day.

INVESTIGATION

The purpose of the Investigation is to point out the ease of counting sets of objects which are grouped by tens as opposed to sets which are not grouped. The groups of ten dots in set A are easily counted. Allow students to use any method they wish to count the dots in set B. One possible method would be to place a piece of tracing paper over set B and ring sets of ten dots.

DISCUSSION

This lesson is important because work with the basic operations is based primarily on a clear understanding of place-value concepts. Also, in working with rational numbers in decimal notation, students must have a very clear understanding of the place-value concepts associated with the whole numbers.

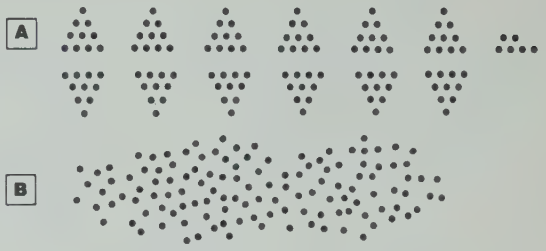
As preparation for the exercises on page E-5, it might be helpful to give your students some experience in writing numerals in expanded notation. For example: $3256 = 3000 + 200 + 50 + 6$.

Decimal Numerals

Investigating the Ideas

We use **whole numbers** {0, 1, 2, 3, 4, 5, 6, . . .} when we count. We use Hindu-Arabic digits to record our counting. Counting is the most basic aspect of mathematics.

Can you count and record the number of dots that are in set A and in set B?

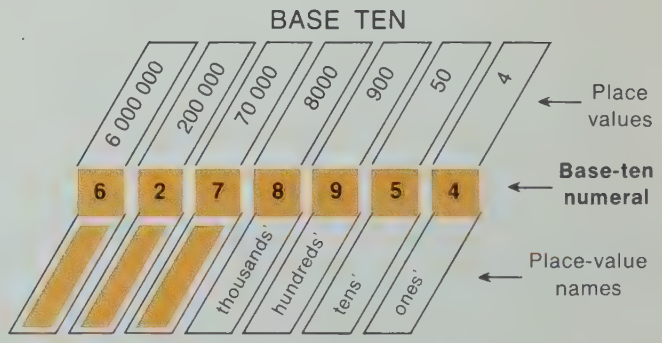


Discussing the Ideas

1. A How many dots are in set A?
   B Why are they easy to count?

2. A How many dots did you find in set B?
   B Can you describe a way to make the counting easier?

3. Our system of writing numerals to represent numbers is called the **decimal system**. The word decimal comes from the Latin word **decem** which means **ten**. By using ten as a base, the idea of **place value**, and the ten Hindu-Arabic digits, we can write numerals for every whole number.



   A Give the names of the next three places beyond the thousands' place in the base-ten diagram above.
   B In the numeral in the diagram, the three digits 2, 7, and 8 tell how many thousands and thus make up the thousands' **period**. Give the names of as many periods beyond the thousands' period as you can.
   C Read the base-ten numeral shown in the diagram.

1. Give the missing number for each ▒.

   A In 6287, the 2 means 200.

   B In 72 964, the 9 means ▒.

   C In 84 760, the 4 means ▒.

   D In 684 219, the 6 means ▒.

   E In 65 728 134, the 5 means ▒.

2. Write a base-ten numeral for each sum.

   A $800 + 70 + 4$

   B $3000 + 600 + 90 + 1$

   C $20\,000 + 8000 + 400 + 30 + 7$

   D $100\,000 + 100 + 1$

   E $60\,000 + 7000 + 80$

   F $3\,000\,000 + 700\,000$

3. The names for the first 12 periods are given in the table.

PERIOD	NAME	PERIOD	NAME	PERIOD	NAME
1st	units, tens, hundreds	5th	trillions	9th	septillions
2nd	thousands	6th	quadrillions	10th	octillions
3rd	millions	7th	quintillions	11th	nonillions
4th	billions	8th	sextillions	12th	decillions

   Use the information in the table to write the Hindu-Arabic numeral for each number name.

   EXAMPLE: 3 trillion ANSWER: 3 000 000 000 000 (5 periods)

   A 6 quadrillion

   B 73 billion

   C 276 septillion

   D 1 decillion

   E 439 octillion

   F 89 million

4. The value of the gross national product in the United States for a certain year was one **trillion dollars**. Write a base-ten numeral for the amount.

think

Suppose that in a certain newspaper the total number of digits used to number the pages of the newspaper was 111. How many pages were in the newspaper?

UTILIZATION

The exercises can be assigned as written work or discussed orally.

Students should be able to name the periods through the billions. Other periods above billions occur infrequently and are given in Exercise 3 for the interest they may generate. Extremely large numbers are often represented in scientific notation.

EXTENSION

Further practice exercises are provided on page 1 of the *Workbook*.

Remedial: If some students have difficulty in reading or writing numerals, some additional practice in this should be provided. Give some number names orally and see if the student can write the correct numeral.

Enrichment: Some students may be interested in doing research concerning the names of the periods of numbers beyond those given in the table in Exercise 3. An excellent reference is Philip J. Davis' *Lore of Large Numbers* (New Haven, Conn.: Yale University Press 1961, page 23). Also, many standard dictionaries include number tables which give the names of the periods beyond decillions.

Think Solution

The newspaper contains 60 pages.

The first 9 pages contain 9 single digits, and $111 - 9 = 102$. Since two digits would be needed for each page between 9 and 100, the newspaper must have $9 + \frac{102}{2}$, or 60 pages in all.

Assignments

Minimum 1, oral; 2; 3. Average 1 and 2 oral; 3. Maximum 1 and 2 oral; 3; 4.

Objectives

The student can identify addends and sums in addition problems, and factors and products in multiplication problems.

The student can demonstrate an understanding of the inverse relationship between addition and subtraction, and multiplication and division.

PREPARATION

The main goal of the lesson is to emphasize the relationship of the addition and multiplication operations and the subtraction and division operations. The function-machine approach to these ideas should provide some novelty for students.

Students who have used the *Investigating School Mathematics* series before will be familiar with the construction and use of the function machine. However, you should check to ascertain that everyone understands the use of the function machine and the meaning of the terms *input*, *function rule*, and *output*.

INVESTIGATION

The Investigation questions involve students in finding sets of related *addends* and *sums*, and *factors* and *products*. Since we are concerned only with pairs of numbers whose sum is 15 or pairs of numbers whose product is 36, the order of the numbers in each pair is unimportant.

There are 8 unordered pairs of whole numbers whose sum is 15 (16 ordered pairs), and there are 5 unordered pairs of whole numbers whose product is 36 (9 ordered pairs).

DISCUSSION

Exercises 1 and 2 emphasize the fact that finding a missing addend in an addition equation is equivalent to finding a difference in the corresponding subtraction equation.

Exercises 3 and 4 emphasize that a similar relationship exists between missing factors and quotients.

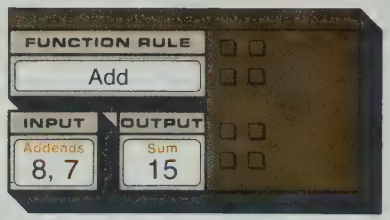
Investigating the Ideas

The **function machines** will help you to review the operations of addition and multiplication for whole numbers.

For each input pair (addends), this **addition** function machine produces a unique output number (sum).

A A S

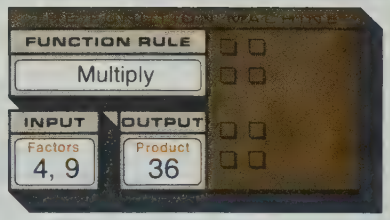
$8 + 7 = 15$



For each input pair (factors), this **multiplication** function machine produces a unique output number (product).

P P F

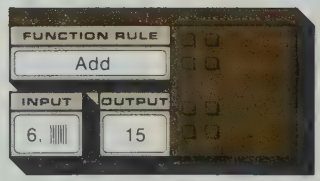
$4 \cdot 9 = 36$



How many different pairs of addends can you find to have a sum of 15? How many different pairs of factors can you find to have a product of 36?

Discussing the Ideas

1. What is the missing addend for this function machine?



2. You can think of **subtraction** as an operation in which you know the sum (S) and one addend (A) and find the other addend. What is the missing addend ***n***?

S A A
$15 - 6 = n$

A A S
$? + 6 = 15$

3. What is the missing factor for this function machine?

4. You can think of **division** as an operation in which you know the product (P) and one factor (F) and find the other factor (F). What is the missing factor ***n***?

P F F
$36 \div 6 = n$

F F P
$? \cdot 6 = 36$

Using the Ideas

1. These function tables show a way of recording input and output numbers for function machines. Complete the tables.

	Function Rule: Add	
	input	output
A	(8, 6)	
B	(9, 4)	
C	(19, 6)	
D	(50, 30)	
E	(99, 101)	
F	(324, 976)	

	Function Rule: Add	
	input	output
G	(▒, 5)	13
H	(7, ▒)	12
I	(▒, 16)	23
J	(14, ▒)	14
K	(▒, 48)	56
L	(186, ▒)	324

	Function Rule: Multiply	
	input	output
M	(5, 7)	
N	(9, 8)	
O	(6, 0)	
P	(9, ▒)	9
Q	(▒, 40)	120
R	(47, ▒)	376

2. Think about finding the missing addend or factor. Then solve each subtraction and division equation.

A $? + 7 = 16$
$16 - 7 = n$

B $? \cdot 9 = 63$
$63 \div 9 = r$

C $? + 29 = 32$
$32 - 29 = c$

D $270 \div 30 = t$

E $105 - 98 = d$

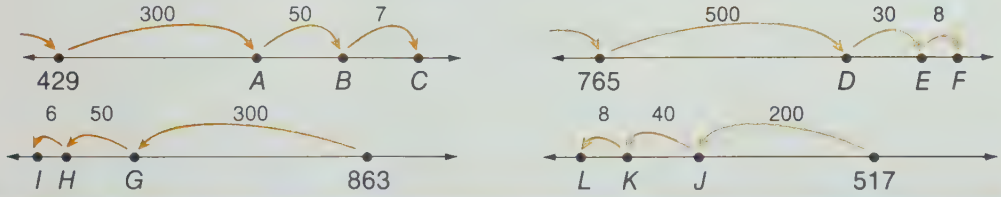
F $3600 \div 40 = w$

G $1000 - 498 = x$

H $10^3 \div 10^2 = s$

I $2268 \div 36 = y$

3. Give the number for the point above each letter.



think

Which of these figures can be traced without lifting your pencil from the paper or retracing?

There is a way to answer this question without tracing. Can you discover it?

UTILIZATION

Depending on the ability of your class, you can assign these exercises as individual written work or use them for group discussion.

EXTENSION

The exercises provided on *Workbook* page 2 and *Duplicator Masters* page 1 would be appropriate for additional practice with the ideas presented in this lesson.

Enrichment: Provide some completed function rule tables, omitting the function rule. Have the students determine the rule by inspection.

Function Rule	
input	output
(9, 6)	3
(15, 5)	10
(37, 19)	18
(100, 54)	46

Function Rule	
input	output
(4, 7)	28
(5, 6)	30
(1, 1)	1
(12, 9)	108

Function Rule	
input	output
(2, 2)	4
(7, 17)	24
(11, 5)	16
(23, 77)	100

Function Rule	
input	output
(30, 3)	10
(42, 3)	14
(69, 23)	3
(17, 1)	17

Think Solution

A principle of the branch of mathematics known as "topology" may help students solve this Think problem: When more than two vertices are formed by an odd number of line segments, the figure is impossible to trace. If the figure contains exactly two vertices formed by an odd number of line segments or if all vertices are formed by an even number of segments, the tracing is possible.

In figure C, four vertices are formed by two or four (even) line segments. Two vertices are formed by three (odd) line segments. Thus, the tracing is possible. (The figure has only two vertices with an odd number of line segments.)

Assignments

Minimum 1, written; 2 and 3, oral.
Average 1, written; 2 and 3, oral.
Maximum 1–3, oral.

Objective

The student can identify the basic principles for addition and multiplication of whole numbers.

PREPARATION

Pages E-8 and E-9 present a departure from the usual Investigation format. Both pages may be covered as one lesson, unless special emphasis is placed on some portions of this material.

DISCUSSION AND UTILIZATION

Page E-8 provides a review of the basic principles for whole numbers.

Have the students study the chart at the top of the page and then complete the exercises that follow it. The answers for these questions are inverted at the bottom of the page for self-checking by the students.

Afterwards, discuss the significant ideas related to the exercises. For example, you should point out that the most valuable use of the commutative and associative principles for addition (multiplication) occurs when we use them in combination to change both the order and the grouping of addends (factors). Thus, the sum $7 + (9 + 3)$ can be written as $(7 + 3) + 9$. You will want to stress the fact that the distributive principle ties the operations of addition and multiplication together. Provide examples such as "Find the product of $9 \cdot 16$." This can be written as $9 \cdot 10 + 9 \cdot 6$

Since these principles will be used in later chapters with integers, rational numbers, and real numbers, emphasis on them at this point will make their use more meaningful.

Assignments

Minimum 1–7, oral.
Average 1–6, oral; 7.
Maximum 1–7.

A Review of Basic Principles

The basic principles for addition and multiplication can be used to explain the computational rules used for the whole numbers. In the table below, the letters a, b, and c refer to any whole number.

Addition	Multiplication
Commutative principle: $a + b = b + a$	Commutative principle: $a \cdot b = b \cdot a$
Associative principle: $a + (b + c) = (a + b) + c$	Associative principle: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Zero principle: $a + 0 = a$	One principle: $a \cdot 1 = a$
Distributive principle: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	

Answer each question below. Then check your answers with the inverted answers at the bottom of the page.

1. Does the commutative principle for addition deal with **order** of addends or with **grouping** of addends?

2. Does the associative principle for multiplication deal with **order** of factors or with **grouping** of factors?

3. The number zero is called the **identity element** for addition. What is the identity element for multiplication?

4. Which principle involves both addition and multiplication?

5. If the answer to **A** is correct, what is the answer to **B**?

6. If the answer to **C** is correct, what is the product for **D**?

A $47 + 86 + 92 + 13 = 238$
B $92 + 47 + 13 + 86 =$
C $(42 \cdot 69) \cdot 37 = 107\,226$
D $(69 \cdot 37) \cdot 42 =$

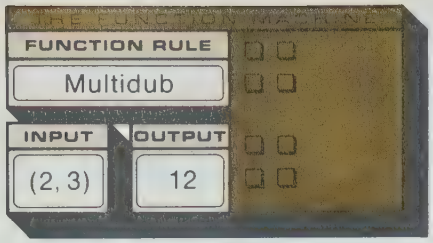
7. Give the number for n and the name of the principle illustrated.

A $89 + n = 46 + 89$
B $97 \cdot 42 = n \cdot 97$
C $n + 0 = 12$
D $9 \cdot (8 \cdot 10) = (9 \cdot 8) \cdot n$
E $57 \cdot 1 = n$
F $(8 \cdot 30) + (8 \cdot 4) = 8 \cdot n$

ANSWERS: 1. order 2. grouping 3. 1 4. distributive 5. 238
6. 107 226 7. A 46, commutative (+); B 42, commutative (×); C 12, zero;
D 10, associative (×); E 57, one; F 34, distributive

Some Special Function Machine Rules

Each function table below shows an interesting operation made up by various students. Complete each table.



1.

Function Rule: Multidub		
	Input	Output
	(2, 3)	12
A	(3, 4)	
B	(5, 12)	
C	(10, 10)	

2.

Function Rule: Doubleadd		
	Input	Output
	(2, 5)	14
A	(4, 5)	
B	(10, 7)	
C	(25, 20)	

3.

Function Rule: Advide 2		
	Input	Output
	(8, 4)	6
A	(5, 7)	
B	(4, 36)	
C	(17, 39)	

4.

Function Rule: Dubfirst, add		
	Input	Output
	(2, 1)	5
	(10, 9)	29
A	(50, 10)	
B	(37, 8)	
C	(98, 4)	

5.

Function Rule: Greaterplus diff		
	Input	Output
	(3, 2)	4
	(100, 2)	198
A	(8, 5)	
B	(25, 20)	
C	(500, 200)	

6.

Function Rule: Admul 3		
	Input	Output
	(2, 3)	15
	(1, 7)	24
A	(3, 4)	
B	(9, 6)	
C	(3, 10)	

★ **7.** Study the function table. Then give a name for the function rule.

Function Rule: [redacted]	
Input	Output
(1, 1)	2
(1, 2)	9
(2, 2)	16
(2, 3)	35

think

Find n and p. Then explain the pattern shown.

1. $3 \cdot 5 = (n \cdot n) - 1 = p$
2. $4 \cdot 6 = (n \cdot n) - 1 = p$
3. $5 \cdot 7 = (n \cdot n) - 1 = p$
4. $9 \cdot 11 = (n \cdot n) - 1 = p$
5. $19 \cdot 21 = (n \cdot n) - 1 = p$
6. $99 \cdot 101 = (n \cdot n) - 1 = p$

Objective

The student can complete a function rule table when given a rule involving a combination of operations.

DISCUSSION AND UTILIZATION

These exercises are designed to provide a stimulating challenge to students. Therefore you may not want to assign the exercises to everyone in your group.

EXTENSION

Enrichment: Have students create their own function rules involving a combination of operations and devise tables without the rule stated (as in Exercise 7). Students could then challenge each other to find the missing function rules.

Think Solution

1. $n = 4$, $p = 15$
2. $n = 5$, $p = 24$
3. $n = 6$, $p = 35$
4. $n = 10$, $p = 99$
5. $n = 20$, $p = 399$
6. $n = 100$, $p = 9999$

The product of any two consecutive odd numbers (or even numbers) is 1 less than the square of the number between them.

Assignments

Minimum 1–3. Average 1–6. Maximum 1–7.

Objective

The student can express appropriate products using exponents and can express a given power as an ordinary numeral.

PREPARATION

A cardboard or wooden model of the 8 by 8 by 8 block building shown in the Investigation may be useful for demonstration and discussion.

INVESTIGATION

The Investigation will require only a short time to complete. Most students will readily see that finding the number of rooms involves computing the product $8 \cdot 8 \cdot 8$. The lesson involves learning to express such products using exponents.

DISCUSSION

Exercise 1 should guide the students in finding some ways of counting the number of rooms in the building. Volume of rectangular parallelepipeds in general need not be discussed.

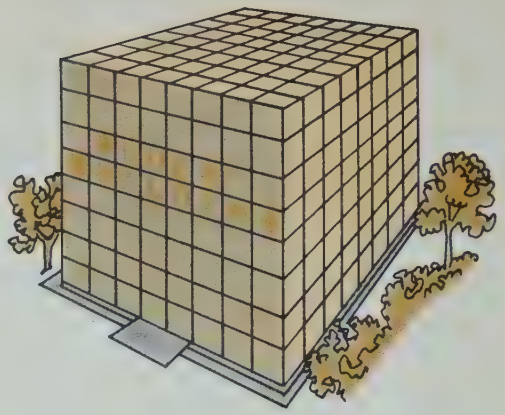
In discussing Exercise 4, encourage students to observe that to find the area of a square 4 units on a side, they would use 4 as a factor 2 times, which can also be expressed by the notation 4^2 ("4 squared"); to find the volume of a cube 4 units on a side, they would use 4 as a factor 3 times, which can be represented by 4^3 ("4 cubed").

In Exercise 5 students should be able to generalize the fact that 10^n, where n is a nonzero whole number, can be written as 1 followed by n zeros.

Exponents

Investigating the Ideas

Each small cube represents a room in the Block Building.



? How many rooms are in the Block Building?

Discussing the Ideas

1. A How many rooms are in each row?
   B How many rooms are on each floor (or story) of the building?
   C How many floors (stories) are in the building?
   D Explain how you found the number of rooms in the building.

2. When the same number is used as a **factor** several times, you can use an **exponent** to show how many times the factor **(base)** is used.

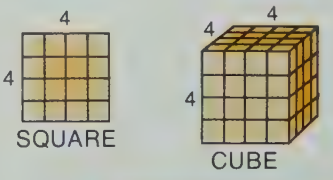
   A How many times is 10 used as a factor in 10^5?

   Exponent → $10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100\,000$ ← Base

   B You can read 10^5 as "ten to the fifth power." How would you read 6^4? 5^8? 9^7?

3. Express the number of rooms in the Block Building using an exponent.

4. The process of finding the area of a square and the volume of a cube suggests a reason why 4^2 is called "four squared" and 4^3 is called "four cubed." Give this reason.



5. You can discover a rule for writing the decimal numeral for any **power of ten**. Explain this rule.

Power of ten		Numeral
10^1	=	10
10^2	=	100
10^3	=	1000
10^4	=	10 000
⋮		⋮

Using the Ideas

1. Write each product using an exponent.

A $7 \cdot 7$ **D** $10 \cdot 10 \cdot 10 \cdot 10$ **G** $5 \cdot 5$

B $3 \cdot 3 \cdot 3$ **E** $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$ **H** $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

C $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ **F** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ **I** $4 \cdot 4 \cdot 4$

2. Tell how many times ten is used as a factor in each part.

A 10^3 **B** 10^2 **C** 10^6 **D** 10^7 **E** 10^{19} **F** 10^{50} **G** 10^{1000}

3. Give the usual numeral. EXAMPLE: $10^2 = 100$

A 10^3 **B** 10^5 **C** 10^6 **D** 10^8 **E** 10^9 **F** 10^{12} **G** 10^1

4. Write the product as a power of ten.

EXAMPLE: $10^2 \cdot 10^1 = (10 \cdot 10) \cdot 10 = 10^3$

A $10^3 \cdot 10^1$ **D** $10^2 \cdot 10^2$ **G** $10^6 \cdot 10^2$

B $10^2 \cdot 10^3$ **E** $10^3 \cdot 10^4$ **H** $10^4 \cdot 10^5$

C $10^1 \cdot 10^4$ **F** $10^5 \cdot 10^1$ **I** $10^5 \cdot 10^8$

Think

The number 10^{100} (1 followed by 100 zeros) is called a **googol**. Which is larger, a googol or 2^{1000}?

5. Write the quotient as a power of ten.

EXAMPLE: $10^4 \div 10^2 = \frac{10\,000}{100} = 100 = 10^2$

A $10^5 \div 10^3$ **B** $10^6 \div 10^3$ **C** $10^3 \div 10^2$ **D** $10^7 \div 10^4$

6. Give the number for each ▨.

A $10^4 \div 10^2 = 10^{4-▨}$ **B** $10^4 \div 10^3 = 10^{4-▨}$ **C** $10^4 \div 10^4 = 10^{4-▨}$

7. Since $10^4 \div 10^4 = \frac{10\,000}{10\,000} = 1$ and $10^4 \div 10^4 = 10^{4-4} = 10^0$,

we define $10^0 = 1$. Similarly, $7^0 = 1$ and $3^0 = 1$.

In general, if n is a non-zero number, then $n^0 = 1$.

The symbol 0^0 is undefined.

Which part is **not** a symbol for 1?

A 8^0 **B** $10^3 \div 10^3$ **C** 1^2 **D** $2^1 \cdot 2^0$

8. Base-ten numerals can be written in **expanded notation**.

EXAMPLE: $3764 = (3 \cdot 10^3) + (7 \cdot 10^2) + (6 \cdot 10^1) + (4 \cdot 10^0)$

Write each base-ten numeral in expanded notation.

A 27 **B** 473 **C** 507 **D** 9625 **E** 87 034 **F** 4 632 470

More practice, page S-2, Set 1

UTILIZATION

Exercise 7 defines n^0 as 1. It may be useful to spend some time discussing this definition and considering examples. The division example of this exercise can be illustrated with bases other than ten.

Exercise 8 illustrates how the powers of ten are utilized in *expanded notation*.

Answers, Exercise 8

A $(2 \cdot 10^1) + (7 \cdot 10^0)$

B $(4 \cdot 10^2) + (7 \cdot 10^1) + (3 \cdot 10^0)$

C $(5 \cdot 10^2) + (0 \cdot 10^1) + (7 \cdot 10^0)$

D $(9 \cdot 10^3) + (6 \cdot 10^2) + (2 \cdot 10^1) + (5 \cdot 10^0)$

E $(8 \cdot 10^4) + (7 \cdot 10^3) + (0 \cdot 10^2) + (3 \cdot 10^1) + (4 \cdot 10^0)$

F $(4 \cdot 10^6) + (6 \cdot 10^5) + (3 \cdot 10^4) + (2 \cdot 10^3) + (4 \cdot 10^2) + (7 \cdot 10^1) + (0 \cdot 10^0)$

EXTENSION

Appropriate exercises for additional practice may be assigned from Supplementary Exercise Set 1 on page S-2, *Workbook* page 3, and *Duplicator Masters* page 2.

Enrichment: Some students may be interested in the laws of exponents for whole numbers.

Multiplication Law: $a^x \cdot a^y = a^{x+y}$

Division Law: (A) $a^x \div a^y = a^{x-y}$, $x \geq y$

(B) $a^x \div a^y = \frac{1}{a^{y-x}}$, $y > x$

Power of a Power Law: $(a^x)^y = a^{xy}$

Power of a Product Law: $(a \cdot b)^x = a^x \cdot b^x$

Students can give numerical examples illustrating each law. Notice that since we are restricted to non-negative exponents at this time, the division law has two cases.

Think Solution

$2^{10} = 1024 \approx 1000$

$100 = 10^3$

$2^{1000} = (2^{10})^{100} \approx (10^3)^{100} \approx 10^{300}$

Therefore $10^{100} = \text{googol} < 2^{1000}$

Assignments

Minimum 1–3, oral; 4–7.

Average 1–3, oral; 4–7.

Maximum 1–3, oral; 4–8.

Objective

The student can round whole numbers to a specified place.

PREPARATION

To introduce the lesson, conduct a short discussion of exact data contrasted to approximate data. The population of Canada from a recent census is an example of a large exact number. The circumference of the Earth at the Equator, which is often given as 40 000 km, represents approximate data derived from rounding more precise data.

INVESTIGATION

The middle number in each given group lies between two successive multiples of some power of ten. The student must find which of the two multiples is nearer to the middle number. This is the basic idea that underlies the rounding process.

DISCUSSION

Rounding plays a key role in estimating and estimation plays a valuable role in problem solving. Therefore the rounding process should be reviewed carefully, even though most of the students are already familiar with the process.

Since the following lesson uses rounding in estimation problems, you may want to cover both lessons in one day.

Exercise 5 points out the agreement to round "up" for numbers halfway between two multiples of 10. This agreement is, of course, an arbitrary one, but it is one that we shall consistently use.

Rounding Whole Numbers

Investigating the Ideas

The middle number of each set of three numbers is always between the other two.

A

280	
278	
270	

C

47 000	
46 083	
46 000	

E

2 500 000	
2 439 637	
2 400 000	

B

3200	
3192	
3100	

D

760 000	
755 298	
750 000	

F

54 000 000	
53 644 139	
53 000 000	

Can you find which of the other two numbers is closer to the middle number in each set?

Discussing the Ideas

1. Because 278 is nearer to 280 than to 270, you can say "278 **rounded to the nearest ten** is 280."
   To round 278 to the nearest **hundred**, you must decide whether 278 is nearer to 200 or 300.
   What is 278 rounded to the nearest hundred?

2. In part **B** above, what is 3192 when rounded to the nearest hundred?

3. Describe the rounding for each of the parts **C** through **F** above.

4. Why do you think it is helpful to be able to round numbers?

5. The number 2500 is halfway between 2000 and 3000.
   In cases such as this, we agree to round "up."
   What is 2500 rounded to the nearest thousand?

6. **A** Estimate the number of students that attend your school.
   B Find the exact number of students in your school and round it to the nearest hundred.
   C How close was your estimate to the rounded number for part **B**?

Using the Ideas

1. **A** Is 6287 nearer to 6000 or 7000?
   B What is 6287 when rounded to the nearest thousand?

2. Choose the number that best completes the sentence.
   A 2 857 456 rounded to the nearest ten is $\begin{cases} 2\,857\,450. \\ 2\,857\,460. \end{cases}$
   B 2 857 456 rounded to the nearest hundred is $\begin{cases} 2\,857\,400. \\ 2\,857\,500. \end{cases}$
   C 2 857 456 rounded to the nearest thousand is $\begin{cases} 2\,857\,000. \\ 2\,858\,000. \end{cases}$
   D 2 857 456 rounded to the nearest ten thousand is $\begin{cases} 2\,850\,000. \\ 2\,860\,000. \end{cases}$
   E 2 857 456 rounded to the nearest hundred thousand is $\begin{cases} 2\,800\,000. \\ 2\,900\,000. \end{cases}$

3. Round each number in the table to the nearest thousand.

	PROVINCE	SCHOOL ENROLLMENT
A	Alberta	437 150
B	Ontario	2 083 951
C	Quebec	1 639 335
D	P.E.I.	30 648

4. The area of Canada is 9 976 185 square kilometres.
   A Round this number to the nearest thousand.
   B Round this number to the nearest million.

5. The highest active volcano is Guallatiri in Chile. It is 6060 metres high.
   A Round the height to the nearest hundred metres.
   B Round the height to the nearest thousand metres.

6. The population of Mexico City in 1970 was 8 541 070.
   A To the nearest hundred thousand, what was the population?
   B To the nearest million, what was the population?

think

The ones' digit of a 3-digit numeral is three times as large as the tens' digit. The hundreds' digit is twice as large as the tens' digit. What is the number if it is greater than 500?

More practice, page S-2, Set 2

UTILIZATION

The exercises on this page can be used for oral discussion. If you wish, you might ask students to round the numbers to places other than those suggested by the text.

EXTENSION

Supplementary Exercise Set 2 on page S-2 and *Workbook* page 4 may be assigned to strengthen the skills of students who would benefit from further practice.

Enrichment: Some students might be challenged to find exact and rounded numbers that occur in the daily newspapers. Scrapbooks, posters, or bulletin board displays could be used to show their findings.

Think Solution

The number is 639. Students can solve the problem by trial-and-error methods. An analysis of the problem follows:

Let d = tens' digits. Then the number, in expanded form, is

$$100 \cdot 2d + 10 \cdot d + 3d \cdot$$

The ones' digit is $3d$ and $3d \leq 9$.
Hence d is 1, 2, or 3. The table shows the number resulting from each choice for d.

	100's digit	10's digit	1's digit	Number
$d = 1$	2	1	3	213
$d = 2$	4	2	6	426
$d = 3$	6	3	9	639

Since the number must be greater than 500, only $d = 3$ satisfies this condition.

Assignments

Minimum 1–6 oral. Average 1–3, oral; 4–6. Maximum 1–6.

Objective

Given problems involving addition, subtraction, multiplication, and division, the student can round the numbers and estimate the answer.

PREPARATION

You may want to discuss estimation before beginning the Investigation. If you do, use examples such as those in Discussion Exercise 4 to avoid preempting the Investigation. Some mental computation using multiples of 10, 100, or 1000 will also help prepare the student for estimating.

INVESTIGATION

Students should choose one or more of the estimation problems. Allow sufficient time to discuss the methods used in making the estimates. It is quite likely that the Investigation will spur some students to seek exact answers rather than estimating answers. Encourage an estimated answer first, and then compare it with the computed answer if such an answer is possible.

Sample answers:

A. For 29 students, each about 13 years old, the total of the ages can be estimated as 30×13, or 390 years.

B. For a 14-year old student, the estimate can be made by finding 10×400 or 15×400, which gives 4000 to 6000 days.

C. For 50-cm steps: 2 km ÷ 50 cm =
200 000 cm ÷ 50 cm = 4000 steps
For 80-cm steps: 2 km ÷ 80 cm =
200 000 cm ÷ 80 cm = 2500 steps

DISCUSSION

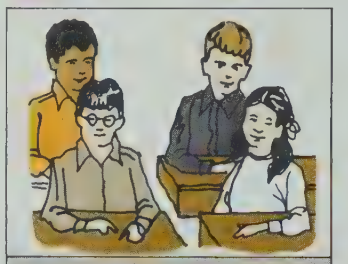
Discuss the role of rounding and mental computation in making estimates. Thus, to estimate the product 98 x 743, we could round each number to the nearest hundred and mentally compute 100 x 700 to get 70 000 as an estimate for the problem. Also, since 98 is very near 100 and 743 is near 740, we could mentally compute 100 x 740 to get 74 000, which is a better estimate.

Estimation

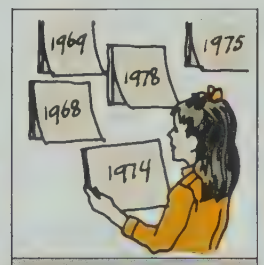
Investigating the Ideas

One of the more useful skills in mathematics is the ability to make a **reasonable estimate** of the answer to a problem before actually computing the exact answer to the problem.

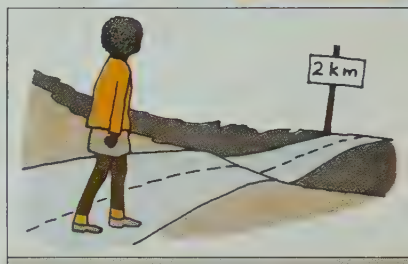
Can you make estimates to one or more of these problems?



A What is the total of the ages of all the students in your math class?



B How many days have you lived?



C About how many steps would you take in walking a distance of two kilometres?

Discussing the Ideas

1. In Bob's eighth grade math class there are 29 students. Bob is 13 years old.
   - **A** Is 300 too low an estimate for the total age of all the students?
   - **B** Is 600 too high an estimate?
   - **C** How would you make an estimate of the total age of all the students in Bob's class?

2. Explain your estimate for Investigation question **B**.

3. Explain how you made your estimate for Investigation question **C**.

4. To make estimates, you can round the numbers to the nearest multiple of 10, 100, or 1000.
   Explain how to round the numbers and estimate the answers to each of these problems.
   A $219 + 398 + 134 + 372$ **B** $63 \cdot 589$ **C** $39\overline{)1611}$

Using the Ideas

1. Use multiples of 10, 100, or 1000 for each ▒.

- **A** To estimate 297 + 609, you can find the sum 300 + ▒.
- **B** To estimate 684 · 9, you can find the product ▒ · 9.
- **C** To estimate 63 · 78, you can find the product 60 · ▒.
- **D** To estimate 2139 ÷ 69, you can find the quotient 2100 ÷ ▒.
- **E** To estimate 692 − 176, you can find the difference ▒ − 200.
- **F** To estimate 6 · 4195, you can find the product 6 · ▒.
- **G** To estimate 524 + 395, you can find the sum ▒ + 400.
- **H** To estimate 63 · 295, you can find the product 60 · ▒.
- **I** To estimate 78 + 63 + 91, you can find the sum 80 + 60 + ▒.
- **J** To estimate 8199 + 2411, you can find the sum ▒ + 2000.
- **K** To estimate 381 · 714, you can find ▒ · 700.

2. Give an estimate for the answer for each part in Exercise 1.

3. Give an estimate for each of the following problems.

A 721 + 695	**D** 32 · 77	**G** 1984 + 6911	**J** 625 + 379
B 843 + 275	**E** 778 ÷ 4	**H** 78 + 24 + 98	**K** 7684 + 3211
C 629 − 179	**F** 71 · 688	**I** 213 · 498	**L** 566 · 32

In Exercises 4 through 6, give an estimate for the answer.

4. The area of Alaska is 1 519 809 square kilometres (km^2). The population in 1970 was 302 173. About how many square kilometres per person are there?

5. In 1971, the average salary of 2 061 000 public school teachers was $10 100. About how much total salary was earned by these teachers in 1971?

6. The USSR has an area of 22 402 200 square kilometres and a population of 242 612 000. Canada has an area of 9 976 185 square kilometres and a population of 21 830 000.

- **A** About how many Russians per square kilometre?
- **B** About how many Canadians per square kilometre?
- **C** Which country is more densely populated?

More practice, page S-3, Set 3

UTILIZATION

The first three exercises could be discussed orally with the students. In Exercise 3, some variation in answers may be expected due to different rounding of the numbers by the students.

EXTENSION

To provide further practice, make selective assignments from Supplementary Exercise Set 3 on page S-3, *Workbook* page 5, and *Duplicator Masters* page 3.

Enrichment: Challenge your students with some difficult estimation problems such as the ones that follow.

1. How many hamburgers are consumed in one day by the people in a given city, town, or province?
2. How many times do you breathe in a day, a month, or a year?
3. How many times has your heart beaten in your lifetime?
4. How far do you walk in a day or a year?

Assignments

Minimum: 1–3, oral. Average 1–3, oral; 4, 5. Maximum 1–3, oral; 4–6.

Objective

The student can add, subtract, multiply, or divide using whole numbers.

PREPARATION

This lesson can be contrasted with the previous lesson. While in the preceding lesson only approximate numbers were expected, in this lesson exact results are expected.

INVESTIGATION

The problems in the Investigation can be used to diagnose students' computing skills. For those students whose computing skills are rusty, the extra practice sets cited at the bottom of page E-17 are recommended.

The coded answers in the Investigation permit the students to check their own answers.

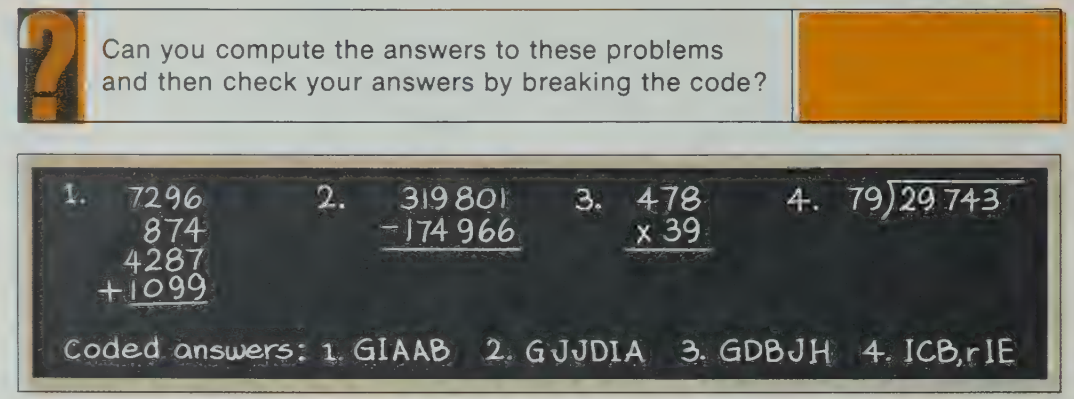
DISCUSSION

During the Discussion, emphasize routine computational shortcuts such as grouping by tens in addition or multiplying by multiples of 10, 100, and 1000. Encourage students to check their results. Exercises 4, 5, and 6 point out the importance of estimation as a rough check on computational accuracy.

Although the word *palindrome* is introduced on page E-17, you might find it convenient to use a discussion of the term's meaning as a bridge between the Discussion and Utilization exercises. Many students may already be familiar with verbal palindromes such as "rotator" and the nonsense sentence "Able was I ere I saw Elba."

Computing with Whole Numbers

Investigating the Ideas


Can you compute the answers to these problems and then check your answers by breaking the code?
1. 7296 + 874 + 4287 + 1099
2. 319 801 − 174 966
3. 478 × 39
4. 79)29 743
Coded answers: 1. GIAAB 2. GJJDIA 3. GDBJH 4. ICB,rIE

Discussing the Ideas

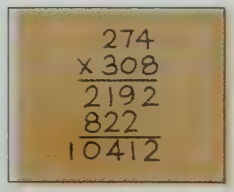
1. What are some shortcuts you can use in addition problems?

2. Explain why the subtraction problem is incorrect even though the screens cover some of the digits.

3. How can you check your work for a subtraction problem?

4. Estimate the product for the multiplication problem 274 · 308.

5. Explain why this student did not get the correct answer.



6. Which is the best estimate of the quotient $64\overline{)38\,954}$?

   A 60 **B** 600 **C** 6000

7. Find the quotient and remainder for Exercise 6.

8. How can you check your work in a division problem?

Using the Ideas

1. A **palindrome** is a numeral which represents the same number if the digits of the numeral are written in reverse order. Examples of palindromes are 171, 2662, and 35653. The example at the right shows how to get a palindrome by reversing the digits of a number and adding. Find the palindrome for each number using the addition method.

A 312 **C** 692 **E** 3587
B 813 **D** 1962 **F** 57919

EXAMPLE:

number →	178
reverse →	+871
sum →	1049
reverse →	+9401
sum →	10450
reverse →	+05401
palindrome →	[redacted]

2. Study and complete part **A**. Then perform the same operations with the numbers given in the other parts.

A

8 7 6	← number
−6 7 8	← reverse
1 9 8	← difference
+8 9 1	← reverse
[redacted]	← sum

B 624 **C** 902 **D** 753

3. Find the products. They may be interesting to compare.

A 96 × 46 **B** 69 × 64 **C** 322 × 669 **D** 223 × 966

4. Find the quotients and remainders.

A $52\overline{)2704}$ **B** $123\overline{)456\,789}$ **C** $1001\overline{)839\,839}$

5. Copy each problem, replacing the screens by digits so that the answers are correct.

A

[redacted] 8 [redacted] 7	
+ 9 6 7 [redacted]	
[redacted] 5 [redacted] 1 5	

B

4 [redacted] [redacted] 7	
− [redacted] 9 8 [redacted]	
1 0 1 8	

C

[redacted] [redacted] [redacted]	
× 9 9 9	
[redacted] [redacted] [redacted] [redacted]	
[redacted] [redacted] [redacted] [redacted]	
[redacted] [redacted] [redacted] [redacted]	
4 7 [redacted] [redacted] 4	

D

[redacted] [redacted]	
$76\overline{)}$ [redacted] [redacted]	
[redacted] [redacted]	
[redacted] [redacted]	
[redacted] 6	
0	

More practice, pages S-3, 4, and 5, Sets 4, 5, 6, and 7

UTILIZATION

These exercises are designed to review addition, subtraction, multiplication, and division in unusual and motivating ways.

In Exercise 1, each given number will produce a *palindrome* in a finite number of reversals and additions according to the procedure shown in the example. However, it should not be thought that the procedure can be applied to all numbers to produce palindromes, at least not in a few steps. For example, it has been shown that the number 196 will not produce a palindrome in 4147 steps. Furthermore, the completed steps revealed no kind of periodicity in the digits which would suggest that a palindrome might ultimately result.

Exercise 2 is based on the fact that any 3-digit numeral of the form *abc*, with $a \neq c$, will produce the number 1089 in the 4 steps shown in part A.

Exercise 3 presents pairs of multiplication problems with reversed digits which produce the same product.

Parts A and C of Exercise 4 are division problems having quotients which are equal to the divisors.

Exercise 5 provides reconstruction problems involving the four basic operations.

Keep in mind that the aim of these problems is to provide some motivation while improving basic computational skills with whole numbers.

EXTENSION

For students who would benefit from further practice, make selective assignments from Supplementary Exercise Sets 4–7 on pages S-3, S-4, and S-5. Additional exercises appropriate for use here are provided on *Workbook* page 6, *Duplicator Masters* pages 4–7, and *Arithmetic Skill Cards* W-1 through W-10.

Enrichment: Have students create their own problem sets similar to the ones on this page and challenge other students to solve their problems.

Assignments

Minimum 1A–C, 2, 3A and B, 4A.
Average 1–4. Maximum 1–5.

Objective

The student can express a power of two with an exponent and can express a base-ten numeral in terms of a sum of powers of two.

PREPARATION

Materials: Index cards, scissors.

As much time as possible should be allowed for the Investigation. Therefore you may want to review only briefly the use of exponents in writing powers of numbers.

INVESTIGATION

The students should work in pairs on the game suggested. If time permits, you might have pairs of students compete in a race to see which ones can construct the correct sum first.

DISCUSSION

The fact that every positive whole number can be expressed as a sum of different powers of two is the basis of the binary numeration system. The purpose of this lesson is to give students some experience with the idea.

This and the three following lessons deal with the binary numeration system. You may regard them as optional lessons; they may be omitted without any loss of continuity or development. The material on the binary system is presented for these reasons:

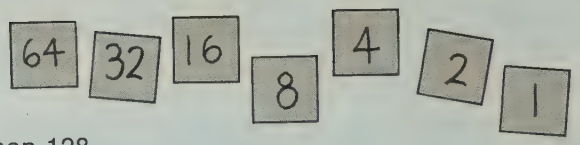
(a) It offers an illuminating comparison and contrast to the familiar base-ten numeration system.
(b) It is highly motivating to many students.
(c) It has some significant computer applications.

You might delay presenting these lessons until later in the school year if you wish.

Powers of Two

Investigating the Ideas

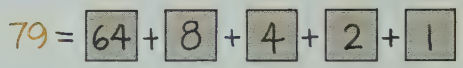
Try this game with a classmate.



RULES: **A** Make seven cards like the ones shown.

B Name any number less than 128.

C Your opponent gets one minute (or less) to choose the cards whose numbers give your number for their sum.



D Your opponent scores one point if he gets the number.

E Your opponent then challenges you with a number.

? Can you play the game to 10 points with a classmate?

Discussing the Ideas

1. Do you think there are some numbers from 1 through 127 that cannot be formed from the seven cards?

2. What is special about the set of cards for these numbers?

   A 63 **B** 31 **C** 15 **D** 7 **E** 3

3. Give the missing exponent for each power.

   A $64 = 2^{\square}$ **B** $32 = 2^{\square}$ **C** $16 = 2^{\square}$ **D** $8 = 2^{\square}$ **E** $4 = 2^{\square}$ **F** $2 = 2^{\square}$ **G** $1 = 2^{\square}$

4. Mary Binary, who was very good at the game, made a table like the one below to help her find the correct cards quickly. She tallied a "1" if a card was needed and a "0" if it was not needed.

   What are the numbers for **A**, **B**, and **C** in the table?

NUMBER	$64 = 2^6$	$32 = 2^5$	$16 = 2^4$	$8 = 2^3$	$4 = 2^2$	$2 = 2^1$	$1 = 2^0$
75	1	0	0	1	0	1	1
46		1	0	1	1	1	0
A	1	1	0	0	1	0	0
B				1	1	1	1
C		1	1	0	1	0	1

Using the Ideas

1. Complete the "tally sheet" for each number.

	NUMBER	64	32	16	8	4	2	1
	26			1	1	0	1	0
A	70							
B	111							
C	49							
D	83							
E	122							
F	98							

2. What is the number indicated by the tallies?

NUMBER	2^6	2^5	2^4	2^3	2^2	2^1	2^0
A	1	1	0	0	1	1	0
B		1	0	0	1	1	1
C	1	0	1	0	1	0	1
D		1	1	1	0	1	1
E		1	0	1	1	0	0
F	1	1	1	1	1	1	0

3. The symbols below represent tallies from the charts above. Give the number that each tally represents.

A

1	0	0	0	0	0	1

B

	1	1	0	1	0	1

C

1	0	1	1	1	1	1

D

1	0	0	1	1	0	1

E

	1	1	1	0	0	0

F

1	1	1	0	1	0	1

4. Make a tally for each of these base-ten numbers.

A	7	**C**	26	**E**	42	**G**	61	**I**	73	**K**	100
B	13	**D**	30	**F**	50	**H**	67	**J**	80	**L**	125

UTILIZATION

In completing the tally sheets for the exercises, students should refer to the completed tally sheet for Discussion Exercise 4. There is no need to introduce binary numerals at this time, even though the tally sheets show base-two numerals. The binary notation will be introduced in the next lesson.

Answers, Exercise 4

	NUMBER	2^6	2^5	2^4	2^3	2^2	2^1	2^0
A	7					1	1	1
B	13				1	1	0	1
C	26			1	1	0	1	0
D	30			1	1	1	1	0
E	42		1	0	1	0	1	0
F	50		1	1	0	0	1	0
G	61		1	1	1	1	0	1
H	67	1	0	0	0	0	1	1
I	73	1	0	0	1	0	0	1
J	80	1	0	1	0	0	0	0
K	100	1	1	0	0	1	0	0
L	125	1	1	1	1	1	0	1

EXTENSION

Enrichment: Suggest that students make tally charts for some body measurements. The tally chart can be extended to include larger numbers.

Sample	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Age					1	1	1	0
Height (cm)	1	0	1	0	1	0	1	0
Weight (kg)			1	0	1	1	1	0
Waist (cm)			1	1	1	1	0	1
Head (cm)			1	1	1	0	1	0
Wrist (cm)				1	0	0	0	1

Assignments

Minimum 1, 2. Average 1–3, 4A–F.
Maximum 1–4.

Objective

The student can write a base-ten numeral as a base-two numeral or a base-two numeral as a base-ten numeral.

PREPARATION

Review the method of preparing tallies for base-ten numerals as presented in the previous lesson.

INVESTIGATION

Have the students study the base-two and base-ten tallies for set A. Discuss the ways that the two tallies are different. Then have the students complete the tallies for set B.

DISCUSSION

The base-two tallies provide a simple transition from base-ten to base-two numerals. The place-value chart above Discussion Exercise 1 should be carefully studied and discussed, for it is essential for understanding how to convert from one base to another.

Converting from base two to base ten and vice versa is not an easy process either to perform or to understand. Do not expect all students to master these skills. However, changing from one base to another does constitute an interesting activity and may prove quite stimulating to your more able students.

You may find it helpful to provide some examples of converting base-two numerals to base-ten numerals and vice versa before assigning the exercises on page E-21.

Answers, Discussion Exercise 3

A. "One, one, base two" means one group of two and one.
B. "One, zero, zero, base two" means one group of four and no twos or ones.
C. "One, one, one, base two" means one group of four, one group of two, and one.
D. "One, zero, one, one, base two" means one group of eight, no group of four, one group of two and one.
E. "One, zero, one, zero, one, base two" means one group of sixteen, one group of four, and one.

The Binary System

Investigating the Ideas

Study the base-two and base-ten tallies that show the number of dots in set A.

A

B

BASE-TWO						
2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	1	1	0	1	1

BASE-TEN		
10^2	10^1	10^0
1	2	3

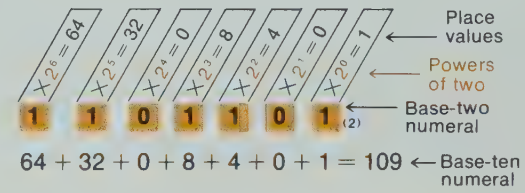
BASE-TWO						
2^6	2^5	2^4	2^3	2^2	2^1	2^0

BASE-TEN		
10^2	10^1	10^0

Can you show base-two and base-ten tallies for the number of dots in set B?

Discussing the Ideas

In the decimal system of numeration, we group by tens in order to write numerals for numbers. In the **binary system** (base-two), we group by twos. The diagram below should help you understand the binary system.



$64 + 32 + 0 + 8 + 4 + 0 + 1 = 109$ ← Base-ten numeral

1. In the decimal numeration system, the place-value names are powers of ten. What powers are the place-value names in the binary system?

2. In the base-ten, or decimal, system of numeration, we use the digits 0 through 9 to write numerals for any whole number. In the base-two, or binary, system, what are the only digits needed?

3. We write $101_{(2)}$ to show $1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$.
The symbol $101_{(2)}$ is read as "One zero one, base two."
This means one group of four, zero groups of two, and one.
Read and give the meaning of each of these base-two numerals.
A $11_{(2)}$ **B** $100_{(2)}$ **C** $111_{(2)}$ **D** $1011_{(2)}$ **E** $10101_{(2)}$

Using the Ideas

1. Complete each sentence concerning the digit printed in color.

EXAMPLE: $1101_{(2)}$ → The 1 in the fours' place represents 1×2^2

A $1011_{(2)}$ → The 1 in the ? place represents ?.

B $110101_{(2)}$ → The 1 in the ? place represents ?.

C $1000001_{(2)}$ → The 1 in the ? place represents ?.

2. Write the base-ten numeral for each base-two numeral.

A $1_{(2)}$	**C** $111_{(2)}$	**E** $101_{(2)}$	**G** $11_{(2)}$	**I** $1001_{(2)}$	**K** $1011_{(2)}$
B $10_{(2)}$	**D** $100_{(2)}$	**F** $110_{(2)}$	**H** $1100_{(2)}$	**J** $1010_{(2)}$	**L** $1111_{(2)}$

3. Copy and complete the table for powers of two. You will need the results in later exercises.

Powers of two	
$2^0 =$	$2^7 =$
$2^1 =$	$2^8 =$
$2^2 =$	$2^9 =$
$2^3 =$	$2^{10} =$
$2^4 =$	$2^{11} =$
$2^5 =$	$2^{12} =$
$2^6 =$	$2^{13} =$

4. Expand each base-two numeral as shown in the example. Then use the table of powers of two and calculate the base-ten numeral that represents the same number.

EXAMPLE: $1101_{(2)} = (1 \cdot 2^3) + (1 \cdot 2^2) + (0 \cdot 2^1) + 1$
$= 8 + 4 + 0 + 1 = 13$

A $1110_{(2)}$	**C** $11100_{(2)}$	**E** $11111_{(2)}$	**G** $1110111_{(2)}$
B $1000_{(2)}$	**D** $10000_{(2)}$	**F** $100000_{(2)}$	**H** $10101010_{(2)}$

5. Write a simple base-two numeral for each part.

A $(1 \cdot 2^1) + (1 \cdot 2^0)$

B $(1 \cdot 2^2) + (0 \cdot 2^1) + (1 \cdot 2^0)$

C $(1 \cdot 2^4) + (1 \cdot 2^3) + (0 \cdot 2^2) + (0 \cdot 2^1) + (0 \cdot 2^0)$

D $(1 \cdot 2^5) + (0 \cdot 2^4) + (1 \cdot 2^3) + (1 \cdot 2^2) + (1 \cdot 2^1) + (0 \cdot 2^0)$

6. Convert these base-ten numerals to base-two numerals. Refer to the powers of two table in Exercise 2.

A 156	**C** 539	**E** 900	**G** 1279
B 372	**D** 684	**F** 1023	**H** 1584

think

Examples **A** and **B** show a shortcut for converting a binary numeral to a base-ten numeral. Can you discover how to use the shortcut and use it to give the base-ten numerals for these numerals?

A $1\ 0\ 0\ 0\ 1\ 1\ 1_{(2)} = 71_{(ten)}$
1 2 4 8 17 35 71

B $1\ 0\ 1\ 1\ 0\ 0\ 0\ 1_{(2)} = 177_{(ten)}$
1 2 5 11 22 44 88 177

A $10111_{(2)}$ **B** $10000101_{(2)}$ **C** $11100001_{(2)}$ **D** $111111111_{(2)}$ **E** $110110001_{(2)}$

UTILIZATION

Exercise 1 can be discussed orally. It is important that students be able to quickly give the place value of any digit in a base-ten numeral.

Selected Answers, Exercise 4

(a) $(1 \cdot 2^3) + (1 \cdot 2^2) + (1 \cdot 2^1) + 0$
$= 8 + 4 + 2 + 0 = 14_{(ten)}$

(c) $(1 \cdot 2^4) + (1 \cdot 2^3) + (1 \cdot 2^2) + (0 \cdot 2^1) + 0$
$= 16 + 8 + 4 + 0 + 0$
$= 28_{(ten)}$

(e) $(1 \cdot 2^4) + (1 \cdot 2^3) + (1 \cdot 2^2) + (1 \cdot 2^1) + 1$
$= 16 + 8 + 4 + 2 + 1$
$= 31_{(ten)}$

EXTENSION

Additional practice exercises are provided on page 7 of the *Workbook*.

Enrichment: This would be an opportune time to suggest that students explore the Mathematical Recreation (page E-28) which involves place value concepts.

Think Solution

The shortcut is performed as follows: Start by counting the left-hand binary digit 1, then double for each place to the right, adding 1 if a 1 occurs in a particular place. The sum obtained when the ones' place is reached is the base-ten numeral for the binary numeral.

To see how the shortcut works consider the base-two numeral $1101_{(2)}$:

$1\ \ 1\ \ 0\ \ 1_{(2)}$
(a) (b) (c) (d)

Successive steps (a), (b), (c), and (d) are

(a) $1 = 1$

(b) $1 \cdot 2 + 1 = 3$

(c) $(1 \cdot 2 + 1) \cdot 2 = 6$

(d) $[(1 \cdot 2 + 1) \cdot 2] \cdot 2 + 1 = 13$

Observe that if the distributive principle is applied to the expression in (d), we get

$1 \cdot 2^3 + 1 \cdot 2^2 + 1 = 1101_{(2)}$
$= 13_{(ten)}$

Assignments

Minimum 1, oral; 2A–D; 3; 4A–C; 5; 6A–C.
Average 1, oral; 2A–H; 3; 4; 5; 6A–F.
Maximum 1, oral; 2–6.

Objective

The student can use binary numerals in problems involving various codes.

PREPARATION

Plan a short review of writing binary numerals for base-ten numerals and vice versa.

INVESTIGATION

While students are completing the five-bit binary code in the Investigation, write some words or some student names in the binary code on the chalkboard. When the students have completed the code table, challenge them to use it to find the words or names you wrote on the chalkboard.

DISCUSSION

You may find it convenient to use the five-bit numeral 00000 to designate a space between two words. Thus, the message I AM COLD, for example, could be coded as follows:

01001	I
00000	(space)
00001	A
01101	M
00000	(space)
00011	C
01111	O
01100	L
00100	D

There is no need to develop any particular skill using the binary code. The main purpose of this lesson is to provide interesting recreational and motivational applications of binary numerals.

Using the Binary System

Investigating the Ideas

Each digit of a binary numeral is called a **bit**. Suppose the letters of the alphabet were coded in order with five-bit binary numerals as shown in the table. For example, the binary code for MATH would be:

01101 → M
00001 → A
10100 → T
01000 → H

A Five-bit Binary Code	
A	0 0 0 0 1
B	0 0 0 1 0
C	0 0 0 1 1
D	0 0 1 0 0
⋮	⋮ ⋮ ⋮ ⋮ ⋮
M	0 1 1 0 1
⋮	⋮ ⋮ ⋮ ⋮ ⋮
Z	1 1 0 1 0

Can you complete the five-bit binary code table and write a word or message using the binary code?

Discussing the Ideas

1. Since E is the fifth letter of the alphabet, what five-bit binary numeral would be the code for E?

2. The code for M is 01101. What base-ten numeral gives its position in the alphabet?

3. Each group of five-bit binary numerals is a coded name. Decode each name.

A

0	0	1	0	0
0	0	0	0	1
1	0	1	1	0
0	0	1	0	1

B

0	1	1	0	1
0	0	0	0	1
1	0	0	1	0
1	1	0	0	1

C

1	0	0	0	0
0	0	0	0	1
1	0	1	0	1
0	1	1	0	0
0	0	0	0	1

4. Explain how this palindromic message could be written in five-bit binary code:

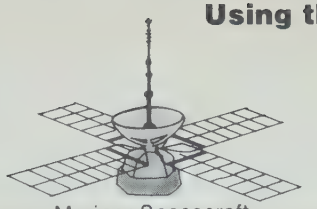
   N E V E R O D D O R E V E N

5. What is the largest number of symbols that could be coded using five-bit binary numerals?

6. How many symbols could be coded with six-bit numerals?

Using the Ideas

The Mariner spacecraft (1965–1971) took pictures of the planet Mars which were radioed back to earth and produced with the help of a computer and binary numerals. The computer received the "pictures" in the form of binary numerals consisting of six bits. Each group of six bits determined the shade of one dot in the final picture. The shading of the dots varied in 64 different shades from completely white to completely black as shown in the table at the right. All binary numerals representing numbers between 0 and 63 represent various shades of gray between white and black.



Mariner Spacecraft

Shade of dot	Six-bit binary numeral	Base-ten numeral
White	000000	0
Almost white	000001	1
⋮	⋮	⋮
Almost black	111110	62
Black	111111	63

1. A Which number from 0 through 63 is represented by 101001?
   B Does this number represent a shade of gray closer to white or to black?

2. In the picture transmission just described, what binary numeral would represent the darkest shade of gray that is not black?

3. Which binary numeral would represent the lighest shade of gray?

4. Give two binary numerals that would represent shades of gray midway between white and black.

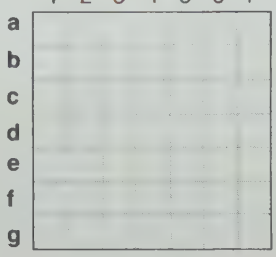
5. Suppose that an imaginary spacecraft, Sparrow I, was sent on a close flight past an imaginary planet Nova. After taking pictures, radio signals were sent back and amplified to give this array of two-bit binary numerals. Use this array, and the code below to convert the numbers into a picture on a 7-by-7 grid as shown.

	1	2	3	4	5	6	7
a	01	11	11	11	11	11	01
b	01	11	00	11	10	11	01
c	11	11	11	11	10	11	01
d	01	11	11	11	10	11	01
e	01	01	11	11	11	11	01
f	01	11	11	11	11	11	01
g	01	01	01	10	10	01	01

Code

Base-ten numeral	Binary numeral	
0 →	00 →	
1 →	01 →	
2 →	10 →	
3 →	11 →	

Grid



UTILIZATION

The exercises on this page deal with a scientific application of binary numeration. Exercises 1, 2, 3, and 4 might be utilized to stimulate discussion. Exercise 5 provides an imaginary simplified example of the Mariner coding scheme.

EXTENSION

Enrichment: Some students may be interested in doing some research on the Mariner space flights. One of the best summary accounts of the Mariner voyages and achievements and of the men behind the missions appeared in *Time* magazine under the title "Portrait of a Planet" (Vol. 86, No. 4, July 23, 1965; pp. 36–44). Enlargements of pictures taken by Mariner can be found in the article "The Terrain of Neighbor Mars" in *Life* (Vol. 59, No. 6, August 6, 1965; pp. 62A–C). An excellent article on the Mariner 6 and Mariner 7 flights can be found in *Saturday Review* (October 11, 1969; pp. 75–77). Some excellent close-up pictures of Mars taken by Mariners 6 and 7 appear in *Science* (Vol. 165, August 15, 1969; pp. 684–690; and Vol. 165, August 22, 1969; pp. 787–795).

The best source of information on this and other space probes is NASA. (Write to NASA, Education Publications Distribution, FAD-1, Washington, D.C. 20546.) Still another source of information concerning the Mariner flights is the Office of Educational Services, Jet Propulsion Laboratory, California Institute of Technology, NASA, Pasadena, California, 91103.

Assignments

Minimum 1–4, oral. Average 1–4, oral; 5. Maximum 1–4, oral; 5.

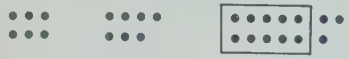
Objective

The student can use binary addition and multiplication tables to compute using binary numerals.

PREPARATION

Review the idea of addition with base-ten numerals using sets of dots. Then present the same problem using base-two groupings as in the example below.

Base ten:



$6_{(ten)} + 7_{(ten)} = 13_{(ten)}$

Base two:

$110_{(2)} + 111_{(2)} = 1101_{(2)}$

INVESTIGATION

The purpose of the Investigation is to have students compare the number of "basic facts" for addition and multiplication in base ten and in base two. You may want to place the completed addition and multiplication charts on the chalkboard for later reference.

DISCUSSION

One of the chief purposes of this lesson is to stimulate interest. We do not intend that students should become efficient at computing in other bases or that they memorize the base-two addition and multiplication tables. However, they should work independently to find sums and products of numbers given in base two.

A point that should be emphasized throughout the study of numbers in other bases is that the numbers themselves do not change; it is only our notation for these numbers that changes.

Provide other examples as needed and discuss them before assigning the exercises on page E-25.

Computing with Base-two Numerals

Investigating the Ideas

In order to compute efficiently using base-ten numerals, you must know a large number of basic addition and multiplication "facts."

To compute using the base-two numerals, you need to know only the eight facts for the base-two tables.

BASE TEN	
(+) *facts*	(×) *facts*
$0 + 0 = 0$	$0 \cdot 0 = 0$
$0 + 1 = 1$	$1 \cdot 1 = 1$
$1 + 1 = 2$	$1 \cdot 2 = 2$
$1 + 2 = 3$	$2 \cdot 2 = 4$
⋮	⋮
$9 + 9 = 18$	$9 \cdot 9 = 81$

BASE TWO						
+	0	1		×	0	1
0				0		
1				1		

Can you copy and complete the base-two addition and multiplication tables?

Discussing the Ideas

1. The only "hard" addition fact is $1_{(2)} + 1_{(2)} = 10_{(2)}$. Are any of the multiplication facts "hard"?

2 Here is the way a student showed an addition problem using base-two numerals. Can you explain how the sum was found?

$$\begin{array}{r} {\scriptstyle 1\ 1\ \ \ \ } \\ 101_{(2)} \\ +11_{(2)} \\ \hline 1000_{(2)} \end{array}$$

3. What is $10_{(2)} - 1_{(2)}$?

4. How would you complete this subtraction problem?

$$\begin{array}{r} 1\ 1\ \not{1}^{\,0}\ \not{0}^{\,10}{}_{(2)} \\ -1\ 1\ 1_{(2)} \\ \hline 1_{(2)} \end{array}$$

5. Explain the steps in this multiplication problem.

$$\begin{array}{r} 1\ 1\ 1_{(2)} \\ \times 1\ 1_{(2)} \\ \hline 1\ 1\ 1 \\ 1\ 1\ 1 \\ \hline 1\ 0\ 1\ 0\ 1_{(2)} \end{array}$$

6. Try this division problem. $10_{(2)}\overline{)10110_{(2)}}$ Check your work by using multiplication.

Using the Ideas

All numerals in the exercises are base-two numerals.

1. Find the sums.

A	B	C	D	E	F
10	11	11	101	100	111
+ 1	+10	+11	110	111	11
			+111	110	+ 1
				+101	

2. Find the differences.

A	B	C	D	E	F
11	10	100	1101	110110	1000
− 1	− 1	− 11	−1011	− 11011	− 1

3. Find the products.

A	C	E	G	I	K
11	10	100	101	100	101
×1	×10	×100	×11	×110	×111

B	D	F	H	J	L
10	100	11	111	111	101
×0	×10	×11	×111	×11	×110

4. Find the quotients.

A $1\overline{)101}$ B $11\overline{)110}$ C $10\overline{)10110}$ D $110\overline{)110000}$

★ **5.** Write a base-two numeral for the number n in each equation.

A $11 + n = 101$	D $101 \cdot n = 1111$	G $11 \cdot n = 1100$
B $111 - n = 110$	E $110 \div 10 = n$	H $n + 100 = 1000$
C $10 \cdot n = 100$	F $n - 11 = 1011$	I $1000 - n = 111$

think

The **duodecimal system** (base twelve) requires two extra digits. Suppose we use the digits and names shown in the diagram.

0	1	2	3	4	5	6	7	8	9	D	ϵ
zero	one	two	three	four	five	six	seven	eight	nine	dek	el
										(from decade)	(from eleven)

1. Count from $1_{(12)}$ to $100_{(12)}$. For example, for the number after $10_{(12)}$, you would say "one one." For the number after $99_{(12)}$, you would say "nine dek."
2. Complete two duodecimal charts (one for multiplication and one for addition) similar to the ones for base two. You will need to use twelve digits $\{0, 1, 3, \ldots, 9, \text{D}, \epsilon\}$ across the top and down the side of each chart.

UTILIZATION

You will need to vary your assignment according to the interest and ability of your students.

EXTENSION

Page 8 of the *Duplicator Masters* provides appropriate exercises for further practice.

Enrichment: Research Projects B and D on page E-27 are recommended for interested students at this time.

Some students might enjoy making a calendar using binary numerals.

September 11110110111						
	1	10	11	100	101	110
111	1000	1001	1010	1011	1100	1101
1110	1111	10000	10001	10010	10011	10100
10101	10110	10111	11000	11001	11010	11011
11100	11101	11110				

Think Solution

1. one, . . . dek, el, one zero,
   one one, . . . one dek, one el, two zero,
   ⋮ ⋮ ⋮ ⋮
   nine one, . . . nine dek, nine el, dek zero,
   dek one, . . . dek dek, dek el, el zero, el one, . . . el dek, el el, one zero zero.

2.

+	0	1	2	3	4	5	6	7	8	9	D	ϵ
0	0	1	2	3	4	5	6	7	8	9	D	ϵ
1	1	2	3	4	5	6	7	8	9	D	ϵ	10
2	2	3	4	5	6	7	8	9	D	ϵ	10	11
3	3	4	5	6	7	8	9	D	ϵ	10	11	12
4	4	5	6	7	8	9	D	ϵ	10	11	12	13
5	5	6	7	8	9	D	ϵ	10	11	12	13	14
6	6	7	8	9	D	ϵ	10	11	12	13	14	15
7	7	8	9	D	ϵ	10	11	12	13	14	15	16
8	8	9	D	ϵ	10	11	12	13	14	15	16	17
9	9	D	ϵ	10	11	12	13	14	15	16	17	18
D	D	ϵ	10	11	12	13	14	15	16	17	18	19
ϵ	ϵ	10	11	12	13	14	15	16	17	18	19	1D

×	0	1	2	3	4	5	6	7	8	9	D	ϵ
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	D	ϵ
2	0	2	4	6	8	D	10	12	14	16	18	1D
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	D	13	18	21	26	2ϵ	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2ϵ	36	41	48	53	5D	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
D	0	D	18	26	34	42	50	5D	68	76	84	92
ϵ	0	ϵ	1D	29	38	47	56	65	74	83	92	D1

Assignments

Minimum 1, 3A–F. Average 1, 2, 3A–H. Maximum 1–5.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The exercises can be assigned as individual written work or discussed in small- or large-group situations. The order of the exercises corresponds to the order of topics in the module. Thus, if you have not covered the binary numeration system you should not assign Exercises 14 through 17.

Most students will benefit from a discussion of the exercises. Give each student an opportunity to clear up any difficulty that arose in connection with the content of the module.

Additional review exercises are presented on page 8 of the *Workbook*.

REVIEWING THE IDEAS

1. Write a Hindu-Arabic numeral for each numeral.

A

B

C MCDLXIV

2. Write a base-ten numeral for each number.

A 3 billion **B** 748 million

3. Which digit in the numeral 634 396 863 represents 30 000 000?

4. Write a subtraction equation that can be used to find the missing addend n in the addition equation $27 + n = 53$.

5. Write a division equation that can be used to find the missing factor y in the multiplication equation $9 \cdot y = 234$.

6. Which principle is illustrated in each of the following equations?

A $7 + 9 = 9 + 7$

B $3462 \cdot 1 = 3462$

C $(2 \cdot 3) \cdot 7 = 7 \cdot (2 \cdot 3)$

D $(9 + 16) + 4 = 9 + (16 + 4)$

E $42 + 0 = 42$

7. Write each product using an exponent.

A $6 \cdot 6 \cdot 6$ **C** $10 \cdot 10$

B $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ **D** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

8. What is the missing exponent? $10^{\square} = 1$

9. Solve the equations.

A $10^2 \cdot 10^3 = n$ **D** $8^2 \cdot 8^1 = n$

B $2^3 \cdot 2^4 = n$ **E** $n \div 7^2 = 7^3$

C $10^4 \cdot 10^0 = n$ **F** $10^4 \div 10^3 = n$

10. Round 7 463 487 to the nearest

A hundred **B** thousand **C** million

11. Which number is the best estimate for $179 \cdot 73$?

A 16 000 **B** 14 000 **C** 12 000

12. Estimate the quotient for $58\overline{)29\,437}$ to the nearest hundred.

13. Compute.

A $\begin{array}{r} 9685 \\ +8796 \\ \hline \end{array}$ **B** $\begin{array}{r} 8007 \\ -2369 \\ \hline \end{array}$ **C** $805 \cdot 496$

D $87\overline{)21\,503}$

14. Write a base-two numeral that tells how many dots there are in this set.

15. Write each base-ten numeral as a base-two numeral.

A 5 **C** 19 **E** 32

B 12 **D** 29 **F** 66

16. Give the base-ten numeral for each base-two numeral.

A $110_{(2)}$ **C** $1000_{(2)}$ **E** $1111_{(2)}$

B $1011_{(2)}$ **D** $11011_{(2)}$ **F** $101011_{(2)}$

17. Compute in base-two.

A $\begin{array}{r} 111_{(2)} \\ +\ 11_{(2)} \\ \hline \end{array}$ **B** $\begin{array}{r} 10000_{(2)} \\ -\ 111_{(2)} \\ \hline \end{array}$ **C** $\begin{array}{r} 111_{(2)} \\ \times 101_{(2)} \\ \hline \end{array}$

D $10_{(2)}\overline{)10010_{(2)}}$

TEST YOURSELF

1. What Hindu-Arabic numeral shows the same number as MCMLXXVI?

2. In the decimal numeral 537 426, the digit 3 stands for 3
   - A thousands.
   - B ten thousands.
   - C hundred thousands.

3. Write the product $6 \cdot 6 \cdot 6 \cdot 6$ using an exponent.

4. Round 347 896 to the nearest ten thousand.

5. Which equation illustrates the distributive principle?
   - A $2 \cdot (7 \cdot 4) = (2 \cdot 7) \cdot 4$
   - B $2 \cdot (7 + 4) = 2 \cdot 7 + 2 \cdot 4$
   - C $(2 \cdot 7) \cdot 4 = 4 \cdot (2 \cdot 7)$

6. Estimate the product $385 \cdot 83$.

7. Compute.

   A
   $$\begin{array}{r} 596 \\ 8129 \\ +\,2748 \\ \hline \end{array}$$

   B
   $$\begin{array}{r} 33\,091 \\ -\,28\,747 \\ \hline \end{array}$$

   C $389 \cdot 76$

   D $83\overline{)2988}$

8. Write a base-two numeral for each base-ten numeral.
   A 13 B 37

9. Write each base-two numeral as a base-ten numeral.
   A $11010_{(2)}$ B $101111_{(2)}$

10. Compute and write the answer using base-two numerals.

    A
    $$\begin{array}{r} 1011_{(2)} \\ 111_{(2)} \\ +\,1101_{(2)} \\ \hline \end{array}$$

    B
    $$\begin{array}{r} 1101_{(2)} \\ \times\ 11_{(2)} \\ \hline \end{array}$$

RESEARCH PROJECTS

A A method of computing called "finger reckoning" was practiced in the 14th century in Europe. Find out how to use your fingers to find products. (See *From Fingers to Computers* by Margaret F. Willerding; Pasadena: Franklin Publications, 1968; Grolier.)

B A binary abacus has only one counter in each place.

Make a model of a binary abacus. Learn how to add and subtract using binary numerals with the abacus.

C The **duodecimal system** (base twelve) requires two extra digits (one for ten and one for eleven). Using ᴅ for ten and ᴇ for eleven (see page E-25), make up some addition and multiplication problems in base twelve. (See *Understanding Arithmetic* by Robert L. Swain; New York: Rinehart and Company, 1957; Holt, Rinehart and Winston.)

D The binary system is utilized by modern electronic computers. Find out about them and make a poster. (See *Mathematics* by David Bergamini and the Editors of *Life*, Life Science Library; New York: Time, Inc., 1963; General Learning Corporation.)

TEST YOURSELF

The ten self-evaluation questions are designed so that students can decide for themselves whether they mastered the objectives listed on the opening page of this module. The number annotated beside each exercise on the student page denotes the objective to which it relates.

Answers to the Test Yourself exercises are printed below the Mathematical Recreation on page E-28. Students should be encouraged to check the problems themselves in order to gain immediate feedback. Remedial work should be provided as needed at this time.

Students who are successful with the Test Yourself questions can expect to be successful on the Module Achievement Tests which are available as part of the testing program designed for use with *School Mathematics*. The Test Yourself questions could also serve as a model for a teacher-made achievement test for the module.

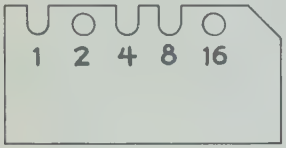
RESEARCH PROJECTS

Encourage as many students as possible to try one or more of these projects. Many of the suggested topics could easily be developed into a project for a mathematics-science fair project.

MATHEMATICAL RECREATION

This Mathematical Recreation can provide stimulating experiences for many students, while strengthening their understanding of place value and base-two numerals. Several worthwhile activities can be built around these cards. For example:

One student can show the back side of a card to another student who, seeing only the holes and slots, must give the base-ten numeral for the front of the card. This involves thinking about the place values in reverse since the ones' place will be on the left of the card.



Thus for the card above the student would think: $1 + 4 + 8 = 13_{(ten)}$.

Here is another surprising activity with the cards:

Shuffle the cards well, keeping the diagonal corners together. With the back side of the stack toward you, put the end of the paper clip through the holes at the 16's place. Lift out all cards that stay on the clip, place them in front of the rest of the cards in the stack. Repeat this step for each of the other places on the cards.

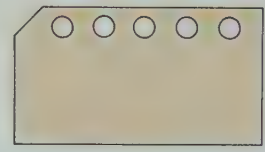
Now examine the cards. You should find that they are stacked in order!

MATHEMATICAL RECREATIONS

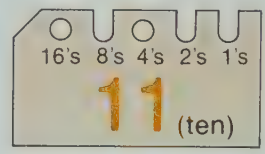
A BASE-TWO CARD SORTER

You will need 32 index cards, a paper punch, and some scissors for this project.

Cut off one corner and punch five holes through an index card as shown. Use this card as a pattern to cut and punch the other 31 cards.

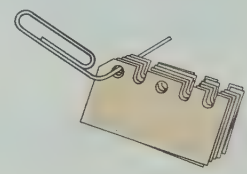


Cut slots to the appropriate holes so that each card represents one of the base-two numerals from $0_{(ten)}$ through $31_{(ten)}$. Number the cards. You can select any number from 0 through $31_{(ten)}$ in five sortings performed as follows:



Suppose that you wish to select card $19_{(ten)}$. $19_{(ten)} = 10011$

Use a paper clip. Put one end through all the holes in the 16's place. Retain the cards that do not stay on the clip. These have a slot in the 16's place. Discard those on the clip. Put the paper clip through the holes in the 8's place. Keep the cards that stay on the clip, discard the others. Put the clip through the hole in the 4's place, retain those cards that stay on the clip. Put the clip through the 2's place. Keep the cards that do not stay on the clip. Put the clip through the 1's place. The card that comes off has a slot in the 16's place, the 2's place, and the ones' place; therefore, it must be $19_{(ten)}$.



Try to locate other card by this method.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 1976 **2. B** **3.** 6^4 **4.** 350 000 **5. B** **6.** 32 000

7. A 11 473; **B** 4 344; **C** 29 564; **D** 36 **8. A** $111_{(2)}$; **B** $10010_{(2)}$

9. A 26; **B** 47 **10. A** $11111_{(2)}$; **B** $100111_{(2)}$

E–28

UNIT E: INTEGERS

Module 2: Sets, Equations, and Inequalities

General Objectives

To review the basic language of sets and the essential symbolism of set theory.

To review and extend the ideas of algebraic expressions equations, and functions using whole numbers.

To increase the students' ability to solve linear equations with whole number solutions.

To strengthen and extend work with inequalities both on the number line and in the positive integral quadrant of the co-ordinate plane.

To provide experience in graphing functions, equations, and inequalities.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
9	**RED** Given basic set terminology and symbolism, the student can understand and use them.	E-30, 31		**SP** b-1, 2	
10	**RED** Given two sets, the student can find the union or intersection of them.	E-32, 33		**SP** b-1, 2	
11	**RED** Given an algebraic expression in one variable and a replacement set, the student can evaluate the expression.	E-34, 35 E-36, 37	**WB** 9, 10 **DM** 9, 10	**SP** b-4	**SWM 2** 247
12	**RED** Given a linear equation, the student can apply the basic principles and use the basic operations to find its solution.	E-38 through 45	**WB** 11-13 **DM** 11, 12	**PD** f-1, 2, 3, 4	
13	**YELLOW** Given a word problem involving whole numbers, the student can solve it by expressing it in flow-chart form or as an equation.	E-46, 47	**WB** 14 **DM** 13	**PD** f-1, 2, 3, 4	**SWM 3** 8, 9, 100
14	**YELLOW** Given inequalities involving "and" and "or" and a replacement set, the student can find their solution sets.	E-48, 49		**SP** b-2	
15	**YELLOW** Given a function and several input numbers, the student can find corresponding output numbers.	E-50, 51		**DS** d-1	
16	**YELLOW** Given a function and several input numbers, the student can find corresponding output numbers and graph the coordinate pairs.	E-52, 53	**WB** 15 **DM** 14	**SP** b-3 **DS** d-1	**SWM 1** 92 **SWM 2** 56, 57
17	***GREEN*** *Given a linear equation involving two variables x and y, the student can find ordered pairs of numbers satisfying the equation and can graph them.*	E-54, 55		**DS** d-1	
18	***GREEN*** *Given an inequality such as $x + y < 4$, the student can graph the inequality.*	E-56, 57		**DS** d-4	
	Reviewing the Ideas	E-58	**WB** 16		

MATHEMATICS

The early lessons of this module review and extend some basic ideas of set theory. Only the necessary language and symbolism that are useful in the work the student will be doing during the course of the year are introduced.

Two sets A and B are said to be in *one-to-one correspondance* if every element of A is paired with exactly one element of B and every element of B is paired with exactly one element of A.

Set B is a subset of set A if every element of B is also an element of A. We write: $B \subset A$.

Each subset of A other than A itself is called a *proper subset* of A.

The kinds of sets that occur in mathematics can be separated into two classes: *finite* and *infinite*.

An infinite set can be defined as follows:

A set A is an infinite set if there is a proper subset of A which can be placed in one-to-one correspondence with A.

For example, let N be the set of whole numbers and M be the set of multiples of 3. Then M is a proper subset of N since every element of M is in N. Furthermore there exists a one-to-one correspondence between the two sets:

$$\begin{array}{l} N: \{0\ 1\ 2\ 3\ 4\ \ 5\ \ldots\} \\ \qquad\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow \\ M: \{0\ 3\ 6\ 9\ 12\ 15\ \ldots\} \end{array}$$

Therefore N is an infinite set.

A finite set is one that is not infinite. It is not possible to establish a one-to-one correspondence between any finite set and one of its proper subsets.

TEACHING THE MODULE

Vocabulary

algebra
algebraic expression
commutative principle
constant function
coordinates
disjoint
distributive principle
element
empty set
equation
finite set
flow chart
function
generalization
graph
identity element
identity function
inequality
infinite set
input
intersection
one principle
open sentence
operation
output
replacement set
set
solution set
subset
union
variable
Venn diagram
zero principle

A great part of the material in this module is algebraic in character. Since only the set of whole numbers are used in this module, it is possible for the student to concentrate upon the essential algebraic ideas without being distracted by computational difficulties. This will pay dividends in later modules in which the domain of discussion will be integers, rational numbers, or real numbers.

You will notice that the approach to solving equations in this module relies upon the use of the basic principles and inverse relations between addition and subtraction and between multiplication and division. Students at this grade level would gain little insight into worthwhile mathematical concepts if they were simply given mechanistic rules or rote methods of finding solutions.

Considerable work with graphing is presented in the chapter. Functions, equations, and inequalities involving whole numbers are graphed. The relation between algebra and the geometric representation of algebraic ideas is one that is emphasized repeatedly throughout the text.

Lesson Schedule

The module includes 14 lessons plus review and evaluation pages. Fast-paced classes may be able to cover more than one lesson per day. The work with graphs at the end of the module may be considered optional, especially for slow-paced classes. About 12 to 15 days should be allowed for adequate coverage.

Evaluation

In evaluating students, you should not expect that they will be able to solve a large number of equations or inequalities with great efficiency. Rather you should expect reasonable competency on a few well chosen problems of the type covered. Short daily quizzes, direct observation of students' work, and student conferences can also aid you in your evaluation.

The self-evaluation test on page E–59 can serve as a model for a teacher-made test covering the module, or you might administer Module Achievement Test 2, Learning Unit E.

RESOURCES FOR ACTIVE LEARNING

Activities

Cloudburst, Vol 5 (logic), Nos. 8148–88, Midwest Publications

ESS: Attribute Games and Problems; Senior Balancing, "Off Centre Boards," Problem Cards M1–M3, Webster, McGraw-Hill

Experiments and Puzzles in Logic, Midwest Publications

Graph Gallery, Creative Publications

Madison Project: *Discovery,* Chapters 1, 3, 24, 30–41, 44–46, Addison-Wesley

Making and Using Graphs and Nomographs, pp. 5–18, 25–34, Rand McNally

Nuffield Project: *Computers and Young Children,* "Flow Charts," pp. 11–27; *Graphs Leading to Algebra,* (open sentences, inequalities, intersections), pp. 11–43; *Logic; Problems* – Red Set, Nos. 1, 2, 7, Wiley

Sourcebook for Substitutes, pp. 76–79, 82–83, Addison-Wesley

For activities on topology, see Resources for Active Learning, Unit H, Module 4

Manipulative Devices

Flow Chart template (Addison-Wesley)

Madison Project: Independent Exploration Kits (Mind/Matter)

Math balance (Creative Publications; Selective Educational Equipment)

Games and Puzzles

Beeline and Hex (Selective Educational Equipment)

Cal-Q-Late (Scott, Foresman)

Checkermatics, Levels 2 & 3 (Selective Educational Equipment)

Chess, Checkers, (3-D) Tic-Tac-Toe (school supplier; Sigma)

Equations (SRA; Wff'N Proof)

FOO (Cuisenaire Co.)

GO (Mind/Matter; Sigma)

Inter-union Logic Game (Sigma)

Kalah (Mancala, Oh-Wah-Ree) (Creative Publications)

Math Match (Creative Publications)

Mudcrack (Creative Publications)

On-Sets (Wff'N Proof)

Operational Systems Games – Set Games (Webster, McGraw-Hill)

Tac-Tickle (Wff'N Proof)

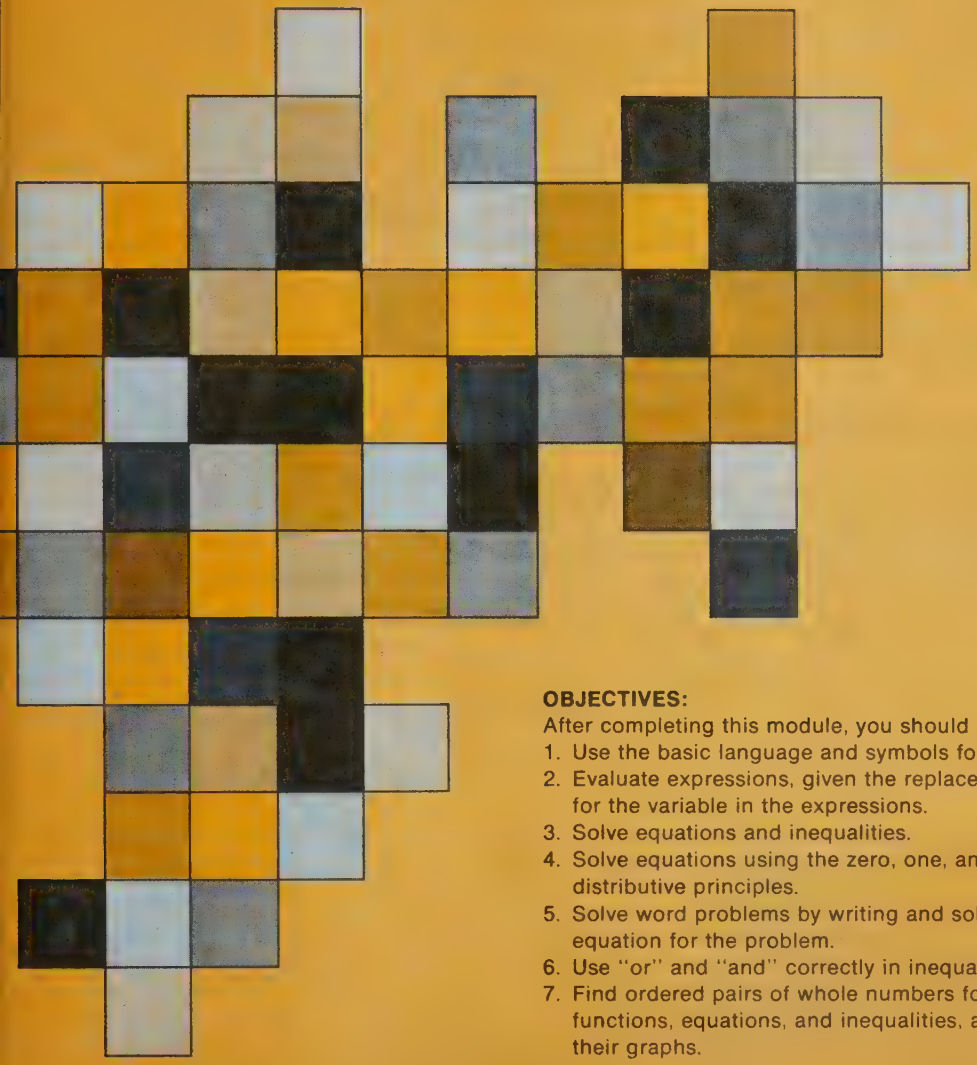
Tri-Nim (Wff'N Proof)

TUF (Cuisenaire Co.; TUF)

Wff'N Proof (Cuisenaire Co.; Wff'N Proof)

UNIT E: *Integers*

MODULE 2: *Sets, Equations, and Inequalities*



OBJECTIVES:

After completing this module, you should be able to:

1. Use the basic language and symbols for sets.
2. Evaluate expressions, given the replacement set for the variable in the expressions.
3. Solve equations and inequalities.
4. Solve equations using the zero, one, and distributive principles.
5. Solve word problems by writing and solving an equation for the problem.
6. Use "or" and "and" correctly in inequalities.
7. Find ordered pairs of whole numbers for functions, equations, and inequalities, and draw their graphs.

Objectives

The student can (A) distinguish between finite and infinite sets; (B) determine if an element belongs to a given set; (C) identify and name subsets of sets; and (D) show relationships of sets by means of Venn diagrams.

PREPARATION

The notion of a set is familiar to most students at this grade level. However, some of the language and notation presented in this lesson is likely to be new to some students. Since there are many ideas contained in this lesson you should cover the material slowly and carefully, without overteaching it. It is important for students to develop good intuitive feelings about set concepts and to be able to use set language and notation well enough to communicate their ideas; there is no need at this level for them to become experts in set theory.

INVESTIGATION

Students can use the given sets A, B, and C as models for the sets described in parts 1, 2, and 3 of the Investigation. We can designate the sets in parts 1 and 3 by listing a few elements.

1. {1, 3, 5, 7, 9, 11, . . .}
3. {0, 1, 4, 9, 16, 25, . . .}

However, for the set of points in the interior of a circle, no simple listing is possible. Students may suggest a picture of a circle, with the interior shaded to represent the set of points.

DISCUSSION

Only intuitive descriptions of finite and infinite sets are given. More precise definitions are given in the Mathematics section of the introduction to this module. Such definitions may be given to students who are capable of understanding them.

Exercises 2 through 5 introduce notations and language for *is an element of* ($\in$), *is a subset of* ($\subset$), and *empty set* (ϕ). You may also want to discuss Venn diagrams (see Exercise 5, on page E-31) before assigning the exercises.

Sets – Language and Symbols

Investigating the Ideas

The language of sets can be found throughout all of mathematics. Some examples are given below.

A The **set** of whole numbers: {0, 1, 2, 3, 4, 5, 6, . . .}

B The **set** of points on a line:

C The **set** of even whole numbers less than ten: {0, 2, 4, 6, 8}

1 The set of odd whole numbers.

2 The set of points inside a circle.

3 The set of numbers that are the squares of the whole numbers.

Discussing the Ideas

1. The three dots in set A indicate that the numbers in the set continue forever in this pattern. The set of whole numbers is an **infinite set**. If a set is not infinite, then it is **finite**. What kind of a set is each of the three sets that you found for the Investigation question?

2. In set C above, we say that 2 is an element of the set and write this as $2 \in C$. The number 3 is not an element of set C and we write $3 \notin C$.
   A Is $8 \in C$? B Is $7 \in C$? C Is $12 \in C$?

3. If $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{2, 4, 6\}$, then Y is a **subset** of X because every element in Y is an element in X. We write: $Y \subset X$
   Which of these sets are subsets of X?
   $A = \{0, 1, 2, 3\}$ $B = \{4, 5\}$ $C = \{3, 5, 7\}$ $D = \{1, 2, 3, 4, 5, 6\}$

4. The set which has no elements is called the **empty set** and is denoted by { } or ϕ. The set of whole numbers greater than 4 and less than 5 is the empty set. Describe the empty set in other ways.

5. The empty set is a subset of every set. Does the empty set have any subsets?

Using the Ideas

1. Tell whether each set is **infinite** or **finite**.
   - **A** The set of multiples of 3.
   - **B** The set of points on the sides of a square.
   - **C** The set of whole numbers that are less than or equal to one trillion.
   - **D** The set of men who have walked on the moon.

2. Suppose $R = \{0, 1, 2, 3, 4, 5, 6\}$, $S = \{2, 4, 6\}$, and $T = \{0, 1, 2, 3, 4\}$. Which statements are true and which are false?

   A $4 \in R$ **C** $5 \in T$ **E** $1 \in T$ **G** $S \subset R$ **I** $T \subset R$

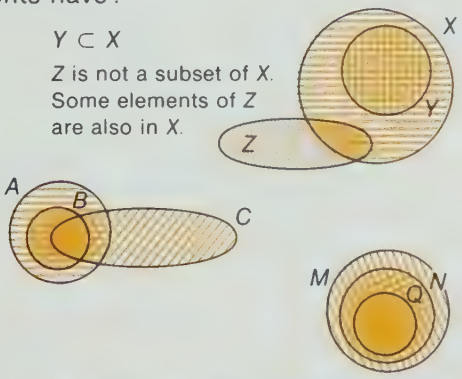
   B $5 \in S$ **D** $0 \in S$ **F** $R \subset S$ **H** $S \subset T$ **J** $T \subset \phi$

3. List all of the subsets of S in Exercise 2.

4. How many subsets does a set of four elements have?

5. A **Venn diagram** is used to show relationships between two or more sets. Think of the ovals or circles as enclosing the elements of the sets.

   Describe three relationships between sets A, B, and C shown at the right.



6. What are the relationships between sets M, N, and Q shown by the Venn diagram at the right?

7. Suppose $J \subset K$ and $K \subset L$. What other relationship must be true?

8. If $A \subset B$ and $B \subset C$, what can you conclude?

★ 9. The set $W = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$, the whole numbers, is an infinite set.
   - **A** Give a subset of W which is also an infinite set.
   - **B** Give a subset of the subset for Part **a** which is also an infinite set.

think

How many different squares can you find in the figure?

UTILIZATION

Exercises 1 and 2 can be discussed orally. It would be useful to discuss all the exercises after the students have completed them, in order to check on any difficulties that students may have encountered.

Solution, Exercise 4

A set having n elements will have 2^n subsets, including the empty set. Thus, a set of 4 elements will have $2^4 = 16$ subsets.

Think Solution

8 small squares
5 (2×2) squares
4 (3×3) squares
1 large square
18 total

EXTENSION

Enrichment: Some students will find the idea of infinite sets interesting to pursue. Infinite sets have some interesting and seemingly paradoxical properties. For example, the one-to-one correspondence between the set of whole numbers and the set of even whole numbers

$$\begin{array}{cccccccc} \{0, & 1, & 2, & 3, & 4, & 5, & \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{0, & 2, & 4, & 6, & 8, & 10, & \ldots\} \end{array}$$

shows that there are as many even whole numbers as there are odd and even whole numbers together. Both of these infinite sets are said to have the same *transfinite cardinal number*. For these sets the number is called *aleph-null* and is denoted $\aleph_0$. The set of rational numbers also has the same transfinite number. However, the set of real numbers, those numbers which can be named by decimals, has a larger transfinite number than $\aleph_0$.

Assignments

Minimum 1 and 2, oral; 3; 5; 6.
Average 1 and 2, oral; 3–6.
Maximum 1 and 2, oral; 3–9.

Objective

Given the elements of two sets, the student can give the elements in the union and intersection of the sets.

PREPARATION

An overhead projector can be used effectively with teacher-made or commercial transparencies to illustrate union and intersection of sets.

DISCUSSION

If your students have had previous experiences with the ideas presented in this lesson, you may want to treat the material as review. However, if the ideas are new to some of the students, you should move more slowly and make certain that students clearly understand the distinction between union and intersection of sets. Use Venn diagrams freely to help illustrate the ideas. At this point we are principally concerned with the intersection and union of sets of numbers, but geometric examples are also useful and sometimes more enlightening.

You may wish to discuss whether or not the operations of union and intersection are commutative or associative and whether there is an analog of the distributive principle using these operations. It is interesting to observe that union and intersection are both commutative and associative operations and we can show that either operation distributes over the other. Thus, there are two forms of a distributive principle:

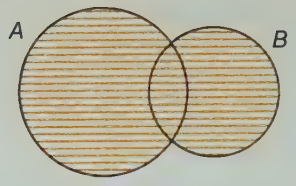
$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

and

$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Union and Intersection of Sets

Discussing the Ideas

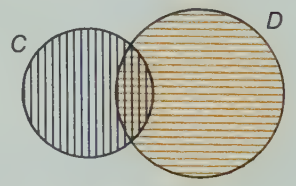


$A \cup B$

1. The union ($\cup$) of set A and set B is the set containing all the elements in A or in B or in both A and B. We write $A \cup B$ for the union of A and B.

   A In the Venn diagram, what parts of circular regions A and B represent the union of set A and set B?

   B If $A = \{5, 6, 7, 8, 9, 10\}$ and $B = \{7, 8, 9, 10, 11, 12\}$, which elements are in set $A \cup B$?

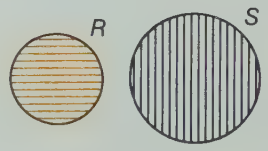


$C \cap D$

2. The intersection ($\cap$) of set C and set D is the set containing all the elements that are common to both C and D. We write $C \cap D$ for the intersection of C and D.

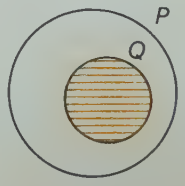
   A In the Venn diagram, what parts of circular regions C and D represent the intersection of set C and set D?

   B If $C = \{5, 6, 7, 8, 9, 10\}$ and $D = \{7, 8, 9, 10, 11, 12\}$ which elements are in $C \cap D$?



$R \cap S = \phi$

3. If set R and set S have no elements in common, their intersection is the empty set. R and S are **disjoint sets**. Suppose $X = \{1\,2, 3, 4, 5\}$, $Y = \{2, 4, 6, 8\}$, and $Z = \{6, 7, 8, 9\}$. Which pair of sets are disjoint?



4. Suppose sets P and Q are such that $Q \subset P$.

   A Describe $P \cup Q$.

   B Describe $P \cap Q$.

Using the Ideas

1. If $A = \{3, 6, 9, 12\}$ and $B = \{1, 3, 5, 7, 9\}$, give the elements in each of the following sets:
   - A $A \cup B$
   - B $A \cap B$

2. If $R = \{0, 1, 2, 3, 4, 5\}$, $S = \{3, 4, 5, 6, 7\}$, and $T = \{7, 8, 9\}$, list the elements in each of the following sets.
   - A $R \cup S$
   - B $R \cup T$
   - C $S \cup T$
   - D $R \cap S$
   - E $R \cap T$
   - F $S \cap T$

3. Use the sets R, S, and T of Exercise 2 to form the following sets.
   - A $(R \cup S) \cap T$
   - B $(R \cap S) \cup T$
   - C $(R \cap S) \cap T$
   - D $(R \cup S) \cap (R \cup T)$
   - E $(R \cap S) \cup (R \cap T)$
   - F $R \cup (S \cap T)$
   - G $(R \cap T) \cup (S \cap T)$
   - H $(R \cup T) \cap (S \cup T)$
   - I $S \cup (R \cap T)$
   - J $R \cap (S \cup T)$
   - K $(R \cap S) \cup (T \cap S)$
   - L $(R \cup T) \cap S$

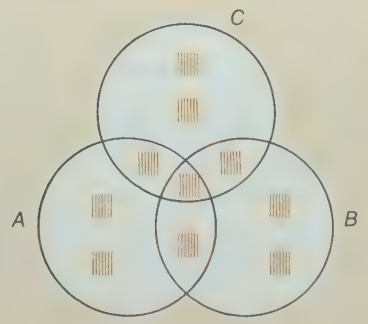
4. Draw a Venn diagram that will show how the three sets of Exercise 2 are related.

5. A Let S be the subset of the whole numbers which are exactly divisible by 4. List several numbers in S.
   - B Let T be the subset of the whole numbers which are exactly divisible by 6. List several elements in this set.
   - C List several numbers in $S \cap T$.
   - D What special properties do the numbers in $S \cap T$ have?

Think

1. Use the numbers in set
   $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
   to replace the ▒ in sets A, B, and C so that
   - A each number in X is used exactly once and
   - B the sum of the numbers in each of the sets A, B, and C is 27.

2. Repeat Exercise 1 making the sum in each set 22.

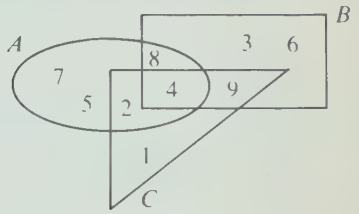


UTILIZATION

More capable students may be interested in observing the two forms of distributivity in Exercise 4. For example parts A and G show that $(R \cup S) \cap T = (R \cap T) \cup (R \cap T)$. Of course, the general proof of this statement should not be presented at this level.

EXTENSION

Enrichment: Duplicate a diagram such as the one below, and distribute copies to the students.



A large variety of problems about sets A, B, and C can be formulated; for example:

1. Find the sum of the numbers in these sets.
   A. $A \cup B$ B. $A \cup C$ C. $A \cap B$
   D. $B \cup C$ E. $B \cap C$ F. $A \cap C$
2. Repeat Exercise 1 but find the products instead of the sum.
3. Find the sum of the numbers in $A \cup B \cup C$.
4. Find the product of the numbers in $B \cap C$, then subtract the sum of the numbers in A ∩ B.

Assignments

Minimum 1–2. Average 1, 2, 3A–H, 4. Maximum 1–5.

Objective

Given algebraic expressions such as $3x - 2$ or $n^2 + 4n + 1$ and a replacement for the variable, the student can evaluate the expressions.

PREPARATION

Review the usual agreements that are made when multiple arithmetic operations are involved in expressions. When no parentheses occur in an expression, we agree to multiply adjoining numbers before adding or subtracting. Thus, $7 \cdot 4 + 3 = 28 + 3 = 31$ and $5^2 - 2 \cdot 8 = 25 - 16 = 9$.

INVESTIGATION

Students should read through the exposition and then evaluate expressions B and C in a manner similar to the example for A.

B. $2x + 3$	C. $x^2 + 5$
$x = 0 \rightarrow 2 \cdot 0 + 3 = 3$	$x = 0 \rightarrow 0^2 + 5 = 5$
$x = 1 \rightarrow 2 \cdot 1 + 3 = 5$	$x = 1 \rightarrow 1^2 + 5 = 6$
$x = 2 \rightarrow 2 \cdot 2 + 3 = 7$	$x = 2 \rightarrow 2^2 + 5 = 9$
$x = 3 \rightarrow 2 \cdot 3 + 3 = 9$	$x = 3 \rightarrow 3^2 + 5 = 14$
$x = 4 \rightarrow 2 \cdot 4 + 3 = 11$	$x = 4 \rightarrow 4^2 + 5 = 21$
$x = 5 \rightarrow 2 \cdot 5 + 3 = 13$	$x = 5 \rightarrow 5^2 + 5 = 30$

DISCUSSION

Most students have evaluated expressions such as the ones in this lesson before, but probably most often on an intuitive level. This lesson is more algebraic in character and forms a basis for later work with simple algebraic equations.

Students should be expected to be able to evaluate expressions such as those in Exercise 4 mentally or with a minimum amount of paper-pencil work.

Evaluating Expressions

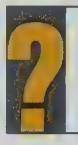
Investigating the Ideas

In algebraic problems, you must often evaluate expressions like these. →

A	$x + 6$
B	$2 \cdot x + 3$
C	$x^2 + 5$

To evaluate the expression, you replace the letter with a number. Suppose that you replaced **x** in expression **A** with each of the numbers in the set {0, 1, 2, 3, 4, 5,}, you would get →

$x = 0 \rightarrow 0 + 6 = 6$
$x = 1 \rightarrow 1 + 6 = 7$
$x = 2 \rightarrow 2 + 6 = 8$
$x = 3 \rightarrow 3 + 6 = 9$
$x = 4 \rightarrow 4 + 6 = 10$
$x = 5 \rightarrow 5 + 6 = 11$



Can you evaluate expressions **B** and **C** using the numbers in the set {0, 1, 2, 3, 4, 5, 6}?

Discussing the Ideas

1. Explain how you evaluated expression **B**.

2. What did you do to evaluate expression **C**?

3. The set of numbers {0, 1, 2, 3, 4, 5} that you used is called a **replacement set**. Each number in the set is a replacement for the **variable x**. Any letter can be used as a variable in an expression.
   What are the variables in each of these expressions?
   A $y + 1$ **B** $3 \cdot n + 7$ **C** $r^2 - 1$ **D** $z \div 4$

4. Evaluate each expression for the replacement given.
   A If $a = 7$, then $2 \cdot a =$ ▒.
   B If $r = 3$, then $11 - r =$ ▒.
   C If $s = 10$, then $3 \cdot s + 13 =$ ▒.
   D If $y = 6$, then $y^2 - 4 =$ ▒.
   E If $t = 8$, then $13 - t =$ ▒.
   F If $q = 35$, then $q \div 7 =$ ▒.
   G If $n = 5$, then $n^3 - 5 =$ ▒.
   H If $w = 15$, then $8 \cdot w =$ ▒.

Using the Ideas

1. Evaluate the expression $x + 9$ for each number in the set $\{1, 2, 3\}$.

2. Evaluate $12 - y$ if the replacement set is $\{5, 6, 7, 8\}$.

3. Evaluate $3 \cdot x - 2$. The replacement set is $\{0, 2, 4, 6\}$.

4. Find the missing number for each part.
   - **A** If $s = 2$, then $8 + s =$ ▒▒.
   - **B** If $w = 9$, then $6 \cdot w =$ ▒▒.
   - **C** If $z = 12$, then $z \div 4 =$ ▒▒.
   - **D** If $x = 7$, then $x \cdot x =$ ▒▒.
   - **E** If $n = 6$, then $(5 \cdot n) - 1 =$ ▒▒.
   - **F** If $t = 2$, then $\frac{12 - t}{5} =$ ▒▒.
   - **G** If $a = 24$, then $a \div 3 =$ ▒▒.
   - **H** If $d = 7$, then $(6 \cdot d) \div 7 =$ ▒▒.

5. Suppose the replacement set for $4 \cdot n - 13$ is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. What is the replacement for n that will make the value of the expression 19?

6. In the expression $x^2 - x$, each x must be replaced with the same number from the replacement set.
   If $x = 3$, then $x^2 - x = 3^2 - 3 = 9 - 3 = 6$.
   Evaluate $n^2 + n + 1$ for each replacement in the set $\{0, 1, 2, 3, 4\}$.

7. **A** Evaluate $y^2 + 2 \cdot y + 1$ using the numbers in set $\{1, 2, 3\}$ as the replacement set.
   B Evaluate $(y + 1)^2$ using the numbers in set $\{1, 2, 3\}$ as the replacement set.

think

Can you find a pattern for the last digit of the numbers that are of the form n^n where n is a whole number?

n	1	2	3	4	5	6	. . .
n^n	$1^1 = 1$	$2^2 = 4$	$3^3 = 27$	$4^4 = 256$	$5^5 = 3125$	$6^6 = ?$	. . .
Last digit	1	4	7	6	5	6	. . .

More practice, page S-5, Set 8

UTILIZATION

You may need to emphasize the fact that when a variable appears more than once in the same expression, as in Exercise 4D, the same number must replace it throughout the expression.

It is expected that students will solve Exercise 5 by substituting each number in the replacement set for n to find when the expression has a value of 19 rather than using any algebraic techniques.

Think Solution

The pattern of digits is not easy to discover. It will surprise many students to discover that the pattern does not start repeating after 10^{10}, but rather after 20^{20}. To find the last digit of a power such as 13^{13}, certain shortcuts can be used. Since the last digit of 3^3 is 7, the last digit of 13^3 is also 7. Then $13^6 = 13^3 \cdot 13^3$ so 13^6 has a last digit of 9 $(7 \cdot 7)$. $13^{12} = 13^6 \cdot 13^6$ so 13^{12} has a last digit of 1 $(9 \cdot 9)$. $13^{13} = 13 \cdot 13^{12}$; hence the last digit of 13^{13} is 3 $(1 \cdot 3)$.

The pattern is:
1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0.

EXTENSION

To provide further practice with the concepts presented in this lesson, make selective assignments from Supplementary Exercise Set 8 on page S-5, *Workbook* page 9, and *Duplicator Masters* page 9.

Enrichment: Challenge your more able students with this "Jet Airliner Expression." Evaluate for $n = \{0, 2, 3, 4\}$.

Expression: $\frac{n^3 - 19n^2 + 196n - 480}{34n - 120}$

This expression will give the number of engines on a 7n7 jet airliner.

Thus: $n = 0 \rightarrow$ engines on a 707 jet (4)
$n = 2 \rightarrow$ engines on a 727 (3)
$n = 3 \rightarrow$ engines on a 737 (2)
$n = 4 \rightarrow$ engines on a 747 (4)

Assignments

Minimum 1–3, 4A–D. Average 1–5. Maximum 1–7.

Objective

Given appropriate equations with the set of whole numbers as the replacement set, the student can find the solution set.

PREPARATION

Some oral practice in finding missing addends and factors could precede the Investigation. Use examples such as these:

"I am thinking of a certain number. If you add 8 to it you will get 13 as the sum."

"I am thinking of a number. When it is multiplied by 12, the product is 48. What is the number?"

INVESTIGATION

The Investigation can be completed quickly. Finding the number covered by the hand introduces students to the basic problem they confront in solving equations.

DISCUSSION

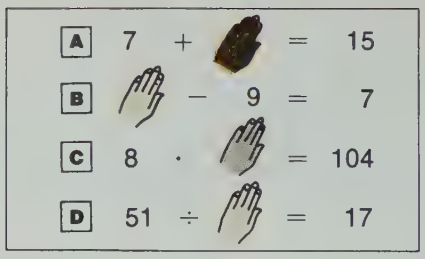
The inverse relation between addition and subtraction and between multiplication and division is the most important mathematical concept students will use in this section. For example, when students think that to solve the equation $x + 5 = 9$, they can write $x = 9 - 5$, they are using the inverse relation between addition and subtraction. Likewise, when they solve $3 \cdot x = 12$ by writing $x = 12 \div 3$, they are applying the inverse relation between multiplication and division.

No formal manipulative rules for solving equations are given at this time. Instead, the equations are kept purposely simple so that the significant ideas are not obscured and the inverse relations will be easily seen.

Solving Equations

Investigating the Ideas

When the replacement set for an open sentence is the set of all whole numbers, then it is impossible to try every number as a possible solution. Instead, you can often find the solutions by thinking about missing addends, sums, factors, or products.



Suppose the hand in each equation covers the number that makes each sentence a true statement.

Can you find the number that is covered by the hand in each equation?

Discussing the Ideas

1. A In equation A above, did the hand cover a missing addend or a sum?
   B What was the number covered by the hand?
   C In what way is equation A like the equation $7 + x = 15$?

2. A In equation B, did the hand cover a missing addend or a sum?
   B What was the number covered by the hand?

3. A In equation C, does the hand cover a factor or a product?
   B In equation D, does the hand cover a factor or a product?
   C What are the numbers covered in equations C and D?

4. We often omit the multiplication symbol between a numeral and a variable or between two variables.
   Instead of $6 \cdot x$, we write $6x$.
   Instead of $r \cdot s$, we write rs.
   Instead of $5 \cdot (a + 4)$, we write $5(a + 4)$.
   How would you write each of the following without using the multiplication symbol?
   A $8 \cdot y$ B $\ell \cdot w$ C $(z + 2) \cdot 7$ D 6 times r

5. Why would you not write a product like $3 \cdot 7$ without using the multiplication symbol?

Using the Ideas

The replacement set for the equations on this page is the set of whole numbers.

1. Give the number for each ▒. Then give the number for n.

A $5 + n = 12$
$n = 12 - ▒$
$n = ?$

B $3n = 21$
$n = ▒ \div 3$
$n = ?$

C $n - 8 = 3$
$n = 3 + ▒$
$n = ?$

D $n \div 6 = 3$
$n = 3 \cdot ▒$
$n = ?$

E $9n = 45$
$n = 45 \div ▒$
$n = ?$

F $13 - n = 6$
$6 + n = ▒$
$n = ?$

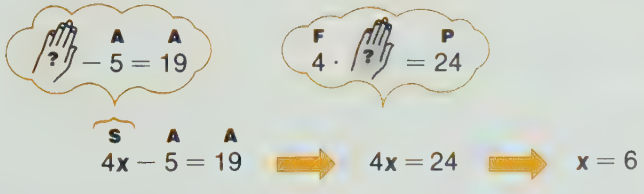
2. Solve the equations.

A $x + 5 = 9$	**E** $7d = 56$	**I** $10 + m = 100$	**M** $r \div 3 = 8$
B $7 + a = 13$	**F** $8 + b = 34$	**J** $10n = 100$	**N** $9 = 28 - a$
C $20 = m + 1$	**G** $56 = 18 + g$	**K** $6a = 72$	**O** $5 = 15 \div s$
D $s + 9 = 10$	**H** $10w = 70$	**L** $49 = 7 + c$	**P** $q - 8 = 4$

3. Study the example.

EQUATION: $4x - 5 = 19$

SOLUTION:



Give the number for the ▒. Then give the number for x.

A $2x + 3 = 11$
$2x = ▒$
$x = ?$

C $2x + 4 = 24$
$2x = ▒$
$x = ?$

E $2x + 3 = 17$
$2x = ▒$
$x = ?$

B $4x - 3 = 17$
$4x = ▒$
$x = ?$

D $3x - 5 = 13$
$3x = ▒$
$x = ?$

F $25 = 5 + 5x$
$▒ = 5x$
$? = x$

4. Solve the equations.

A $4x + 1 = 21$	**C** $5x - 1 = 14$	**E** $2x + 9 = 11$
B $3x + 7 = 37$	**D** $8x + 3 = 3$	**F** $6x - 5 = 31$

More practice, page S-6, Set 10

UTILIZATION

At this stage in students' development, it is probably more valuable for them to attack problems by intuitive methods than to use the more mechanical procedures that might be introduced. Thus, the "hand-covering" technique illustrated in Exercise 3 is easily understood by students and is recommended, especially in illustrating two-step solutions.

Students can be asked to "check" their solutions to some of the equations by substituting their solutions for the variables in the original equation.

It should be pointed out to the students that equations often are given without a specifically denoted replacement set. In such cases, we agree that the replacement set is the largest set available – in this module, the set of whole numbers. After integers are studied, the replacement set may be the set of all integers.

EXTENSION

Additional practice with the concepts presented in this lesson is provided by Supplementary Exercise Set 10 on page S-6 and *Workbook* page 11.

Enrichment: Equations such as the ones below may offer extra challenge to your more capable students. (Solutions are given in parentheses.)

A. $\frac{x+2}{5} = 2$
$x = ▒$ (8)

B. $\frac{432}{x} = 8$
$▒ = x$ (54)

C. $\frac{x+3}{8} = 1$
$x = ▒$ (5)

D. $\frac{x-3}{2} = 4$
$x = ▒$ (11)

E. $\frac{x-1}{3} = 1$
$x = ▒$ (4)

F. $\frac{12}{x+1} = 2$
$x = ▒$ (5)

★G. $2(3x - 5) = 14$
$x = ▒$ (4)

★H. $\frac{2x+3}{5} = 3$
$x = ▒$ (6)

Assignments

Minimum 1, 2A–H, 3A–C.
Average 1, 2A–L, 3, 4A–C.
Maximum 1, 2I–P, 3, 4.

Objective

The student can solve appropriate equations by applying the zero and one principles.

PREPARATION

Review the fact that 0 is the identity element for addition and 1 is the identity element for multiplication in the set of whole numbers.

INVESTIGATION

Equations A through E are generalizations for the set of whole numbers. However, equation F is not a generalization for the set of all whole numbers, but only for the nonzero whole numbers. This is a subtle point but one that needs emphasis.

The generalizations in A through E are theorems in a formal course of algebra, but their proofs would be out of place at this grade level. Generalizations A and D are direct consequences of the property of 0 and the definition of subtraction:

If $x + 0 = x$, then $x - x = 0$.
Or $x - 0 = x$.

Generalization E is a consequence of the one principle and the definition of division:

If $x \cdot 1 = x$, then $x = x \div 1$.

Equation B is a generalization based on the zero and distributive principles. Equation C is a generalization based on the one and distributive principles.

DISCUSSION

Give careful attention to Discussion Exercises 3 and 4, making certain that students understand the role of the zero and one principles in these problems. Give additional examples as needed, such as these:

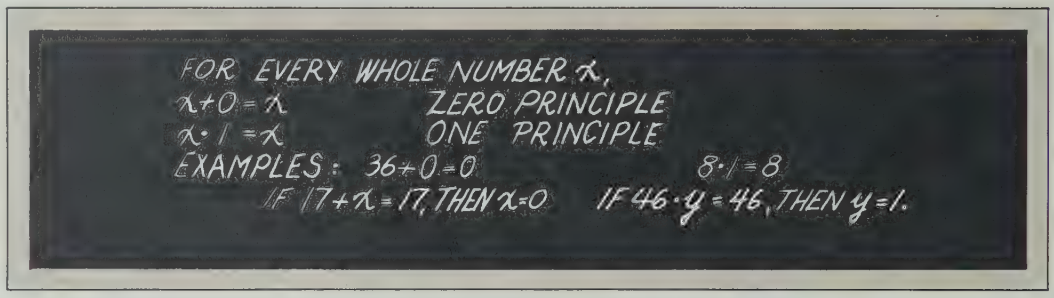
$9 + (x - 3) = 9$
$(x - 3) = 0$ (Zero principle: $9 + 0 = 9$)
$x = 3$

$12 \cdot (9 - x) = 12$
$(9 - x) = 1$ (One principle: $12 \cdot 1 = 12$)
$x = 8$

Using the Zero and One Principles

Investigating the Ideas

The zero principle and the one principle are examples of generalizations because they are true statements for every whole number.



?

How many of the these equations are generalizations?

A $x - x = 0$ **C** $x + x = 2x$ **E** $x \div 1 = x$

B $x \cdot 0 = 0$ **D** $x - 0 = x$ **F** $x \div x = 1$

Discussing the Ideas

1. Are there any equations in **A** through **F** above that are not generalizations? Explain.

2. **A** In equation **F**, if **x** is replaced by 0, you have $0 \div 0 = 1$. Is this a true statement?
   B Is equation **F** true if **x** is replaced by any whole number **except** 0?

3. Explain how to find the number for each screen. Think about the zero principle.

   If $21 + (x - 5) = 21$,
   then $x - 5 =$ ▒▒
   and then $x =$ ▒▒

4. Explain how to find the number for each screen. Think about the one principle.

   If $8 (x - 3) = 8$,
   then $x - 3 =$ ▒▒
   and then $x =$ ▒▒

Using the Ideas

1. Give the number for each ▒. Then find the number for x.

A $6 + (x - 1) = 6$
$x - 1 =$ ▒
$x =$?

B $7(3 - x) = 7$
$3 - x =$ ▒
$x =$?

C $4(x - 2) = 0$
$x - 2 =$ ▒
$x =$?

D $9(5 - x) = 9$
$5 - x =$ ▒
$x =$?

E $13 + (x - 11) = 13$
$x - 11 =$ ▒
$x =$?

F $7 - (x + 3) = 0$
$x + 3 =$ ▒
$x =$?

G $8 \div (x + 2) = 1$
$x + 2 =$ ▒
$x =$?

H $6x + 18 = 18$
$6x =$ ▒
$x =$?

I $(x - 8) - 3 = 0$
$x - 8 =$ ▒
$x =$?

J $(x + 3) \div 10 = 1$
$x + 3 =$ ▒
$x =$?

K $(x + 3) + 10 = 25$
$x + 3 =$ ▒
$x =$?

L $9(12 \div x) = 54$
$12 \div x = 6$
$x =$?

2. Copy and solve the equations.

A $17 - (8 - x) = 17$

B $6(x - 4) = 6$

C $3(x - 5) = 0$

D $7(3x - 6) = 0$

E $(x + 3) \div 1 = 10$

F $3(x - 5) = 3$

G $5(2x - 10) = 0$

H $9 + (2x - 12) = 9$

I $8 - (5 - x) = 7$

think

Try this Addition Mystery Trick on a friend. Ask your friend to write a four-digit number. You look at the number and then write the Mystery Sum on another piece of paper. Ask your friend to write two more four-digit numbers. Then you quickly write the last two numbers to give the Mystery Sum.

4728	Friend
3524	Friend
6953	Friend
6475	You
3046	You
24726	Mystery Sum

1. Study the example and explain the trick.

2. What would the Mystery Sum be if the first number were 2369?

UTILIZATION

Students are given considerable guidance in organizing their work in Exercise 1. In the first step, encourage students to think of the expression in the parentheses as a single number. Once this number is known, the solution to the equation can be quickly found.

Students will benefit from having some parts of Exercise 2 displayed on the chalkboard and discussed.

EXTENSION

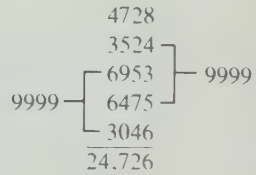
Page 12 of the *Workbook* and page 11 of the *Duplicator Masters* offer exercises that are suitable for additional practice with the concepts of this lesson.

Enrichment: Ask students to solve these equations. (For your convenience, the solutions are given in parentheses.)

1. $3(x - 4) = 0$ (4)
2. $18(3x - 5) = 18$ (2)
3. $6 + (2x - 12) = 6$ (6)
4. $7(2x - 10) = 0$ (5)
5. $9(x - 5) = 9$ (6)
6. $(x + 4)(x - 1) = 0$ (1)

Think Solution

The mystery sum is always the friend's first number plus 20 000 minus 2, or his number plus 19 998. This sum can always be obtained if the friend's second and third number and the two numbers you write have a sum of 19 998. This is easy to do if each of the numbers you write when paired with your friend's second and third numbers gives a sum of 9999. In the case illustrated note the pairs that give the sum of 9999.



Assignments

Minimum 1A–H. Average 1, 2A–C.
Maximum 1, 2.

Objective

The student can find sums such as $3x + 2x$ and $9x + 5x$ by applying the distributive principle.

PREPARATION

Write some expressions such as $4x + 3$, $19 - 3x$, $6x^2$ on the chalkboard. Practice mental evaluations of the expressions for x using a small subset of whole numbers as a replacement set for x.

INVESTIGATION

Some students will quickly discover that a simple way to evaluate $2x + 3x$ is to evaluate the expression $5x$. Other students are likely not to notice this. You may extend the replacement set to 10 or more numbers for added emphasis.

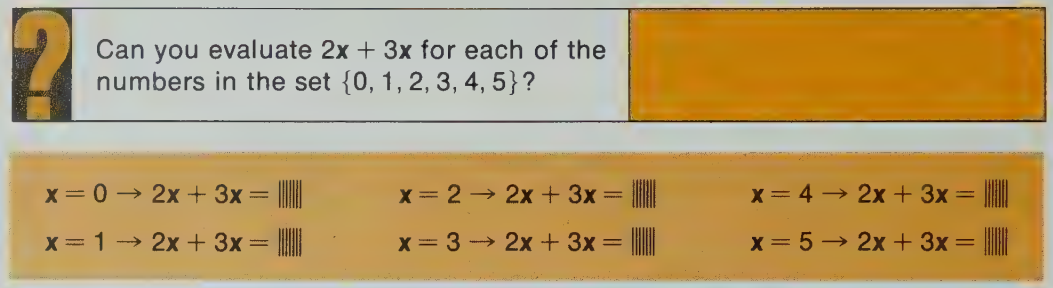
DISCUSSION

The distributive principle, which is exemplified in Exercise 3, plays an important role in algebra. By applying the principle to equations such as $2x + 3x = 30$, students can extend their equation solving ability.

Discussion Exercise 5 is concerned with an extended form of the distributive principle in which there are more than two addends. Be sure to emphasize Discussion Exercise 6, which states that multiplication distributes over subtraction.

The Distributive Principle

Investigating the Ideas



Discussing the Ideas

1. Describe any pattern that you see in the set of numbers you found when you evaluated $2x + 3x$.

2. **A** Do you think $2x + 3x = 5x$ is a generalization for the set $\{0, 1, 2, 3, 4, 5\}$?
   B Do you think $2x + 3x = 5x$ is a generalization for all whole numbers?

3. The **distributive principle** can help you understand why $2x + 3x = 5x$ is a generalization. Use this principle to give the number for each screen.

   The Distributive Principle: $ba + ca = (b + c)a$

   $2x + 3x = (\quad + \quad)x = \quad x$

4. Use the distributive principle to give the number for ▒.
   A $6x + 2x =$ ▒x **C** $4x + 5x =$ ▒x **E** $1x + 4x =$ ▒x
   B $3x + 7x =$ ▒x **D** $7x + 6x =$ ▒x **F** $8x + 3x =$ ▒x

5. The distributive principle can be extended to more than two addends. Thus, $ax + bx + cx = (a + b + c)x$. Use this *extended form of the distributive principle* to find the sums.
   A $2x + 5x + 6x$ **B** $3x + 4x + 10x$ **C** $9x + 9x + 9x$

6. Multiplication also distributes over subtraction:
   $$ba - ca = (b - c)a$$
   A What is $5x - 3x$? **B** What is $10x - 4x$?

Using the Ideas

1. Find the sum or difference.

A $5x + 7x$	E $10x - 3x$	I $6x + 6x + 6x$
B $2x + 9x$	F $7x + 1x$	J $7x + 10x + 4x$
C $8x - 2x$	G $6x - 5x$	K $8x + 3x + 12x$
D $4x + 8x$	H $2x + 4x + 3x$	L $(7x - 2x) + 3x$

2. Since $x \cdot 1 = 1 \cdot x = x$ for all x, we often omit the numeral 1 before the variable. Find each sum or difference.

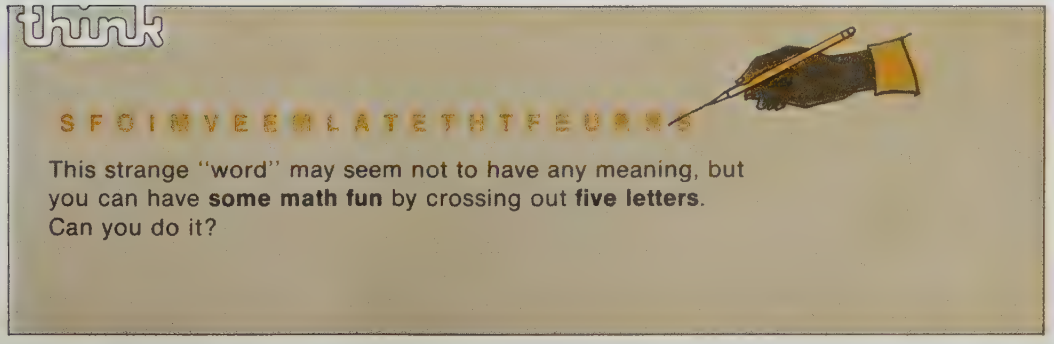
A $x + 2x$	E $10x - x$	I $(7x - 3x) - x$
B $x + x + x$	F $x + x + x + x$	J $x + (8x - 5x)$
C $3x - x$	G $x + x + 6x$	K $x + 2x + 3x + 4x$
D $5x + x$	H $(x + 9x) - 2x$	L $9x - (10x - x)$

3. Give the expression for each ▒.

A $8x = 3x +$ ▒	D ▒ $+ 6x = 13x$	H $3x -$ ▒ $= 2x$
ANSWER: $5x$	E $4x = 6x -$ ▒	I $6x +$ ▒ $+ 3x = 15x$
B $9x =$ ▒ $+ 5x$	F $9x -$ ▒ $= x$	J ▒ $+ x + 2x = 4x$
C $2x +$ ▒ $= 10x$	G $x +$ ▒ $= 2x$	K $(5x -$ ▒ $) + 2x = 6x$

4. Evaluate each expression.

A $7x + 2x$ when $x = 5$	G $10x - 3x$ when $x = 2$
B $4x + 6x$ when $x = 9$	H $x + 5x$ when $x = 10$
C $x + 8x$ when $x = 7$	I $9x - x$ when $x = 0$
D $11x - 5x$ when $x = 8$	J $10x - 10x$ when $x = 3$
E $23x + 7x$ when $x = 4$	K $5x - 4x$ when $x = 5$
F $15x - x$ when $x = 10$	L $(3x - 2x) + x$ when $x = 1$



This strange "word" may seem not to have any meaning, but you can have **some math fun** by crossing out **five letters**. Can you do it?

UTILIZATION

Part or all of Exercises 1 and 2 might be treated orally. In Exercise 2, be sure to emphasize the statement of the agreement that the factor of 1 with a variable is usually omitted: instead of $1 \cdot x$ or $1x$, we simply write x.

EXTENSION

Remedial: For students who find it difficult to understand and use the distributive principle, you might try examples like these:

1A. Two x's and three x's = how many x's?
B. $2x + 3x =$ ▒x

2A. Five x's and four x's = how many x's?
B. $5x + 4x =$ ▒x

Think Solution

Cross out the letters in the expression that spell FIVE LETTERS. The remaining letters spell SOME MATH FUN.

Assignments

Minimum 1, oral; 2.
Average 1 and 2, oral; 3; 4A–F.
Maximum 1 and 2, oral; 3; 4.

Objective

The student can use the distributive principle in solving appropriate equations.

PREPARATION

Conduct a brief oral review of the use of the distributive principle in expressions such as $9x + 2x$, $4x - 3x$, and $x + 4x$.

INVESTIGATION

The Investigation presents the kind of equation students will work with throughout this lesson. By applying the distributive principle to $7x + 2x$, students can see that a must be 9. Hence $9x = 72$ and therefore $x = b = 8$.

DISCUSSION

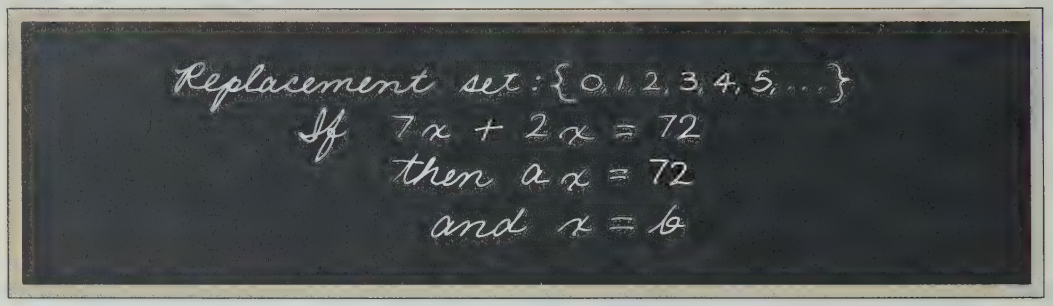
Discuss the steps that are followed in order to solve the equations in Exercises 2, 3, and 6. Provide additional examples if necessary.

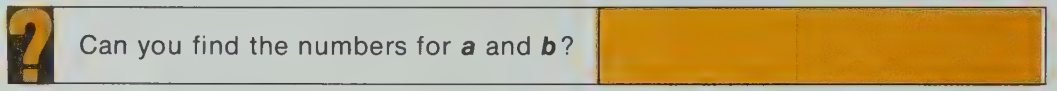
The usual substitution method of "checking" an apparent solution to an equation is actually an important, logical part of the equation-solving process. In more advanced algebraic work, students may encounter equations that yield apparent solutions which, if substituted in the equation, do not satisfy the equation; that is, they do not result in a true statement. Therefore the check is a vital step.

Solving Equations Using the Distributive Principle

Investigating the Ideas

Study the equations below.





Discussing the Ideas

1. A How did you find the number for a?
   B How did you find the number for b?

2. Complete each step in the solution of this equation:

$$5x + 3x = 56$$
$$8x = \blacksquare$$
$$x = \blacksquare$$

3. Explain each step in the solution of this equation:

$$9x - 4x = 40$$
$$5x = 40$$
$$x = 8$$
CHECK: $9 \cdot 8 - 4 \cdot 8 = 72 - 32 = 40$

4. Explain how the check of the solution was made.

5. Check the solution you found for the equation in the Investigation.

6. Explain or show how you would solve and check this equation. $(5x + x) + 1 = 19$

Using the Ideas

1. Copy the equations, giving the correct number for each ▒.

A
$2x + 4x = 18$
$6x = 18$
$x = ▒$

D
$4x + 3x = 7$
$▒x = 7$
$x = ▒$

G
$2x + 3x + 2 = 22$
$5x = ▒$
$x = ▒$

B
$2x + 3x = 30$
$5x = 30$
$x = ▒$

E
$5x - 2x = 12$
$▒x = 12$
$x = ▒$

H
$4x + x - 5 = 30$
$5x = ▒$
$x = ▒$

C
$10x - 4x = 42$
$6x = 42$
$x = ▒$

F
$(2x + 3x) + 4x = 27$
$9x = ▒$
$x = ▒$

I
$(11x - 4x) - 4 = 10$
$7x = ▒$
$x = ▒$

2. Copy and solve the equations.

A $3x + 4x = 14$

B $5x + 3x = 32$

C $x + 5x = 24$

D $7x - 6x = 5$

E $(8x - 3x) - 2x = 15$

F $11x - 4x = 14$

G $9x - 2x = 21$

★**H** $x + (3x - 3) = 17$

★ **I** $5x + (2 + 3x) = 18$

3. Copy and solve the equations.

A $(2x + 3x) + 2 = 17$

B $4x - x = 9$

C $(3x + 4x) + 1 = 29$

D $(x + 5x) + 3 = 39$

E $(8x - 3x) - 2x = 15$

F $(10x - 5x) + 3 = 18$

★ **G** $6x = 15 + 3x$

★ **H** $8x = 2x + 30$

★ **I** $26 + 3x = 2 + 7x$

think

The number 5 can be partitioned as a sum of whole numbers in the seven ways shown at the right. Do not count the different orders of addends due to commutativity, but include the number itself as one partition.
How many partitions does 6 have?
How many partitions does 7 have?

Partitions of 5

$1 + 1 + 1 + 1 + 1$
$1 + 1 + 1 + 2$
$1 + 1 + 3$
$1 + 2 + 2$
$1 + 4$
$3 + 2$
5

More practice, page S-7, Set 11

UTILIZATION

The format of Exercise 1 gives students guidance in solving the equations. You might find it beneficial to have students show a check of their solutions on some or all of the equations. While the check may seem laborious to many students, it not only provides a check on their work but gives additional practice in evaluating algebraic expressions.

EXTENSION

Extra practice is provided by Supplementary Exercise Set 11 on page S-7, *Workbook* page 13, and *Duplicator Masters* page 12.

Think Solution

11 Partitions of 6:	15 Partitions of 7:
6	7
5 + 1	6 + 1
4 + 2	5 + 2
4 + 1 + 1	5 + 1 + 1
3 + 3	4 + 3
3 + 2 + 1	4 + 2 + 1
3 + 1 + 1 + 1	4 + 1 + 1 + 1
2 + 2 + 2	3 + 3 + 1
2 + 2 + 1 + 1	3 + 2 + 2
2 + 1 + 1 + 1 + 1	3 + 2 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1	3 + 1 + 1 + 1 + 1
	2 + 2 + 2 + 1
	2 + 2 + 1 + 1 + 1
	2 + 1 + 1 + 1 + 1 + 1
	1 + 1 + 1 + 1 + 1 + 1 + 1

Assignments

Minimum 1A–F, 2A–F, 3A–C.
Average 1, 2A–G, 3A–F. Maximum 1–3.

Objective

Given an appropriate word problem, the student can write and solve a related equation.

PREPARATION

Students who have used other books in the *Investigating School Mathematics* series will be familiar with flow charts such as those found in this Investigation. Students who have had little or no experience with flow charts may need some preliminary instruction. Point out the shape of the input box, the output box, and the rectangular instruction boxes. Often, flow charts are shown with "start" and "stop" symbols. We omit them here and present only the essential elements of flow charts.

INVESTIGATION

The flow charts in the Investigation present word problems in a step-by-step sequence. After studying the completed example at the top of the page, students should be able to write equations for flow charts A, B, and C.

DISCUSSION

Student responses for Discussion Exercise 1 should be similar to those given below.

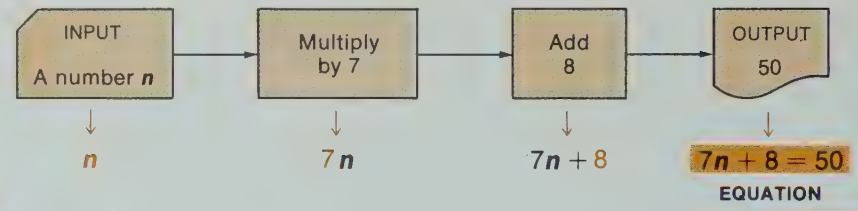
A. If a number x is multiplied by 3 and then 5 is subtracted, the result is 7.
B. If 12 is added to a certain number and the sum is multiplied by 2, the product is 36.
C. If a number n is multiplied by 6 and the same number is then subtracted from the product, the difference is 25.

The purpose of having students construct a flow chart in Exercise 2 is to help them analyze the problem in terms of the steps needed to write the equation. Once this skill is mastered, the construction of flow charts can be discontinued. Present other word problems for similar analyses as needed before assigning the exercises on page E-47.

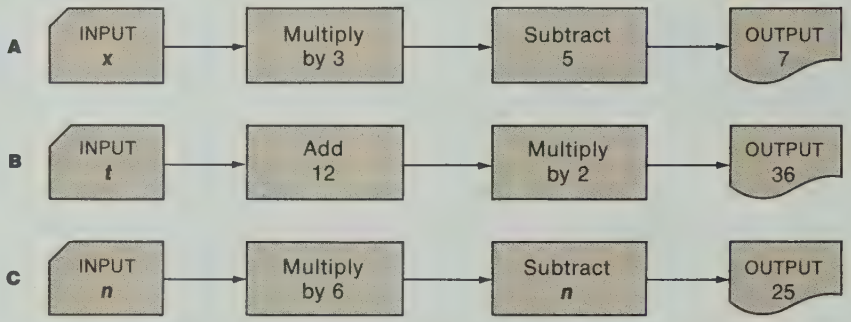
Writing and Solving Equations

Investigating the Ideas

The flow chart below suggests an equation.



Can you write an equation for each of these flow charts?



Discussing the Ideas

1. The equation for the first flow chart above could be stated in words as, "If a certain number n is multiplied by 7 and then 8 is added to the product, the result is 50." How would you state the equation for flow charts A, B, and C?

2. A Make a flow chart suggested by the word problem below.

   If 3 is subtracted from a certain number and the difference multiplied by 7, the product will be 28.

   B Which equation is correct for the word problem?

   A Let n = the number
   $(3 - n)\,7 = 28$

   B Let x = the number
   $(x - 3)\,7 = 28$

   C Let t = the number
   $t - 3 \cdot 7 = 28$

3. What are the solutions for the equations that you wrote for the flow charts in the Investigation?

Using the Ideas

1. Write the equation suggested by each flow chart.

A

B

C

2. Solve each equation for Exercise 1.

In Exercises 3 through 7, design a flow chart to illustrate each problem. Then write and solve an equation for the flow chart.

3. Five times a certain number plus 6 is 46. What is the number?

4. If 6 is subtracted from a certain number and the difference is multiplied by 10, the product is 70. What is the number?

5. Margaret said, "If you multiply my age by 3 and add 1, you will get my mother's age." If Margaret's mother is 40 years old, how old is Margaret?

6. Rex had 87 cents. After paying for five candy bars he had 12 cents left. How much did each bar cost?

7. Diane multiplied a certain number by 5. Karl multiplied the same number by 3. Then they added the two products together and got a sum of 56. What was the number?

★ **8.** Beth is 5 years older than Tina. The sum of their ages 10 years from now will be 53.

A What is the sum of their ages now?

B How old is each girl?

★ **9.** Dave is 3 years older than Chuck. When Chuck is as old as Dave is now, the sum of their ages will be 39. How old is each one now?

think

Mark four points on a piece of paper with no three of the points on the same line. Now connect each point to the other three so that six segments are formed. Show how you can arrange the points so that none of the segments cross each other.

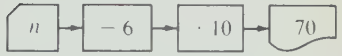
More practice, page S-7, Set 12

UTILIZATION

You might find it worthwhile to have the flow charts for Exercises 3–7 shown on the chalkboard or overhead projector and discussed. With more capable students, however, you might choose to omit the requirement that flow charts be constructed for Exercises 3–7.

Flow Charts, Exercises 3–7

3.

4. 

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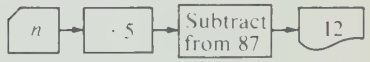
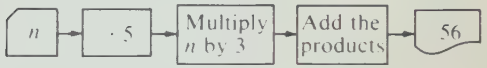
6. 

7. 

Solutions, Exercises 8 and 9

8. Let t = Tina's age now.
$t + 5$ = Beth's age now,
$t + 10$ = Tina's age in 10 years
$t + 5 + 10$ = Beth's age in 10 years
Then $(t + 10) + (t + 5 + 10) = 53$
$2t = 28$
$t = 14$ (Tina)
$t + 5 = 19$ (Beth)
$14 + 19 = 33$

9. Let c = Chuck's age and $c + 3$ = Dave's age.
Then $(c + 3 + 3) + (c + 3) = 39$
$2c = 30$
$c = 15$ (Chuck)
$c + 3 = 18$ (Dave)

EXTENSION

To provide further practice, make assignments from Supplementary Exercise Set 12 on page S-7, *Workbook* page 14, and *Duplicator Masters* page 13.

Assignments

Minimum 1–5. Average 1–7. Maximum 1–9.

Objective

Given inequalities using the conjunctions "or" and "and" and the replacement set, the student can find the solution set.

DISCUSSION

The main thrust of the lesson is to encourage students to understand and use the connectives *and* and *or* correctly in mathematics. The lesson points out that to find the solution set of two open sentences joined by *and*, one must find the *intersection* of the solution sets for each open sentence. The solution set for two open sentences joined by the word *or* is found by forming the *union* of the solution sets of the two open sentences.

In logic, if p and q are statements, then the *disjunction*, p *or* q, is true except when both p and q are false. The *conjunction*, p *and* q, is true only when both p and q are true. Write several simple conjunctions and disjunctions on the chalkboard and have the students decide which are true and which are false. Examples such as

$2 \cdot 2 = 4$ and $3 \cdot 5 = 8$ (False)

$2 \cdot 2 = 4$ or $3 \cdot 5 = 8$ (True)

are quite adequate.

Note that the symbols $\leq$ and $\geq$ are introduced in Discussion Exercise 3.

"Or" and "And" in Inequalities

Discussing the Ideas

The examples in the display below will help you understand the logical use of the words "and" and "or" in algebra and how these two words are related to the idea of union and intersection of sets. The replacement set for these examples is {1, 2, 3, 4, 5, 6, 7, 8, 9}.

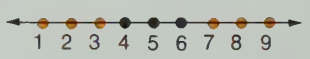
Inequality	Solution set
$x < 7$	$A = \{1, 2, 3, 4, 5, 6\}$
$x > 4$	$B = \{5, 6, 7, 8, 9\}$
$x > 4$ and $x < 7$	$A \cap B = \{5, 6\}$

Inequality	Solution set
$x < 4$	$C = \{1, 2, 3\}$
$x > 6$	$D = \{7, 8, 9\}$
$x < 4$ or $x > 6$	$C \cup D = \{1, 2, 3, 7, 8, 9\}$

1. The inequality $x > 4$ and $x < 7$ is often written as $4 < x < 7$. You can think of this as meaning "all numbers x from the replacement set that are between 4 and 7."
What does each of these inequalities mean?

A $1 < x < 5$ **B** $3 < x < 9$ **C** $7 > x > 2$

2. Does the set of points shown in color on the number line represent the solution set to inequality A or B?



A $x < 4$ and $x > 6$ **B** $x < 4$ or $x > 6$

3. An inequality such as $x < 3$ or $x = 3$ is often denoted by the special symbol $\leq$ and written $x \leq 3$. It is read "x is less than or equal to 3." The symbol $\geq$ means "is greater than or equal to."

A Using {1, 2, 3, 4, 5, 6, 7, 8, 9} as the replacement set, what is the solution set for $x \leq 3$?

B What is the solution set for $x \geq 3$?

4. How could you write each inequality in a shorter way?

A $x < 7$ or $x = 7$ **C** $x > 5$ or $x = 5$ **E** $x < 5$ and $x > 3$

B $x < 8$ and $x > 6$ **D** $x > 1$ and $x < 7$ **F** $x = 6$ *or* $x > 6$

Using the Ideas

1. The replacement set for this exercise is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Give the solution set for each inequality.

A	$x < 8$ $A = \{?\}$	$x > 5$ $B = \{?\}$	$x < 8$ and $x > 5$ $A \cap B = \{?\}$
B	$x > 2$ $C = \{?\}$	$x < 9$ $D = \{?\}$	$x > 2$ and $x < 9$ $C \cap D = \{?\}$
C	$x < 3$ $E = \{?\}$	$x > 7$ $F = \{?\}$	$x < 3$ or $x > 7$ $E \cup F = \{?\}$
D	$x > 5$ $G = \{?\}$	$x = 2$ $H = \{?\}$	$x > 5$ or $x = 2$ $G \cup H = \{?\}$

2. Write the shortened version for each inequality.

A $x > 2$ and $x < 10$
B $x < 19$ and $x > 12$
C $x < 11$ and $x > 0$
D $x > a$ and $x < b$
E $x = 9$ or $x < 9$
F $x > 3$ or $x = 3$

3. Give the solution set for each inequality. The replacement set is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

A $5 < x < 10$
B $2 < x < 7$
C $x < 3$ or $x > 7$
D $x \geq 7$
E $x \leq 4$
F $x \geq 6$
G $1 \leq x \leq 7$
H $5 \leq x \leq 7$
I $8 \leq x \leq 9$

★ **4.** Give the solution set for each inequality. If the solution set is large, give only a description of it. Replacement set: All whole numbers.

A $23 < x < 31$
B $4 \leq x \leq 19$
C $4997 < x < 5002$
D $10^0 \leq x \leq 3$
E $10^2 < x < 10^3$
F $2^3 < x < 3^2$
G $3^5 < x < 7^3$
H $24^2 < x < 26^2$
I $3^3 \leq x \leq 3^4$

★ **5.** Find the solution set for each inequality. Replacement set: All whole numbers.

A $6 < x - 2 < 10$
B $3 < x + 4 < 12$
C $0 < 2x < 10$
D $10 < x - 2 < 12$
E $8 \leq 3x \leq 27$
F $169 < x^2 < 225$
G $3 < x - 4 < 9$
H $10 < 4 - x < 17$
I $0 < x - (x - 1) < 10$

UTILIZATION

Note the structure of Exercise 1. You may find it helpful to discuss this problem or present examples similar to it before assigning the remainder of the exercises.

In Exercises 4 and 5 students may indicate the solution sets for some parts in different ways. Thus, for part E of Exercise 4, the solution set could be indicated by {101, 102, 103, . . . , 997, 998, 999} or by the verbal description, "The set of all whole numbers that are between 100 and 1000."

EXTENSION

Remedial: Ask students to tell whether each conjunction or disjunction is true or is false. If it is false, they should explain why it is false.

1. 128 is an even number *and* 259 is an odd number. (True)
2. 37 is an even number *or* 39 is an even number. (False)
3. 37 is an even number *and* 38 is an even number. (False; both statements must be true.)
4. There are 10 cm in every metre *or* there are 10 dm in every metre. (True)
5. 28 is a multiple of 7 *or* 20 is a multiple of 6. (True)
6. 28 is a multiple of 7 *and* 20 is a multiple of 6. (False; both statements must be true.)

Enrichment: Each exercise shows a conjunction or disjunction. Find the solution set. Replacement set: All whole numbers.

1. $x < 10$ and x is a multiple of 3. (0, 3, 6, 9)
2. $r + 2 < 8$ and r is an odd number. (1, 3, 5)
3. $m + 2 = 7$ or $m - 3 = 3$ (5, 6)
4. $6 + x < 10$ or $2x < 10$ (0, 1, 2, 3, 4)
5. $x < 9$ and $x^2 < 9$ (0, 1, 2)
6. $x = 23$ or $x = 16$ (23, 16)

Assignments

Minimum 1–3. Average 1–3, 4A–C.
Maximum 1–5.

Objective

Given a function rule and input numbers, n, the student can find the corresponding output numbers, $f(n)$.

PREPARATION

Finding the output number for a function, given the input number and rule, is much like evaluating an algebraic expression, which was studied earlier in this module on pages E-34 and 35. A short review of evaluation of algebraic expressions will help prepare the student for the Investigation.

INVESTIGATION

Discuss the meaning of the symbol associated with functions. Although any letter can denote a function, we use f. If n denotes the input number, the corresponding output number is denoted by $f(n)$. When students have completed the Investigation, you might have some of them show or explain how they computed particular output numbers.

DISCUSSION

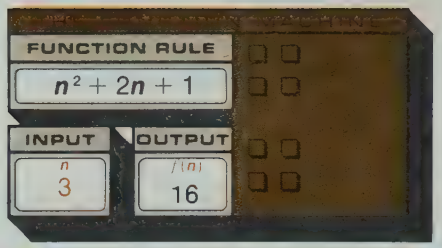
When discussing Exercise 2, students will probably point out that each output number is a square. This is true because $n^2 + 2n + 1 = (n + 1)^2$ is an identity for every number n. Thus the output numbers are squares of one more than the input number.

The function rule for Exercise 3 is interesting because if n is an odd number, $n^2 - 2$ is divisible by 8. This statement is easily proved algebraically, but the proof probably should be omitted except for more able students (see Extension). The differences of successive pairs of output numbers for this function increase by 1. The set of output numbers $\{1, 3, 6, 10, 15, \ldots\}$ is the set of *triangular numbers*.

Have some students write their function rule for Exercise 4 on the chalkboard and have their classmates give the output numbers for selected input numbers.

Investigating the Ideas

For each **input number** ***n*** from a set of numbers, the function machine applied a **function rule** to produce an output number. The **output number** is denoted by $f(n)$. This symbol is read: "f of ***n***." The function machine shows that when the input number is 3, the output number is $f(3) = 3^2 + 2 \cdot 3 + 1 = 9 + 6 + 1 = 16$.



Can you use this function rule to find the output numbers below?

A $f(1)$ B $f(2)$ C $f(10)$ D $f(15)$ E $f(21)$ F $f(42)$ G $f(96)$ H $f(100)$

Discussing the Ideas

1. You can show the input-output pairs for a function by a **function table**.

   How would you complete this function table for the other input numbers in the Investigation?

Function Rule: $n^2 + 2n + 1$	
n	$f(n)$
1	4
2	9
3	16
4	
10	

2. What pattern can you see concerning the output numbers?

3. The function rule has odd numbers as inputs. Find the missing numbers in the table. What do you notice about the output numbers?

Function Rule: $(n^2 - 1) \div 8$	
n	$f(n)$
1	0
3	1
5	
7	
9	
11	

4. Invent a function rule of your own. Make a function table for your function.

Using the Ideas

In Exercises 1 through 6, copy and complete the function tables.

1.

Function Rule	
$(n-1)^2$	
n	$f(n)$
7	36
3	4
A 10	
B 4	
C 1	

2.

Function Rule	
$7n-n$	
n	$f(n)$
1	6
4	24
A 7	
B 12	
C 36	

3.

Function Rule	
$3(5n-2)$	
n	$f(n)$
1	9
2	24
A 3	
B 4	
C 5	

4.

Function Rule	
n^3-n^2	
n	$f(n)$
2	4
A 3	
B 5	
C 8	
D 10	

5.

Function Rule	
$\frac{n^2-n}{n}$	
n	$f(n)$
3	2
A 4	
B 5	
C 10	
D 100	

★ **6.**

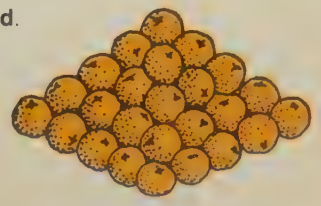
Function Rule	
2^n+1	
n	$f(n)$
1	3
2	5
A 3	
B 4	
C 5	
D 6	

★ **7.** What is the function rule for this table?

Function Rule	
▮▮▮▮▮▮▮▮▮▮▮▮	
n	(n)
1	9
2	19
3	29
4	39
5	49

think

Dan piled some oranges in the shape of a **triangular pyramid**. The bottom layer was a triangle with five oranges on each side.



How many oranges were in the pile?

UTILIZATION

If students can complete these exercises quickly, you might consider moving on to the next lesson on graphing functions, which is closely related to this lesson. Thorough understanding of this lesson is essential for understanding the succeeding lesson.

EXTENSION

Remedial: Provide extra practice with simple function rules such as the following:

A. $f(n) = n + 7$
B. $f(n) = 2n - 1$
C. $f(n) = n^2$
D. $f(n) = 100 - n$
E. $f(n) = n + 2n$
F. $f(n) = (n + 3) - n$

Enrichment: Some good students will be able to follow a proof that $n^2 - 1$ is divisible by 8 for every odd whole number n. If n is odd, then $n = 2w + 1$, where w is any whole number. Thus $n^2 - 1 = (2w + 1)^2 - 1 = (4w^2 + 4w + 1) - 1 = 4w^2 + 4w$. $4w^2 + 4w = 4w(w + 1)$, so 4 is a factor of $n^2 - 1$. But either w or $w + 1$ will be an *even* whole number and will have 2 as a factor, hence $n^2 - 1$ must have 8 as a factor for every odd whole number n.

Think Solution

The lowest layer of oranges contains 5^2 oranges the next layer 4^2, the next 3^2, then 2^2 then 1. Hence there are $5^2 + 4^2 + 3^2 + 2^2 + 1 = 55$ oranges in the pile. If n represents the number of oranges on one side the bottom layer, then the number of oranges in the pyramidal pile is given by the function rule

$$f(n) = \frac{n(n+1)(2n+1)}{6}$$

Some students will be interested in checking this function rule with their solution.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Objective

Given input numbers and a function rule, the student can compute the corresponding output numbers and then graph the input-output pairs of the functions.

PREPARATION

Materials: Graph paper or co-ordinate grid paper (*Duplicator Masters*, pages 83, 86).

Locating the point in the first quadrant for an ordered pair of whole numbers should not be a new experience for very many students at this level. However, a brief review of this skill may be in order. Provide graph paper for students for the Investigation if they do not have their own.

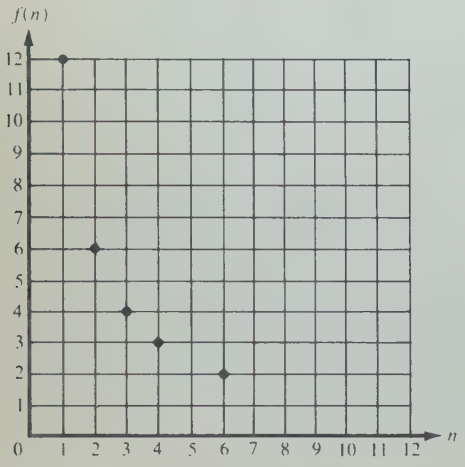
INVESTIGATION

The Investigation may help you determine the amount of review of graphing your students may need before continuing with the lesson.

DISCUSSION

Notice that the graphs of the functions consist of distinct points rather than all the points in a line or curve, because the input numbers are restricted to whole numbers.

Solution, Discussion Exercise 2B



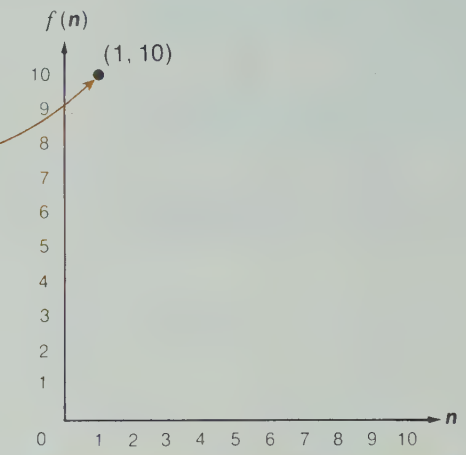
Graphs of Functions

Investigating the Ideas

The input-output numbers for a function can be used as the **co-ordinates** of points in the plane. When you graph the points for the co-ordinates, you have **graphed** the function.

Function Rule: $12 - (2n)$

n	$f(n)$	Co-ordinates
1	10	→ (1, 10)
2	8	→ (2, 8)
3	[illegible]	→ (3, [illegible])
4	[illegible]	→ (4, [illegible])
5	[illegible]	→ (5, [illegible])
6	[illegible]	→ (6, [illegible])



Can you complete the function table and then graph the points for the function rule above?

Discussing the Ideas

1. Do the points for the input-output pairs of the function fall in a pattern on the graph? Describe the pattern.

2. **A** Find the missing numbers in the function table.

Function rule: $f(n) = 12 \div n$

n	1	2	3	4	6
$f(n)$	12	6	[illegible]	[illegible]	[illegible]

B Draw the graph for the pairs of this function.

C Do the points in the graph for the function fall on a line?

3. Invent a function rule of your own. Find some input-output pairs and then draw a graph of the function.

Using the Ideas

In Exercises 1, 2, and 3, complete the function tables and then draw the graph of each function.

1.

Function Rule: $n^2 - n + 1$

	n	$f(n)$
	0	1
	1	1
A	2	
B	3	
C	4	

2.

Function Rule: $(n^2 + 17) - 8n$

	n	$f(n)$
	1	10
A	2	
B	3	
C	4	
D	5	

3.

Function Rule: $6n - n^2$

	n	$f(n)$
A	0	
B	1	
C	2	
D	3	
E	4	

4. The function given by the rule $f(x) = x$ is called an **identity function**. Draw a graph of this function using the whole numbers 0 through 10 as input numbers.

5. A **constant function** is a function whose output is just one number. Draw a graph of the constant function given by the rule $f(n) = 7$ using the whole numbers 1 through 8 as input numbers.

★**6.** Air temperature can be thought of as a function of the time of day. Make a graph of the temperature function given in the table below.

Function Rule: Temperature

n (time)	$f(n)$ (temperature °C)
11:00 A.M.	20
12:00 Noon	22
1:00 P.M.	25
2:00 P.M.	28
3:00 P.M.	29
4:00 P.M.	27

think

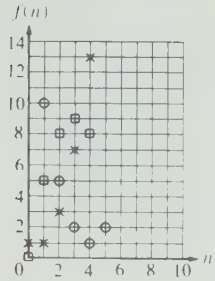
Suppose that in Speed City, four policemen can write four tickets for speeding in four minutes. If the entire police force can write eighty tickets in one hour and 20 minutes, how many persons are on the police force in Speed City?

UTILIZATION

Graph paper should be available for students to use for the exercises on this page.

Solutions, Exercises 1–6.

1, 2, 3.



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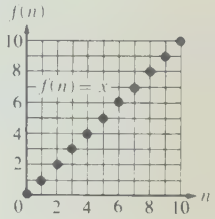
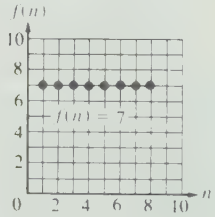


Exercise 1 graph: ×
Exercise 2 graph: ○
Exercise 3 graph: □

5.



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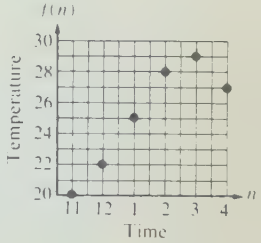


EXTENSION

Exercises appropriate for extending practice with the ideas presented in this lesson are provided on *Workbook* page 15 and *Duplicator Masters* page 14.

Think Solution

If 4 policemen can write 4 tickets in 4 minutes, it must take 4 minutes for 1 policeman to write 1 ticket. In 1 hour 20 minutes, or 80 minutes, 1 policeman can write 20 tickets. Since 80 tickets were written, there must be a total of 4 policemen.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

Given a linear equation such as $x - y = 5$ the student can find ordered pairs of whole numbers (x, y) satisfying the equation and can graph the points for the pairs.

PREPARATION

Materials: Graph paper or first quadrant grids (*Duplicator Masters*, pages 83, 86).

To introduce the lesson you might ask the students a question such as this: "I am thinking of two whole numbers. Their difference is 2. What are some of the pairs of numbers I might be thinking of?" Have the students list several such pairs.

INVESTIGATION

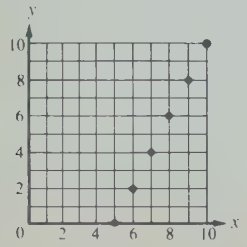
Encourage students to use their own methods to decide which statement describes the set of points shown on the graph. Only statement C is true for every number pair shown on the graph.

DISCUSSION

For most students, graphs of equations are somewhat more difficult than graphs of functions, because of the problem posed by two variables and the uncertainty of knowing what numbers to substitute for the variables. Therefore it may be necessary for you to move slowly and provide extra examples before assigning the exercises on page E-55.

In the table for Discussion Exercise 2, the row labeled $2x$ is given to aid students in finding the corresponding value for y. This extra step could be done mentally when students are more familiar with the process.

Solution, Discussion Exercise 2C



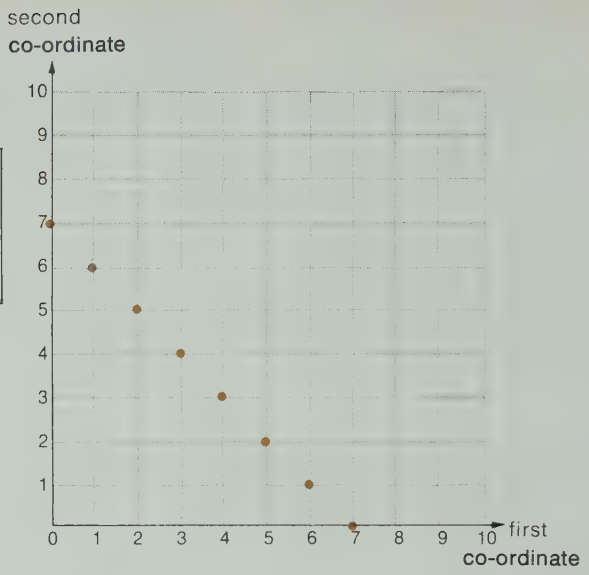
Graphs of Equations

Investigating the Ideas

The co-ordinates for the points shown graphed at the right are related in a special way.

Which one of the statements below best describes the relation between the numbers for the points?

- **A** The first co-ordinate is greater than the second.
- **B** The difference of the two co-ordinates in each pair is 1.
- **C** The sum of the two co-ordinates in each pair is 7.
- **D** The product of the two co-ordinates in each pair is 7.



Discussing the Ideas

1. Suppose x designates the first co-ordinate and y designates the second co-ordinate, then the points graphed in the Investigation consist of those ordered pairs of whole numbers (x, y) such that $x + y = 7$.

   For $(7, 0)$, we have $7 + 0 = 7$.
   For $(6, 1)$, we have $6 + 1 = 7$.
   For $(5, 2)$, we have $5 + 2 = 7$.

   What are some other replacements for x and y in the equation $x + y = 7$?

2. To draw the graph of the equation $2x - y = 10$, you can make a table of the pairs for x and y and then graph the points for the pairs.

   - **A** Find the missing numbers in the table.
   - **B** List the pairs (x, y) in the table.
   - **C** Graph the pairs (x, y) in part **B**.

x	5	6	7	8	9	10
$2x$	10	12				
y	0	2				

Using the Ideas

1. Complete the table of numbers for the equation $x - y = 2$.

x	10	9	8	7	6	5	4	3	2
y	8	7							

2. Make a graph of the equation $x - y = 2$ using the pairs (x, y) in the table for Exercise 1.

3. Complete the table for the equation $x + 2y = 15$.

x	1	3	5	7	9
2y	14	12			
y	7	6			

4. Graph the equation $x + 2y = 15$ using the pairs (x, y) for Exercise 3.

5. Complete the table for the equation $2x - 3 = y$.

x	2	3	4	5	6
2x	4	6	8		
y	1	3			

6. Graph the equation $2x - 3 = y$ using the pairs (x, y) for Exercise 5.

7. Make a table of pairs (x, y) for each equation. Then graph the equation.

- **A** $x - y = 4$
- **B** $x + y = 9$
- **C** $x = y$
- **D** $x + 2 = y$
- **E** $x - y = 0$
- **F** $2x - y = 0$

Think

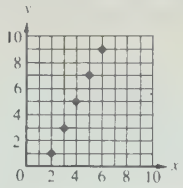
After a set of tennis, Chris said to Juanita, "If I had won three more games than I did, I would have beaten you by the same score that you beat me."
"That's right," said Juanita, "but, of course, I won twice as many games as you did." How many games did each person win?

More practice, page S-8, Set 13

UTILIZATION

Solutions, Exercises 6 and 7

6.



7. Pairs (x, y) may be chosen arbitrarily.

A. $x - y = 4$

x	y
8	4
7	3
5	1
4	0

B. $x + y = 9$

x	y
0	9
3	6
5	4
9	0

C. $x = y$

x	y
0	0
3	3
7	7

D. $x + 2 = y$

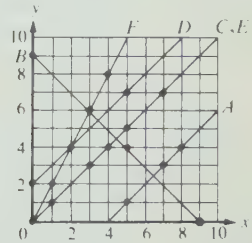
x	y
0	2
2	4
5	7
8	10

E. $x - y = 0$

x	y
3	3
5	5
1	1
4	4

F. $2x - y = 0$

x	y
1	2
2	4
3	6
4	8



EXTENSION

The game of "Battleship" is a good game to practice naming co-ordinates of points. Two or more players, can play the game. Each player uses a 10 by 10 co-ordinate grid and secretly "hides" a "battleship" by designating 3 consecutive points on the grid, horizontally, vertically, or diagonally. Each player in turn calls out the co-ordinate of some point in the plane. Thus this point has been "bombed" and each player should place a small x on that point. If the point was one of the points on a player's ship, he or she must announce a "hit." When all three points have been hit, that player is eliminated from the game.

Assignments

Minimum 1–4. Average 1–6. Maximum 5–7.

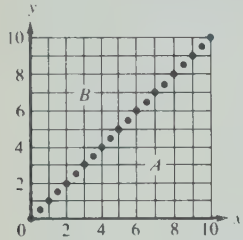
Objective

The student can draw graphs of inequalities which involve ordered pairs of whole numbers.

PREPARATION

Materials: Graph paper or first quadrant grids (*Duplicator Masters*, pages 83, 86).

To introduce the lesson, display the graph of $x = y$ on the overhead projector or chalkboard.



Use A and B to denote the half-planes on either side of the set of points $x = y$. Any point in either half-plane can be described by writing $x \neq y$. Ask the students to choose several points in half-plane A. Then ask: "What is the relation between x and y?" $(x > y)$ "What is the relation between x and y in half-plane B?" $(x < y)$

INVESTIGATION

Students may use trial-and-error techniques in the Investigation. The pairs with 5 as their sum are easily found. A few pairs whose sums are less than 5 will enable students to discover the portion of the plane in which the points for all such pairs lie.

DISCUSSION

Graphs of inequalities in the co-ordinate plane will be a topic that is new to nearly all of your students. Therefore you will need to proceed slowly and supply additional examples if needed.

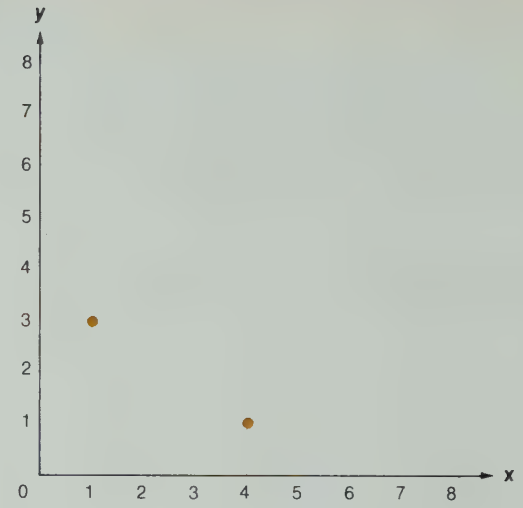
In discussing Exercise 2, point out that the graph of $x < 4$ and $y < 4$ is the intersection of the solution set for $x < 4$ and the solution set for $y < 4$. Intersections of solution sets were studied earlier in this module but were related to the number line—a one-dimensional space. This lesson extends the idea of inequalities to the plane—a two-dimensional space.

Graphing Inequalities

Investigating the Ideas

How many ordered pairs of whole numbers can you find which have a sum that is less than or equal to 5? Two such pairs are (1, 3) and (4, 1).

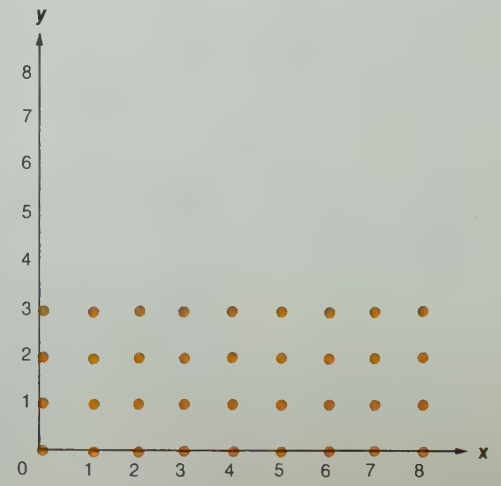
Can you mark the points for the ordered pairs on a co-ordinate grid?



Discussing the Ideas

1. The set of all ordered pairs of whole numbers whose sum is less than or equal to 5 is the **solution set** for the inequality $x + y \leq 5$.
   The pair (2, 1) is a **solution** because $2 + 1 \leq 5$ is **true**.
   The pair (5, 1) is not a solution because $5 + 1 \leq 5$ is **false**.
   How many ordered pairs are in the solution set for the inequality $x + y \leq 5$?

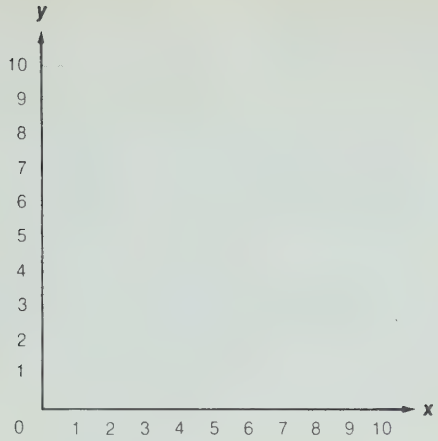
2. The graph at the right shows the set of all ordered pairs of whole numbers (x, y) such that $y < 4$. Any ordered pair in which the second number is less than 4 is a solution for this inequality.
   - **A** What does the graph of $x < 4$ look like?
   - **B** What pairs (x, y) are in the solution set for $x < 4$ **and** $y < 4$?



Using the Ideas

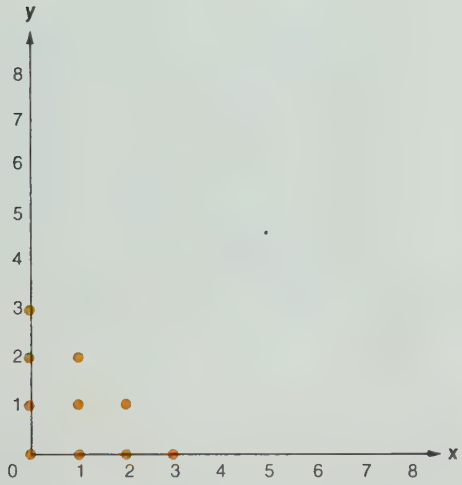
1. Draw graphs of these inequalities. The replacement set for x and y is the set of whole numbers.

- **A** $x + y \leq 7$
- **B** $x + 2y < 6$
- **C** $x < 5$
- **D** $y < 5$
- **E** $x < 5$ and $y < 5$
- **F** $x - y < 2$
- **G** $x < 4$ and $x + y < 7$
- **H** $x + y < 4$ and $x + y < 8$

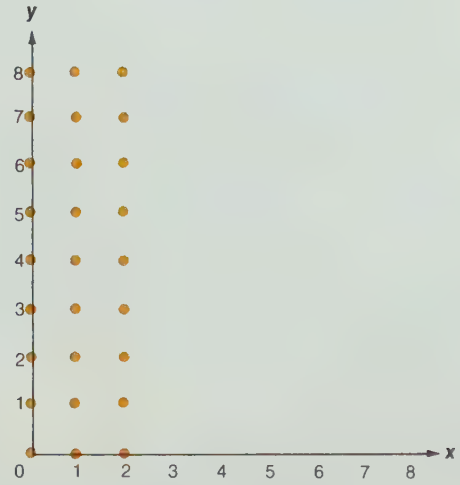


2. Write an inequality for the set of points shown on each graph.

A



B



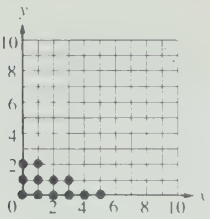
★ **3.** Draw a graph of all ordered pairs of whole numbers x and y such that $2 < x < 6$ and $3 < y < 7$.

★ **4.** Draw a graph of all ordered pairs of whole numbers x and y such that $x = 5$ and $4 \leq y \leq 9$.

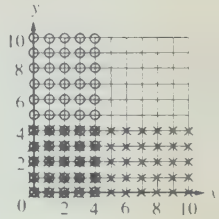
UTILIZATION

Solutions, Exercises 1, 3, and 4

1B.

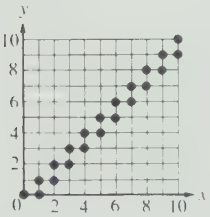


1C, D, E.

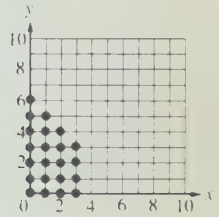


Graph for C: ○
Graph for D: ×
Graph for E: ⊠

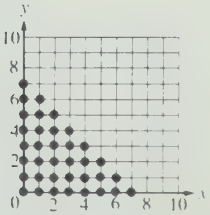
1F.



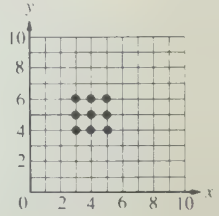
1G.



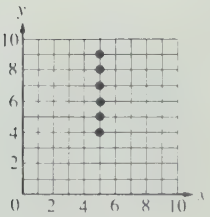
1H.



3.



4.



Assignments

Minimum 1A–F. Average 1, 2. Maximum 1–4.

Objective

The student can demonstrate the ability to work with concepts developed in this module.

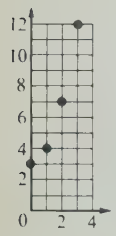
UTILIZATION

The review exercises should be utilized in whatever way is best suited to your teaching methods and classroom structure. Students will benefit by a discussion of the exercises and checking of their answers.

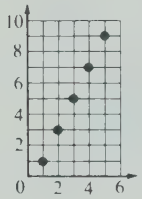
Additional review exercises are provided on page 16 of the *Workbook*.

Solutions, Exercises 16B, 17, and 18

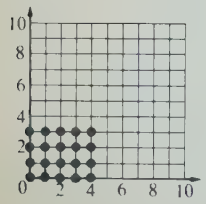
16B.



17.



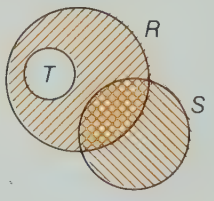
18.



REVIEWING THE IDEAS

1. $X = \{0, 2, 4, 6\}$ and $Y = \{0, 1, 2, 3, 4, 5, 6\}$
   - A Is $2 \in X$?
   - B Is $6 \in Y$?
   - C Is $5 \in X$?
   - D Is $X \subset Y$?

2. Study the Venn diagram, then decide whether each statement is true or false.



   - A $T \subset R$
   - B $T \subset S$
   - C $T \cup R = R$
   - D $S \cap T = \phi$
   - E $R \cap T = R$

3. $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, and $C = \{1, 3\}$. Find the elements in each of the following sets.
   - A $A \cup B$
   - B $A \cap B$
   - C $A \cup C$
   - D $A \cap C$
   - E $B \cup C$
   - F $B \cap C$

4. Draw a Venn diagram which shows the relationships between the three sets in Exercise 3.

5. Evaluate $(n + 7) \div 4$ when $n = 17$.

6. Evaluate $5(x^2 + 1)$ when $x = 3$.

7. Find the solution set for each open sentence. Use $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the replacement set.
   - A $x + 17 = 26$
   - B $6n = 48$
   - C $x < 4$
   - D $y + 2 > 9$

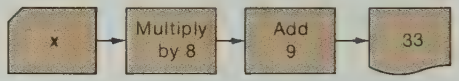
8. Is $x \cdot 1 = x$ a generalization for the set of whole numbers?

9. Solve the equations.
   - A $10 + (n - 7) = 10$
   - B $5(4 - x) = 5$

10. Give the correct number for each ▨.
    - A $3r + 8r = ▨r$
    - B $17m - 8m = ▨m$

11. Give the correct number for each ▨ in the equations.
    - A $5x + 6x = 44$; $▨x = 44$; $x = ▨$
    - B $6x - 2x = 20$; $▨x = 20$; $x = ▨$

12. Write an equation for this flow chart.



13. Solve the equation for Exercise 12.

14. Write and solve an equation for this problem:
    A certain number is added to 5 and the sum is multiplied by 4. The product is 48. What is the number?

15. Using $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the replacement set, find the solution set for each part.
    - A $x < 8$ and $x > 5$
    - B $x < 2$ or $x > 9$
    - C $x < 5$ or $x > 5$
    - D $3 \leq x \leq 8$

16. A Complete the function table.
    B Draw a graph of the function.

	Function Rule $n^2 + 3$	
	n	$f(n)$
A	0	▨
B	1	▨
C	2	▨
D	3	▨

17. Draw a graph of the equation $2x - y = 1$.

18. Graph the ordered pairs of whole number (x, y) such that $x \leq 4$ and $y \leq 3$.

TEST YOURSELF

1. $C = \{0, 1, 2, 3\}$ and $D = \{1, 3, 5, 7\}$
   - A List the elements in $C \cup D$.
   - B List the elements in $C \cap D$.

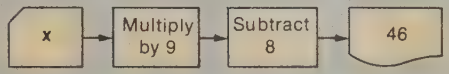
2. Give the number for the ▒.
   - A If $x = 4$, then $x + 7 =$ ▒.
   - B If $y = 7$, then $2y - 5 =$ ▒.

3. What is the solution set for the open sentence $x < 5$? The replacement set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

4. Solve the equations using the set of whole numbers as the replacement set.
   - A $4(x - 5) = 4$
   - B $9 + (x - 3) = 9$

5. Solve the equations. The replacement set is the set of whole numbers.
   - A $5x + 2x = 35$
   - B $9x - x = 56$

6. Write an equation for this flow chart:



7. Write and solve an equation for this problem:
   If 3 is subtracted from a certain number and the difference is multiplied by 8, the product is 16. What is the number?

8. Suppose the replacement set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, what is the solution set for each part?
   - A $x < 3$ or $x > 8$
   - B $x > 4$ and $x < 9$
   - C $x \leq 6$
   - D $x > 10$

9. Complete the function rule table.

	Function Rule $4n^2 - 3$	
	n	$f(n)$
	2	13
A	3	▒
B	4	▒
C	5	▒

10. Give the missing numbers in the table for the equation $3x + y = 12$.

x	0	1	2	3	4
y	12	9	A	B	C

RESEARCH PROJECTS

A The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, . . ., has many interesting properties. Find some of them and make a report to the class. (See *MATHEMATICS, A Human Endeavor* by Harold R. Jacobs; San Francisco: W. H. Freeman and Company, 1970.)

B One application of set theory and logic involves switching networks. Read about these, and make and interpret some of your own. (See *Mathematics and the Modern World* by Mario F. Triola; Menlo Park, California: Cummings Publishing Company, 1973; Addison-Wesley.)

TEST YOURSELF

The self-evaluation test covers the main topics of the module. If some portions of the module were omitted, students should not be asked to answer the questions dealing with those topics.

Note that the correspondence between the test items and the module objectives is denoted by annotations on the student book page.

RESEARCH PROJECTS

Although sequences are not introduced until the next module, Project A, which concerns the Fibonacci sequence, could be an interesting research topic any time during the year. Each number of the sequence, after the first two numbers, is the sum of the two numbers immediately preceding that number. Applications to botany, Pascal's triangle, music, and many other topics can be found.

The topic of Project B is presented briefly in Learning Unit H, Module 4. However, some students may be interested in studying it at this time.

MATHEMATICAL RECREATION

This topological puzzle can be solved by turning the rectangular hole in the strip "inside out." Bend the sides of the rectangular hole downward and bring them up through the hole, turning the sides completely over. When the twists are straightened out, the end will be looped through the hole.

Students who enjoy this topological puzzle may want to pursue topology in more detail. The references listed below are highly recommended.

Johnson, Donovan A., and William H. Glenn, *Topology* (Manchester, Mo.: Webster Publishing Co., 1960; available from McGraw-Hill Ryerson).

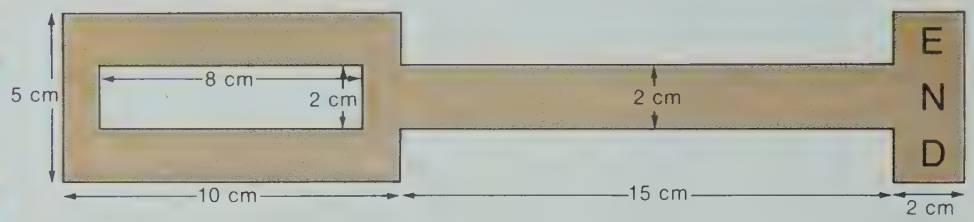
Newman, James R., ed., *The World of Mathematics*, 4 vols. (New York: Simon and Schuster, 1956; paper, 1962), Vol 1, part 4. (available from Musson Book Co.)

Steinhaus, H., *Mathematical Snapshots*, 3rd American ed., rev. and enl. (New York: Oxford University Press, 1969).

MATHEMATICAL RECREATIONS

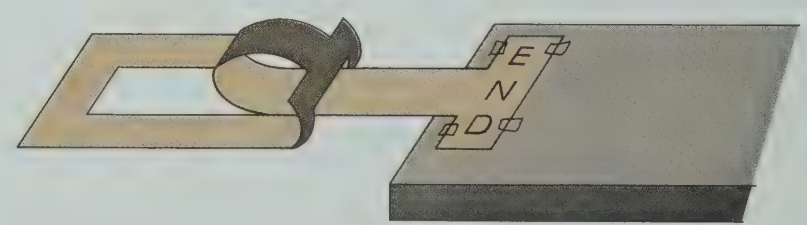
A TOPOLOGICAL PUZZLE

Topology is a kind of geometry which concerns those properties of figures which are not changed when the figures are twisted, stretched, or deformed in any way except by cutting or tearing. The figure below can be used for an interesting topological puzzle.



Cut out a shape like this from a strip of paper. (The dimensions do not have to be exactly the same as shown.)

It is easy to put the END through the hole in the strip to form the shape below.



Here is the puzzle:

Try to make the strip look like this when the END is first taped to the edge of a table.

RECREATIONS MATHEMATICAL

TEST YOURSELF Answers

1. A {0, 1, 2, 3, 5, 7}; B {1, 3} **2.** A 11; B 9 **3.** {0, 1, 2, 3, 4}
4. A $x = 6$; B $x = 3$ **5.** A $x = 5$; B $x = 7$ **6.** $9x - 8 = 46$
7. $8(x - 3) = 16$; $x = 5$ **8.** A {0, 1, 2, 9, 10}; B {5, 6, 7, 8}; C {0, 1, 2, 3, 4, 5, 6}; D ϕ **9.** A 33; B 61; C 97
10 A 6; B 3; C 0

UNIT E: INTEGERS

Module 3: Number Theory

General Objectives

To review and extend work with factors and with prime and composite numbers.

To review the greatest common factor of two numbers.

To review the least common multiple of two numbers.

To stimulate interest in mathematics through the study of divisibility rules and sequences.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
19	**RED** Given a whole number, the student can list all the factors of the number.	E-62, 63	**WB** 17	**SP** g-1	
20	**RED** Given a whole number less than or equal to 100, the student can state whether it is a prime or a composite number.	E-64, 65	(**WB** 18) **DM** 15	**SP** g-1	**SWM 1** 152 **SWM 2** 315 **SWM 3** 20, 21, 310
21	**RED** Given a whole number, the student can give the prime factorization of the number.	E-68, 69	**WB** 19 **DM** 16	**SP** g-1, 4	
22	**RED** Given two whole numbers, the student can find their greatest common factor.	E-70, 71	**WB** 20 **DM** 17	**ASC** W-11 **SP** g-2	
23	**RED** Given two whole numbers, the student can find their least common multiple.	E-74, 75	**WB** 22 **DM** 18	**ASC** W-12 **SP** g-3	
24	**YELLOW** Given a whole number, the student can determine some factors of the number by applying various divisibility rules.	E-76, 77 E-78, 79	**WB** 23		
25	***GREEN*** *Given two large whole numbers, the student can use the Euclidean Algorithm to find the greatest common factor of the two numbers.*	E-72, 73	**WB** 21	**SP** g-2	
26	***GREEN*** *Given the n^{th} term of a sequence, the student can find particular terms of the sequence.*	E-80, 81			**SWM 3** 22-27
27	***GREEN*** *Given appropriate examples, the student can identify square and triangular numbers.*	E-82, 83			**SWM 3** 18, 19
	Reviewing the Ideas	E-86	**WB** 24		

MATHEMATICS

This module develops a number of important concepts basic to topics encountered later in the text. Certainly, the concepts of GCF and LCM are very useful in working with rational numbers. Aside from developing the more obvious applications of these topics, this module gives students a chance to work with patterns and relationships between numbers and sets of numbers that are not often found in work with algorithms. In keeping with this emphasis on the discovery of patterns and relationships, the module should be treated as a mathematical adventure.

A *sequence* is a function whose domain is the set of natural numbers, $\{1, 2, 3, 4, 5, \ldots\}$. All of the sequences presented in the module have simple rules of formation.

TEACHING THE MODULE

Vocabulary

complete factorization
composite number
divisible
Euclidean Algorithm
factor
Fibonacci sequence
Fundamental Theorem of Arithmetic
greatest common factor (GCF)
least common multiple (LCM)
multiple
perfect number
prime factorization
prime number
relatively prime number
sequence
Sieve of Eratosthenes
square number
term
triangular number
twin primes
Unique Factorization Theorem

Much of the material in this module will be review material for most students. Therefore you may find the Extension sections of the lessons particularly helpful in maintaining student interest. In addition, several lessons are suggested as optional lessons. For those students who need the motivation provided by a special challenge or a change of pace, these optional lessons may be especially useful.

The topic of divisibility included in two lessons is optional. Also, the three lessons on sequences should be regarded as optional.

In teaching the module, keep in mind that discovery of patterns and insight into the nature of numbers and number relationships are the main goals of this module.

Lesson Schedule

Your lesson schedule will depend largely on the amount of material you choose to cover. A few of the lessons can be covered in less than a full class period with able students, but many of the other lessons merit at least one period for full utilization. In general, however, you should plan to allow about 10 days to cover the module.

Evaluation

The two most important objectives for student mastery are those related to finding the GCF and the LCM of two numbers. The work with factors and prime numbers is of secondary importance. The other material should be regarded as optional and should be given less emphasis in your evaluations of students' work.

One of the more general objectives of this module is to increase student interest in mathematics. Your daily observations and discussions with students will help you evaluate how well this objective is being met.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 3, pp. 55, 66, 81; *Aftermath 4,* pp. 49, 91, 94, Creative Publications

Cloudburst, Vol. 5 (divisibility, figurate and perfect numbers, patterns), Nos. 2438, 2448, 2818–68, 2918–98, 3438, 3448, 8338–78, Midwest Publications

Math Experiments with Pentominoes, HR8–11, Midwest Publications

Mathematics: Man's Key to Progress, Book A (number patterns), pp. 41–43, 58; Book B, pp. 14–16, 20–21, Rand McNally

Nuffield Project: *Computation and Structure 4* (triangular and square numbers), pp. 62–63; *Number Patterns 2; Problems*—Red Set, Nos. 4A, 9, 9A, Wiley

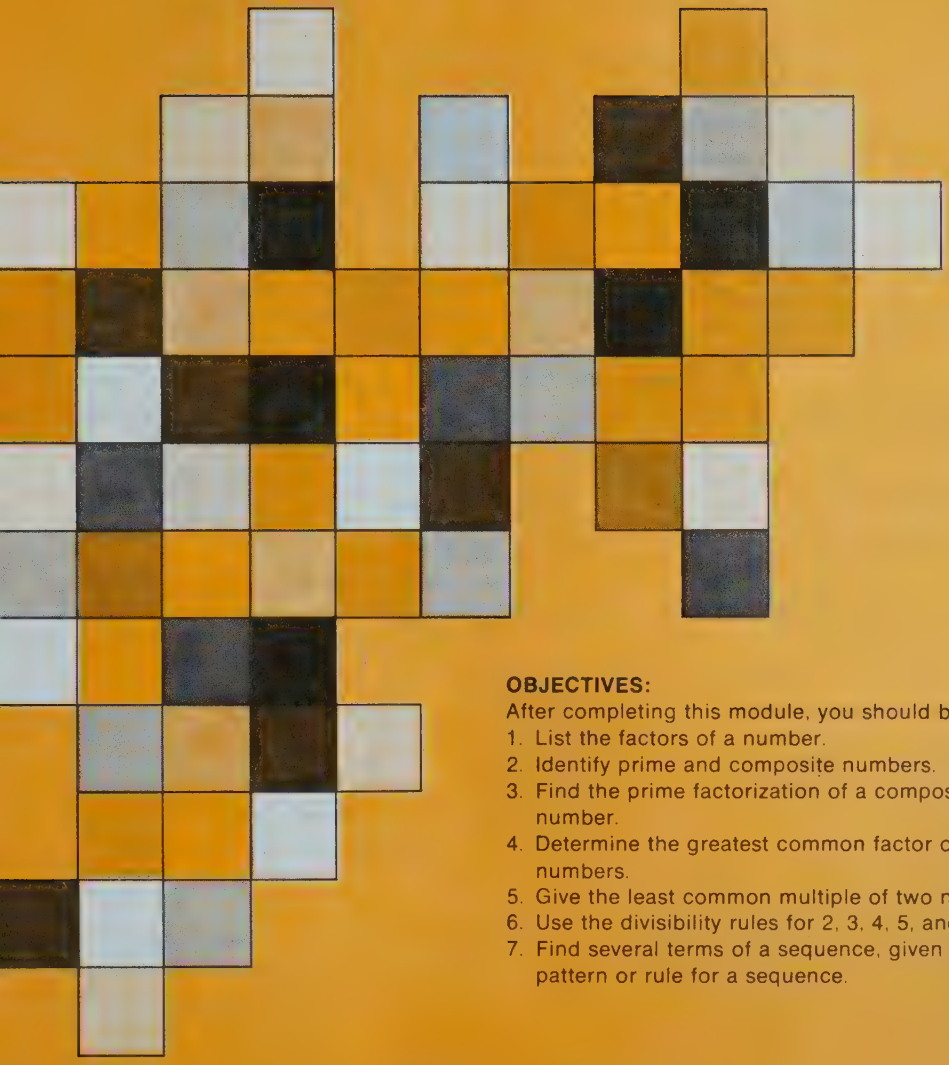
Powers, Roots and Logarithms, Set 5E, Cards 31–41, Herder and Herder

Sourcebook for Substitutes, pp. 73–75, Addison-Wesley

For some other activities with geometric progressions, refer to Resources for Active Learning, Unit H, Module 1.

Games and Puzzles

Multifacto Producto (Scott, Foresman)
Numo (Gamco; Midwest Publications)
Prime Drag (Creative Publications)



OBJECTIVES:
After completing this module, you should be able to:

1. List the factors of a number.
2. Identify prime and composite numbers.
3. Find the prime factorization of a composite number.
4. Determine the greatest common factor of two numbers.
5. Give the least common multiple of two numbers.
6. Use the divisibility rules for 2, 3, 4, 5, and 9.
7. Find several terms of a sequence, given the pattern or rule for a sequence.

Objective

The student can list all the factors of certain numbers and can use division to decide if one number is a factor of another number.

PREPARATION

Review the definitions of *factor* and *product* before beginning the Investigation.

INVESTIGATION

The Investigation should not require much time to complete. Have a student place the factors for each number on the chalkboard and use the results as an aid during the Discussion.

1: {1}
2: {1, 2}
3: {1, 3}
4: {1, 2, 4}
5: {1, 5}
6: {1, 2, 3, 6}
7: {1, 7}
8: {1, 2, 4, 8}
9: {1, 3, 9}
10: {1, 2, 5, 10}
11: {1, 11}
12: {1, 2, 3, 4, 6, 12}
13: {1, 13}
14: {1, 2, 7, 14}
15: {1, 3, 5, 15}
16: {1, 2, 4, 8, 16}
17: {1, 17}
18: {1, 2, 3, 6, 9, 18}
19: {1, 19}
20: {1, 2, 4, 5, 10, 20}

DISCUSSION

Exercises 1, 2, and 3 may suggest additional discussion questions such as these:

A. Which numbers have an odd number of factors? (The square numbers: 1, 4, 9, 16 . . .)
B. What number has only 1 factor? (1)
C. What is the smallest number with at least 4 factors? (6)
D. What is the smallest number with 6 factors? (12)

Exercise 4 illustrates that every whole number is a factor of zero because $0 \cdot n = 0$ for every whole number n. However, zero is a factor only of itself because if zero is chosen as one factor the product will always be zero.

Exercises 5 and 6 illustrate how division may be used to decide whether a given number is a factor of another. In Exercise 6, the students should be able to see that if the dividend were 841 the remainder would be zero, hence 29 is a factor of 841.

Factors

Investigating the Ideas

If a, b, and c are whole numbers and $a \cdot b = c$, then a and b are **factors** of c.

2 and 7 are factors of 14 because $2 \cdot 7 = 14$

1 and 14 are factors of 14 because $1 \cdot 14 = 14$

No other whole numbers are factors of 14.
The set of all factors of 14 is {1, 2, 7, 14}

Number	Factors
1	{1}
2	{1, 2}
3	{1, 3}
4	{1, 2, 4}
⋮	⋮
20	⋮

Can you find the set of all factors of each whole number from 1 through 20?

Discussing the Ideas

1. What number is a factor of every number?

2. Which numbers from 1 through 20 had six factors?

3. Which numbers from 1 through 20 had only two factors?

4. A Is every whole number a factor of zero?
   B Is zero a factor of any whole number? Which one?

$0 \cdot 0 = 0$
$0 \cdot 1 = 0$
$0 \cdot 2 = 0$
$0 \cdot 3 = 0$
⋮

5. A Is 43 a factor of 731? Why?
   B Is 17 a factor of 918? Explain.
   C Is 17 a factor of 497? Explain.

$$\begin{array}{r} 43 \\ 17\overline{)731} \\ 68 \\ \hline 51 \\ 51 \\ \hline 0 \end{array}$$

6. The division shows that 29 is not a factor of 843. Can you decide if 29 is a factor of 841 without dividing? Explain.

$$\begin{array}{r} 29 \\ 29\overline{)843} \\ 58 \\ \hline 263 \\ 261 \\ \hline 2 \end{array}$$

Using the Ideas

1. Solve each equation.

A $1 \cdot x = 84$ **C** $3 \cdot z = 84$ **E** $6 \cdot s = 84$

B $2 \cdot y = 84$ **D** $4 \cdot r = 84$ **F** $7 \cdot t = 84$

2. Use the results of Exercise 1 to list all the factors of 84.

3. The whole number 9 has exactly three factors: 1, 3, and 9. Find a number smaller than 9 that also has exactly three factors.

4. Find the smallest number with exactly four factors.

5. Give the set of all factors of each number.

A 70 **B** 66 **C** 48 **D** 150 **E** 420

6. Use division, if necessary, to answer each question **yes** or **no**.

A Is 7 a factor of 91?

B Is 12 a factor of 148?

C Is 37 a factor of 111?

D Is 89 a factor of 3115?

E Is 123 a factor of 1371?

F Is 143 a factor of 1001?

★**7.** The number 6 is called a **perfect number** because the sum of its factors other than itself is 6: $1 + 2 + 3 = 6$.

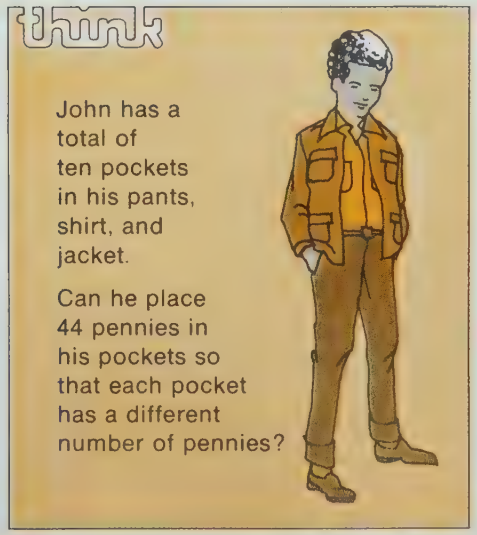
A Another perfect number is found between 20 and 30. What is this number?

B Another perfect number is found between 495 and 500. What is this number?

★**8.** A number is known to have 2, 3, and 5 among its factors. Give five more factors of this number.

★**9.** **A** The number $1365 = 3 \cdot 5 \cdot 7 \cdot 13$ has 16 factors. What are these factors?

B The number 2310 has 32 factors. What are these factors?


think
John has a total of ten pockets in his pants, shirt, and jacket.
Can he place 44 pennies in his pockets so that each pocket has a different number of pennies?

UTILIZATION

In discussing Exercise 8, point out that 2, 3, and 5 have no common factors, so any product of 2 or more of these factors will give another factor of the number. Of course, 1 is a factor of any number and should be included. Note that if 2 and 4 are factors of some number, one cannot be sure that 8 is a factor of the number because 2 and 4 are not relatively prime to each other.

Answers, Exercises 5 and 9

5A. {1, 2, 5, 7, 10, 14, 35, 70}

B. {1, 2, 3, 6, 11, 22, 33, 66}

C. {1, 2, 3, 4, 6, 8, 12, 16, 24, 48}

D. {1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150}

E. {1, 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 105, 140, 210, 420}

9A. {1, 3, 5, 7, 13, 15, 21, 35, 39, 65, 91, 105, 195, 273, 455, 1365}

B. {1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, 30, 33, 35, 42, 55, 66, 70, 77, 105, 110, 154, 165, 210, 231, 330, 385, 462, 770, 1155, 2310}

EXTENSION

Exercises for additional related practice are provided on *Workbook* page 17.

Enrichment: Some students may be interested in a formula for perfect numbers. A number N is perfect if $N = (2^n - 1) \cdot 2^{n-1}$, provided $2^n - 1$ is a prime number. When $n = 2$, 3, 5, or 7, $2^n - 1$ is equal to 3, 7, 31, and 127, respectively, all of which are prime numbers. These values for n give the first four perfect numbers: 6, 28, 496, and 8128.

Think Solution

If John put 1 penny in the first pocket, 2 in the second, 3 in the third, and so on, this would give the series $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, with 8 pennies left for the other two pockets. If he puts no pennies in the first pocket, he can use 9 pockets $(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36)$, with 8 pennies left for the last pocket.

Assignments

Minimum 1–4, 5A–C, 6. Average 1–6. Maximum 1–9.

Objective

The student can identify the prime numbers that are less than 100.

PREPARATION

Since the Investigation involves listing the set of factors of a number, you may wish to review this idea briefly.

INVESTIGATION

When listing the numbers less than 80 which have exactly 2 factors, the students are finding the prime numbers that are less than 80. Most students will realize this although they may not remember the precise definition of a prime number.

The numbers students should list for the Investigation are {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79}.

DISCUSSION

It is interesting to note that the definitions of prime and composite numbers do not divide the set of whole numbers into two subsets, but rather into three. The numbers zero and 1 are neither prime nor composite (Exercise 4). The main objection to including 1 in the set of primes is that if we do so, we must give up the so-called Fundamental Theorem of Arithmetic, which asserts that if a whole number is not prime, then it is a product of a unique set of prime numbers. That is, if we allow 1 to be a prime number, we could have many different prime factorizations of 12, such as $12 = 1 \cdot 2 \cdot 2 \cdot 3$, $12 = 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3$, $12 = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3$, and so forth.

Prime and Composite Numbers

Investigating the Ideas

The set of whole numbers can be separated into different subsets according to the number of factors each number has. The table at the right shows that 0 has every whole number as a factor. The number 1 has only one factor. Numbers such as 2, 3, and 5 have exactly two factors. Other numbers have more than two factors.

Whole number	Set of all factors
0	{0, 1, 2, 3, 4, 5, 6, . . .}
1	{1}
2	{1, 2}
3	{1, 3}
4	{1, 2, 4}
5	{1, 5}
6	{1, 2, 3, 6}
⋮	⋮

How many numbers less than 80 can you find that have exactly two factors?

Discussing the Ideas

1. The whole numbers which have exactly two factors are **prime numbers**. How many prime numbers are there that are less than 80?

2. Is 57 a prime number? Why or why not?

3. A whole number, other than 0, is **composite** if it has more than two factors.
   - **A** Which numbers in the table are composite?
   - **B** Which are prime?

	Whole number	Set of all factors
A	51	{1, 3, 17, 51}
B	109	{1, 109}
C	187	{1, 11, 17, 187}
D	1001	{1, 7, 11, 13, 77, 91, 143, 1001}
E	1009	{1, 1009}

4. **A** Is 0 either a prime or a composite number?
   B Is 1 either a prime or a composite number?

5. Are there any even prime numbers?

6. Which numbers in the set below are not prime numbers?
   {7, 17, 27, 37, 47, 57, 67, 77, 87, 97}

Using the Ideas

Eratosthenes, a Greek mathematician who lived over 2000 years ago, invented a "sieve" for finding prime numbers. The exercises below illustrate the **Sieve of Eratosthenes**.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1. Make a table of whole numbers 1 through 100 in six columns.

2. The first number, 1, is not prime. Mark it out. The number 2 is the smallest prime. Circle it. Then strike out all other numbers that have 2 as a factor. What is the first number greater than 2 to eliminate from the table?

1	2	3	4	5	6
7	8	9	10	11	12

3. **A** Circle the first number after 2 that has not been eliminated before. What prime number is it?

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22		
25	26	27			

B Now strike out all the numbers that have 3 as a factor. Some of these numbers will have been eliminated before. What is the first number to be eliminated from the table twice? What are the prime factors of this number?

C What is the first "new" number to be eliminated from the table?

D What is the next number larger than 3 that has not been eliminated from the table?

4. **A** Circle the number 5 in the table, for it is prime, and then strike out each of the numbers that has 5 as a factor.

B What is the first new number eliminated from the table this time?

5. Continue the process above until you reach a point where no new number is eliminated from the table. The numbers remaining are the primes less than 100.

★ **6.** Extend the sieve to find all primes less than 500.

UTILIZATION

You may want to check with your class to see if they are familiar with the Sieve of Eratosthenes. If your students have used the Sieve previously, the experiences provided by these exercises may not be particularly rewarding. Hence, you might omit this page altogether and move to the next lesson. However, the method illustrated by the Sieve is well worth the time it may take if it is a new experience for students.

If possible, supply a duplicated set of numbers arranged in columns for students to use in their work. For Exercise 3C, note that, when we strike out the multiples of any number, the first "new" number we strike out is the square of that number.

EXTENSION

The exercises on page 15 of the *Duplicator Masters* can be assigned to provide appropriate further practice.

Enrichment: Some students might be interested in exploring a different set of "prime" numbers.

Let $E' = \{1\} \cup \{\text{even whole numbers}\}$, that is, $E' = \{1, 2, 4, 6, 8, 10, 12, \ldots\}$. Consider the following definition:

A number in E' is "prime" if it has exactly two different factors in E'.

Questions

1. What are the "primes" in E′?
2. What are the "composite" numbers in E′?
3. Construct a sieve for E′ similar to the Sieve of Eratosthenes for the natural numbers.

Answers

1. $\{2, 6, 10, 14, 18, \ldots\}$
   (Note that the only factors of 6 in E' are 1 and 6.)
2. $\{4, 8, 12, 16, 20, \ldots\}$
3. By striking out every other number starting with 1, the "primes" in E′ will remain.

Assignments

Minimum 1–5. Average 1–5. Maximum 1–6.

Objective

The student can use a table of primes to solve problems.

PREPARATION

The preparation period could well be spent examining and discussing the tables of primes given in the Investigation section.

This lesson is optional. If you choose to cover the material, then it should be in the spirit of an informal investigation into interesting ideas concerning prime numbers.

INVESTIGATION

Some students may observe that there appears to be a gradual decrease in the number of primes between successive multiples of 100. This may lead them to the erroneous conclusion that eventually there may be no more primes. It can be proved that there are infinitely many primes, that is, that there is no largest prime number. (See Research Project D on page E-87.) That proof, purported to be given by Euclid (*c*300 B.C.), is a model of elegance in mathematical proof but is probably beyond the ability of most students at this level.

DISCUSSION

The Discussion should be directed toward developing student understanding of the use of the table displayed in the Investigation. You may want to include questions such as this: "Why are 2 and 13 the only two primes in the table with a difference of 11?" (2 is the only even prime and the difference between an odd and even number is an odd number; the difference of any other two primes will always be an even number.)

A Short Table of Primes

Investigating the Ideas

The table below gives the first 199 prime numbers. Using this table, you can find that the 43rd prime is 191, for it is found in the row headed "4" and the column headed "3."

Can you use the table of prime numbers to help you complete the chart below?

A short table of prime numbers

	0	1	2	3	4	5	6	7	8	9
0		2	3	5	7	11	13	17	19	23
1	29	31	37	41	43	47	53	59	61	67
2	71	73	79	83	89	97	101	103	107	109
3	113	127	131	137	139	149	151	157	163	167
4	173	179	181	191	193	197	199	211	223	227
5	229	233	239	241	251	257	263	269	271	277
6	281	283	293	307	311	313	317	331	337	347
7	349	353	359	367	373	379	383	389	397	401
8	409	419	421	431	433	439	443	449	457	461
9	463	467	479	487	491	499	503	509	521	523
10	541	547	557	563	569	571	577	587	593	599
11	601	607	613	617	619	631	641	643	647	653
12	659	661	673	677	683	691	701	709	719	727
13	733	739	743	751	757	761	769	773	787	797
14	809	811	821	823	827	829	839	853	857	859
15	863	877	881	883	887	907	911	919	929	937
16	941	947	953	967	971	977	983	991	997	1009
17	1013	1019	1021	1031	1033	1039	1049	1051	1061	1063
18	1069	1087	1091	1093	1097	1103	1109	1117	1123	1129
19	1151	1153	1163	1171	1181	1187	1193	1201	1213	1217

Numbers between	How many primes?
0 and 100	
100 and 200	
200 and 300	
300 and 400	
400 and 500	
500 and 600	
600 and 700	
700 and 800	
800 and 900	
900 and 1000	
1000 and 1100	
1100 and 1200	

Discussing the Ideas

1. Between which pair of numbers in the chart did the greatest number of primes occur? How many?

2. What was the least number of primes occurring between any pair of the numbers in the chart?

3. Using the table of primes, find the following:
   - **A** 125th prime
   - **B** 11th prime
   - **C** 147th prime
   - **D** 53rd prime
   - **E** 80th prime
   - **F** 182nd prime
   - **G** 127th prime
   - **H** 155th prime

4. Using the table, determine the relative position of each of the following primes. EXAMPLE: 89 is the 24th prime.
   - **A** 431
   - **B** 719
   - **C** 101
   - **D** 607
   - **E** 1181
   - **F** 523
   - **G** 157

Using the Ideas

1. Use the table of primes to find the following:

A 22nd prime **B** 41st prime **C** 75th prime **D** 92nd prime

2. Is 279 a prime? If not, list all of its factors.

3. Using the table, give the relative position of each prime.

A 311 **B** 461 **C** 11 **D** 659 **E** 1213 **F** 991

4. Which numbers in the set {1, 13, 113, 1113} are prime?

5. Which numbers in this set are prime?
{101, 201, 301, 401, 501, 601, 701, 801, 901, 1001, 1101, 1201}

6. Pairs of primes, such as 11 and 13 or 29 and 31, are called **twin primes** because they have a difference of 2.

A List all the pairs of twin primes that are less than 100.

B How many pairs of twin primes are there between 1000 and 1100?

★**C** Show that the prime 1217 does not have a twin.

Think

List all the whole numbers that are less than 60 and have an odd number of factors. What property does the set of numbers have?

7. There are three consecutive odd numbers which are prime. What are these numbers?

8. The three primes 11, 13, and 17 are three consecutive primes. They have the form p, $p + 2$, and $p + 6$ where p stands for a prime. Find another set of primes with this form.

★**9.** For each input number n, the function machine multiplies the input number by 1 less than the input number and then adds 11 to the product.

A Copy and complete the function table.

B Are all of the output numbers $f(n)$ prime numbers?

Function Rule: $n\ (n-1) + 11$

n	$f(n)$	n	$f(n)$
1	11	7	
2	13	8	
3		9	
4		10	
5		11	
6		12	

★**10.** Using the function rule $n\ (n-1) + 17$ and the input numbers 1 through 20, find the numbers for $f(n)$. In each case, do you get a prime number for $f(n)$?

UTILIZATION

You will find that some of these exercises will generate interesting class discussions. Note that much of the material involves a minimum of computation.

Certain formulas, such as that given in Exercise 10, generate many primes, but eventually they yield numbers that are not primes. No simple polynomial formula can be found which will produce only primes. The formula given in Exercise 10 is called a "prime-rich" formula because of the yield of primes.

Answers, Exercise 10

Function Rule: $n(n-1) + 17$

n	$f(n)$	n	$f(n)$
1	17	11	127
2	19	12	149
3	23	13	173
4	29	14	199
5	37	15	227
6	47	16	257
7	59	17	289
8	73	18	323
9	89	19	359
10	107	20	397

289 and 323 are not prime numbers.

EXTENSION

For additional practice exercises, refer to page 18 of the *Workbook*.

Enrichment: Students might enjoy exploring the famous unsolved problem in mathematics called Goldbach's Conjecture. (See Research Projects A and C on page E-87.) This conjecture concerns whether or not every even number greater than 2 can be expressed as the sum of two primes. For example, $24 = 11 + 13$, and $44 = 31 + 13 = 27 + 7$. Ask students to write each number below as the sum of two primes.

A. 20 B. 40 C. 100 D. 80
E. 66 F. 72 G. 32 H. 54

Think Solution

{1, 4, 9, 16, 25, 36, 49} all are square numbers.

Assignments

Minimum 1–5. Average 1–7. Maximum 1–10.

Objective

Given a composite number, the student can give the prime factorization of the number.

PREPARATION

Most students will have constructed factor trees before. However, it would be well to review the construction of factor trees before starting the Investigation.

INVESTIGATION

The purpose of the Investigation is to help students discover that the "top row" of a factor tree for a composite number such as 48 will contain the same prime factors regardless of which nonprime factors were used in the lower rows. This illustrates the Fundamental Theorem of Arithmetic, which will be presented in Discussion Exercise 2.

DISCUSSION

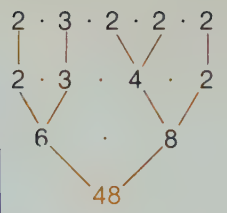
This lesson has several useful applications. It will be related to finding the greatest common factor of two composite numbers and to determining the lowest-terms fraction for a given fraction.

Note that the expressions *prime factorization* and *factor completely* are used interchangeably.

Prime Factorization

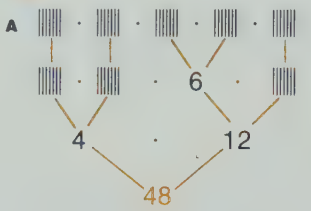
Investigating the Ideas

The **factor tree** at the right shows that 48 can be expressed as the product $2 \cdot 3 \cdot 2 \cdot 2 \cdot 2$.

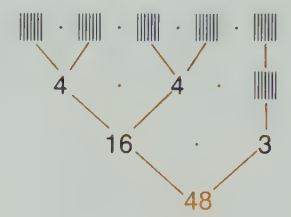


Can you complete these other factor trees for 48?

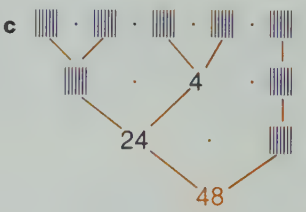
A



B



C



Discussing the Ideas

1. A Are all of the factors in the top row of each factor tree for 48 prime numbers?
   B Except for the order in which the factors occur, are the same prime factors found in the top row of each tree?

2. The factor trees illustrate an important theorem in mathematics called the **Fundamental Theorem of Arithmetic**. It is also known as the **Unique Factorization Theorem**.

   Each composite number can be expressed as a product of primes in exactly one way, except for the order of the factors.

   Make a factor tree for 36. What are the prime factors of 36?

3. When a number is written as a product of prime numbers, we say that we have given a **prime factorization** of the number, or that we have **factored the number completely**. Give the missing number for each ▥ so that a prime factorization of each number is found.

   A $14 = 2 \cdot$ ▥
   B $20 = 5 \cdot 2 \cdot$ ▥
   C $26 = 2 \cdot$ ▥
   D $8 = 2 \cdot 2 \cdot$ ▥
   E $18 = 3 \cdot 2 \cdot$ ▥
   F $30 = 2 \cdot 3 \cdot$ ▥
   G $42 = 2 \cdot 3 \cdot$ ▥
   H $27 = 3 \cdot 3 \cdot$ ▥
   I $44 = 2 \cdot 2 \cdot$ ▥

4. Factor each number completely. Use a factor tree if necessary.

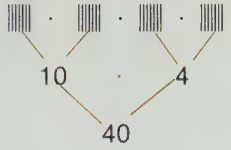
   A 60 B 42 C 140 D 300 E 625

Using the Ideas

1. Complete each factor tree.

A

B



C

2. Factor each composite number completely.

A 15	**C** 22	**E** 50	**G** 45	**I** 81	**K** 66
B 12	**D** 39	**F** 70	**H** 16	**J** 60	**L** 100

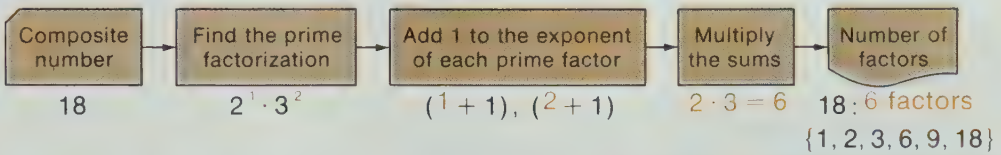
3. Each composite number is written as the product of two factors. Use these factors to help you give prime factorizations of the numbers. EXAMPLE: $140 = 4 \cdot 35$, $140 = 2 \cdot 2 \cdot 5 \cdot 7$

A $98 = 2 \cdot 49$	**E** $196 = 14 \cdot 14$	**I** $240 = 15 \cdot 16$
B $63 = 9 \cdot 7$	**F** $150 = 10 \cdot 15$	**J** $189 = 9 \cdot 21$
C $60 = 6 \cdot 10$	**G** $132 = 11 \cdot 12$	**K** $144 = 16 \cdot 9$
D $156 = 12 \cdot 13$	**H** $72 = 9 \cdot 8$	**L** $490 = 10 \cdot 49$

4. Use exponents to give the prime factorizations of the numbers. For example, $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$.

A 24	**D** 81	**G** 120	**J** 484	**M** 441
B 72	**E** 216	**H** 48	**K** 288	**N** 612
C 250	**F** 1000	**I** 108	**L** 225	**O** 1089

★ **5.** The flow chart below can help you find the number of factors of any composite number.


Composite number
Find the prime factorization
Add 1 to the exponent of each prime factor
Multiply the sums
Number of factors
18
$2^1 \cdot 3^2$
$(1 + 1), (2 + 1)$
$2 \cdot 3 = 6$
18: 6 factors
{1, 2, 3, 6, 9, 18}

Give the number of factors of each number.

A $24 = 2^3 \cdot 3^1$	**D** $72 = 2^3 \cdot 3^2$
B $30 = 2^1 \cdot 3^1 \cdot 5^1$	**E** $1000 = 2^3 \cdot 5^3$
C $100 = 2^2 \cdot 5^2$	**F** $1001 = 7^1 \cdot 11^1 \cdot 13^1$

★ **6.** List the set of all factors of the numbers given in Exercise 5.

Think

1. Divide the figure into three pieces that are exactly alike.
2. Divide the figure into four pieces that are exactly alike.

More practice, page S-8, Set 14

UTILIZATION

Exercises 5 and 6 are optional. The flow chart gives the steps used to determine the number of prime factors of a number. For example, the prime factorization of 120 is $2^3 \cdot 3^1 \cdot 5^1$. Adding 1 to each exponent we get the numbers 4, 2, 2. The product $4 \cdot 2 \cdot 2 = 16$ is the number of factors of the number. As a check, the students can list the factors:
{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120}

Answers, Exercise 6

A. {1, 2, 3, 4, 6, 8, 12, 24}
B. {1, 2, 3, 5, 6, 10, 15, 30}
C. {1, 2, 4, 5, 10, 20, 25, 50, 100}
D. {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}
E. {1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500, 1000}
F. {1, 7, 11, 13, 77, 91, 143, 1001}

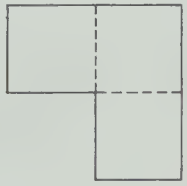
EXTENSION

To provide students with extra practice make assignments from Supplementary Exercise Set 14 on page S-8, *Workbook* page 19, and *Duplicator Masters* page 16.

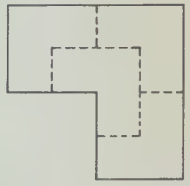
Think Solution

Part 2 of the problem may cause difficulty. If each square in part 1 relinquishes its inside one quarter, those three quarters combined form a figure of the same size and shape as the remaining portions of the three squares. (By "inside one quarter," we mean the quarter of the square opposite the outside corners.)

1.



2.



Objective

Given two whole numbers, the student can find their greatest common factor.

PREPARATION

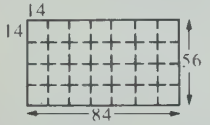
If possible, have available fine-ruled graph paper on which students can draw a rectangle 84 units long and 56 units wide for use in the Investigation.

INVESTIGATION

Make certain that students clearly understand the problem posed by the Investigation question. The squares that are chosen must fit exactly with no overlapping. If fine-ruled graph paper is available, let students use this to draw a rectangle 84 by 56 units to represent a desk top. Then they can use whatever method they choose to find the largest possible squares into which the rectangular region can be evenly partitioned.

DISCUSSION

Have a student use the chalkboard or overhead to demonstrate the solution to the Investigation.



Discussion Exercise 1 brings out the idea that the solution to the Investigation is the square with sides whose measure is the largest factor of both 84 and 56.

Exercise 2 illustrates how intersection of the sets of factors of two numbers can help a student determine the greatest common factor of the pair.

Fraction notation is introduced in Discussion Exercise 3 to show the usefulness of the GCF concept, but actual development of fraction notation and any reference to rational numbers should be deferred until Unit F, Module 1. The two important ideas to be emphasized here are, first, what the GCF of two numbers is, and second, what it means to say that two numbers are *relatively prime* to each other. The latter term is introduced in Exercise 5 on page E-71, but might be discussed at this time.

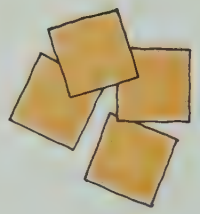
Greatest Common Factor

Investigating the Ideas

Jody's desk top is a rectangle 84 centimetres long by 56 centimetres wide. She wants to cover the desk top exactly with squares. Of course, 1 by 1-centimetre squares could be used or 2 by 2-centimetre squares would work also.

What is the largest size square that could be used to completely cover the desk top?

Discussing the Ideas



1. Peggy said, "The squares that fit must have sides whose lengths are factors of both 56 and 84."
   - **A** List the factors of 56.
   - **B** List the factors of 84.
   - **C** What are the common factors of 56 and 84?
   - **D** What is the largest factor common to 56 and 84?

2. The **greatest common factor (GCF)** of two numbers is the largest number that is a common factor of both numbers.

   Study the information at the right. What is the GCF of 18 and 24?

   The set of factors of 18: $A = \{1, 2, 3, 6, 9, 18\}$
   The set of factors of 24: $B = \{1, 2, 3, 4, 6, 8, 12, 24\}$
   The set of common factors of 18 and 24: $A \cap B = \{1, 2, 3, 6\}$
   The GCF of 18 and 24 is ▒.

3. To find the lowest-terms fraction for a given fraction, divide the numerator and the denominator by their GCF.

   EXAMPLE: $\frac{18}{24}$ The GCF of 18 and 24 is 6. $\frac{18 \div 6}{24 \div 6} = \frac{3}{4}$ ← **Lowest-terms fraction**

   What is the GCF of the numerator and denominator of each fraction?

   A $\frac{12}{18}$ **B** $\frac{9}{12}$ **C** $\frac{50}{60}$ **D** $\frac{15}{25}$ **E** $\frac{12}{14}$

Using the Ideas

1. **A** List the set of factors of 12.
B List the set of factors of 18.
C Find the intersection of the sets in Parts **A** and **B**.
D What is the GCF of 12 and 18?

2. **A** List the set of factors of 36.
B List the set of factors of 54.
C Find the intersection of the sets in Parts **A** and **B**.
D What is the GCF of 36 and 54?

3. Find the GCF of each pair.

A 12, 20	**C** 21, 56	**E** 12, 16	**G** 14, 63	**I** 34, 51
B 8, 18	**D** 30, 45	**F** 20, 30	**H** 24, 72	**J** 180, 250

4. What is the GCF of the numerator and the denominator of each fraction?

A $\frac{10}{25}$ **B** $\frac{16}{24}$ **C** $\frac{27}{30}$ **D** $\frac{24}{72}$ **E** $\frac{56}{98}$ **F** $\frac{48}{120}$

5. If the GCF of two whole numbers is 1, we say that the numbers are **relatively prime**. Using this definition, decide which pairs of numbers are relatively prime. If the numbers are not relatively prime, give the GCF.

A 4, 15	**C** 8, 27	**E** 20, 33	**G** 26, 51	**I** 249, 303
B 15, 18	**D** 14, 15	**F** 33, 144	**H** 87, 74	**J** 221, 247

6. The examples show how prime factorizations can be used to find the GCF of a pair of numbers.

EXAMPLES:

A
$20 = 2 \cdot 2 \cdot 5$
$28 = 2 \cdot 2 \cdot 7$
$\downarrow$
$GCF = 2 \cdot 2 = 4$

B
$60 = 2 \cdot 2 \cdot 3 \cdot 5$
$126 = 2 \cdot 3 \cdot 3 \cdot 7$
$\downarrow$
$GCF = 2 \cdot 3 = 6$

Give the GCF of each pair.

A $30 = 2 \cdot 3 \cdot 5$
$105 = 3 \cdot 5 \cdot 7$

B $44 = 2 \cdot 2 \cdot 11$
$42 = 2 \cdot 3 \cdot 7$

C $14 = 2 \cdot 7$
$84 = 2 \cdot 2 \cdot 3 \cdot 7$

D $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
$858 = 2 \cdot 3 \cdot 11 \cdot 13$

E $540 = 2^2 \cdot 3^3 \cdot 5$
$90 = 2 \cdot 3^2 \cdot 5$

F $55\,125 = 3^2 \cdot 5^3 \cdot 7^2$
$514\,080 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$

Think

Use the digits 2, 3, and 7 to write all the 3-digit numerals you can with none of the digits repeated in the same numeral. Then find the GCF of these numbers.

More practice, page S-9, Set 15

UTILIZATION

Exercises 1 and 2 might be treated orally if you prefer.

The term *relatively prime*, defined in Exercise 5, may need some elaboration. This term describes a relation between certain pairs of numbers. The statement that "8 is relatively prime to 9" means that the greatest common factor of 8 and 9 is 1. Another way of thinking about this is to look at the prime factorization of 8 and 9:

$8 = 2^3$, $9 = 3^2$. None of the prime factors of 8 are factors of 9 or vice versa. Hence, the two numbers are "relatively prime" with respect to each other.

EXTENSION

Extra practice assignments may be made from Supplementary Exercise Set 15 on page S-9, *Workbook* page 20, *Duplicator Masters* page 17, and *Arithmetic Skill Card* W-11.

Think Solution

The three-digit numbers that can be formed are 237, 273, 327, 372, 723, and 732. If we understand the divisibility property of 3 (the sum of the digits of a number, if divisible by 3, will be divisible by 3 no matter what the order of the digits), we know that 3 will be a factor in finding the GCF of these numbers.

Now consider one of the numbers, for example, 237. It can be completely factored into $237 = 3 \cdot 79$. Checking one other number, for example, $273 = 3 \cdot 7 \cdot 13$, we immediately see that the only common factor of 237 and 273 is 3. Thus, the GCF for all the numbers must be 3.

Assignments

Minimum 1 and 2, oral; 3.
Average 1 and 2, oral; 3; 4.
Maximum 1 and 2, oral; 3–6.

Objective

Given two numbers of two or more digits, the student can find the GCF of the numbers by using the Euclidean Algorithm.

PREPARATION

This section on the Euclidean Algorithm is optional enrichment material.

Flow charts have been used earlier in this learning unit. The flow chart for the Euclidean Algorithm contains a question box. Since this is a feature that some of your students may not have seen before, explain that all questions asked in a question box are answerable by either "yes" or "no." If a question is answered "yes" the arrow labelled "yes" is followed to the next step; otherwise, the "no" arrow is followed.

INVESTIGATION

To introduce the Investigation, you might ask your students this question: "Can you find the GCF of 1 483 417 and 397 801?" (Answer: 2153.) Most students will agree that this seems to be a difficult problem. Point out that the flow chart in the Investigation will enable them to solve this "hard" problem by a method used by Greek mathematicians more than 2000 years ago.

Have the students try the flow chart for 1104 and 2507. A volunteer might show or describe his method for finding the GCF before you move on to the Discussion Exercises.

DISCUSSION

Relate the division arrangement shown in Discussion Exercise 3 to the flow chart in the Investigation. After discussing Exercise 4, supply additional practice problems as needed. For example:

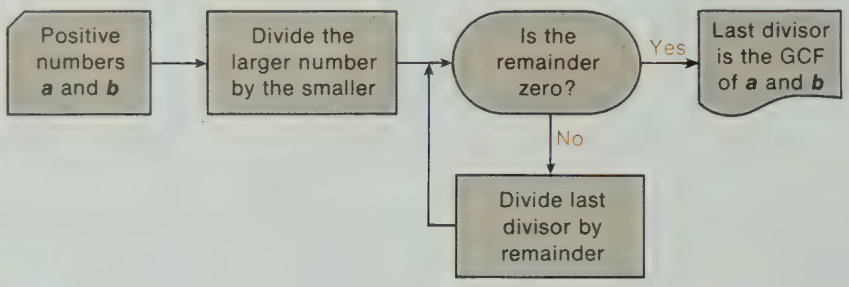
A. 247 and 513 (GCF: 19) B. 594 and 1430 (GCF: 22) C. 6077 and 10 403 (GCF: 103)

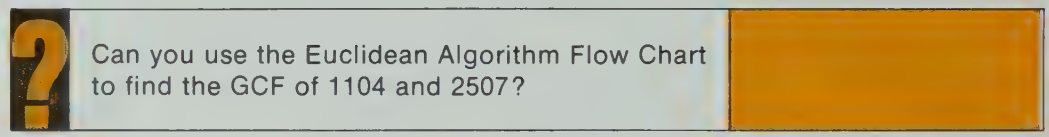
The Euclidean Algorithm

Investigating the Ideas

If you wish to find the GCF of two rather large numbers, you may find the problem difficult. The flow chart below shows the steps of the **Euclidean Algorithm**, a method which simplifies the problem of finding the GCF of two numbers.

EUCLIDEAN ALGORITHM FLOW CHART


Positive numbers *a* and *b*
Divide the larger number by the smaller
Is the remainder zero?
Yes
Last divisor is the GCF of *a* and *b*
No
Divide last divisor by remainder



Discussing the Ideas

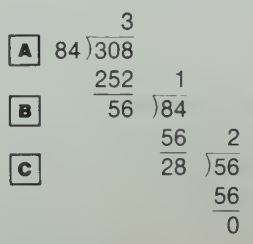
1. Explain how to use the flow chart when $a = 12$ and $b = 8$.

2. What is the GCF of 12 and 8? Did you get this number as the output number for the flow chart?

3. The repeated division at the right shows a shortcut way of arranging the division problems in order to find the GCF of 84 and 308.

   A Explain what was done in each step.

   B What is the GCF of 84 and 308?



4. Find the GCF of 141 and 376 using the Euclidean Algorithm.

Using the Ideas

1. Find the GCF of each pair by using the Euclidean Algorithm.

A 14, 210	**D** 135, 351	**G** 368, 80	**J** 475, 1501
B 184, 69	**E** 70, 525	**H** 259, 888	**K** 7560, 1232
C 115, 391	**F** 64, 128	**I** 2599, 2825	**L** 38 127; 31 737

2. Prove that 340 and 819 are relatively prime; that is, show that their GCF is 1 using the Euclidean Algorithm.

3. Any two consecutive whole numbers such as 8 and 9 or 342 and 343 are relatively prime.

A What is the first remainder using the Euclidean Algorithm?

B What is the second remainder?

4. Are 1001 and 100 001 relatively prime? What is their GCF?

★ **5.** There are 256 seventh grade students at Edishtron Junior High School. There are 224 eighth grade students in the school also. The principal of the school arranged the homerooms so that both grades have the same number of students in each homeroom. Furthermore, the number of students in each homeroom is as large as possible.

A How many students are in each homeroom?

B How many homerooms are needed?

think

1. A student chooses a whole number between 1 and 1000. You are to discover the chosen number by asking ten or fewer questions of the student. The questions must be stated so that they can be answered by Yes or No. Can you find the number?
2. How many questions would you need to find any whole number less than a million by asking Yes or No questions?

UTILIZATION

Note that exercise 1 consists of 12 parts. You may not find it necessary to assign all parts.

In Exercise 3, the first remainder will be 1, and using 1 as the next divisor will always result in a zero remainder. Hence 1 is the GCF of any two consecutive whole numbers.

EXTENSION

Workbook page 21 is recommended for students who would benefit from additional practice exercises.

Think Solution

1. Suppose the number selected is 745. The questions must be asked so as to successively halve the sets of numbers containing the chosen number. The following sequence of questions could be used.

	Question: Is the number larger than	*Answer*	*Number lies between*
(1)	500	Yes	(500–1000)
(2)	750	No	(500–750)
(3)	625	Yes	(625–750)
(4)	687	Yes	(687–750)
(5)	718	Yes	(718–750)
(6)	734	Yes	(734–750)
(7)	742	Yes	(742–750)
(8)	746	No	(742–746)
(9)	744	Yes	(744–746)
(10)	745	No	The number is 745.

2. As observed in part 1, we can halve the size of the set of numbers containing the chosen number with each question asked; therefore we need to know what power of 2 will give a number at least as large as a million.

$$2^{10} = 1024 > 1000,$$

therefore $2^{20} > 1000^2 > 1\,000\,000$.

Thus, we can locate any number less than 1 000 000 in as few as 20 questions.

Assignments

Minimum 1A–H. Average 1A–J, 2, 3.
Maximum 1G–L, 2–5.

Objective

Given two whole numbers, the student can find the least common multiple of the numbers.

PREPARATION

The material presented in this lesson may be largely review for the students. Therefore you should determine the time to be spent on it accordingly.

INVESTIGATION

The Investigation presents a problem which will lead students to use intuitive reasoning that involves the basic idea of the least common multiples of two numbers. Some students may reason that while Alpha makes 1 revolution, Beta makes only $\frac{3}{4}$ of a revolution. Alpha will make 4 revolutions while Beta is making 3. Since $4 \cdot 90 = 3 \cdot 120$, it will take 360 minutes for the satellites to arrive back in their original positions.

DISCUSSION

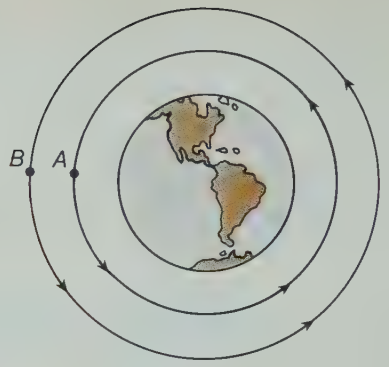
Discussion Exercise 3 presents a basic approach to finding the least common multiple. Exercise 4 illustrates a shortcut for finding the LCM that can be used with suitable pairs of numbers.

Observe that zero is a multiple of every nonzero number. Thus, in defining the LCM, we must specify that we are speaking of the smallest *nonzero* multiple common to both given numbers.

Least Common Multiple

Investigating the Ideas

Suppose two satellites, Alpha (*A*) and Beta (*B*), are orbiting the earth. Satellite Alpha makes one complete orbit in 90 minutes and Beta makes one complete orbit in 120 minutes. The illustration shows the present position of Alpha and Beta.



Can you find how many minutes it will take for Alpha and Beta to get back together in the same position?

Discussing the Ideas

1. Explain how you found your answer to the Investigation question.

2. Marty said, "I counted multiples of 90 until I reached a multiple of 120." What number did Marty find?

Multiples of 90: {90, 180, 270, ▒▒, ▒▒, . . .}
Multiples of 120: {120, 240, ▒▒, ▒▒, ▒▒, . . .}

3. Marty found the least common multiple (LCM) of 90 and 120. It is the smallest nonzero common multiple of two numbers. Study the example below. Give the number for the ▒▒.

Set of nonzero multiples of 8: $A = \{8, 16, 24, 32, 40, 48, 56, 64, \ldots\}$
Set of nonzero multiples of 12: $B = \{12, 24, 36, 48, 60, \ldots\}$
Common nonzero multiples of 8 and 12: $A \cap B = \{24, 48, \ldots\}$
The smallest nonzero common multiple of 8 and 12 is ▒▒.
The LCM of 8 and 12 is ▒▒.

4. Study the example and then find the LCM for the pairs of numbers below.

EXAMPLE: Find the LCM of 8 and 10.
Nonzero multiples of 8 → {8, 16, 24, 32, 40, 48, . . .}
↑
40 is the first nonzero multiple of 8 that is also a multiple of 10.
The LCM of 8 and 10 is 40.

A 6 and 15 **B** 12 and 20 **C** 9 and 12

Using the Ideas

1. **A** List the nonzero multiples of 6 that are less than 72.

B List the nonzero multiples of 20 that are less than 80.

C List the set of nonzero common multiples of 6 and 20 that are less than 72.

D What is the LCM of 6 and 20?

2. Find the LCM of each pair of numbers by thinking about multiples of the larger number until you reach a nonzero multiple of the smaller number.

EXAMPLE: Find the LCM of 6 and 8.
Nonzero multiples of 8 → {8, 16, . . .}
The LCM of 6 and 8 is 24

A 4, 6	**C** 9, 12	**E** 8, 7	**G** 8, 10	**I** 6, 21
B 6, 15	**D** 10, 15	**F** 8, 20	**H** 10, 16	**J** 15, 24

3. To find the LCM of 2, 3, and 5, first find the LCM of 3 and 5 (which is 15). Now find the LCM of 2 and 15 (which is 30). The LCM of 2, 3, and 5 is 30. Find the LCM of each group.

A 4, 6, 10	**D** 6, 10, 12	**G** 8, 10, 12	**J** 10, 15, 20
B 2, 3, 4	**E** 4, 6, 8	**H** 9, 12, 15	**K** 6, 12, 18
C 4, 6, 9	**F** 3, 4, 6	**I** 2, 4, 6	**L** 30, 40, 60

4. If the prime factors of two numbers are known, it is easy to choose the LCM. Study the example.

EXAMPLE: Find the LCM of 18 and 15.

$18 = 2 \cdot 3 \cdot 3$
$15 = 3 \cdot 5$

The LCM of 18 and 15 must contain all the factors of 18 as well as those factors of 15 which are not factors of 18.

The LCM of 18 and 15 is $2 \cdot 3 \cdot 3 \cdot 5 = 90$

Find the LCM of each pair.

A $4 = 2 \cdot 2$, $14 = 2 \cdot 7$

B $20 = 2 \cdot 2 \cdot 5$, $25 = 5 \cdot 5$

C $42 = 2 \cdot 3 \cdot 7$, $70 = 2 \cdot 5 \cdot 7$

D $8 = 2 \cdot 2 \cdot 2$, $9 = 3 \cdot 3$

E $33 = 3 \cdot 11$, $15 = 3 \cdot 5$

F $18 = 2 \cdot 3^2$, $60 = 2^2 \cdot 3 \cdot 5$

G $27 = 3^3$, $36 = 2^2 \cdot 3^2$

H $48 = 2^4 \cdot 3$, $54 = 2 \cdot 3^3$

think

Two cars, *A* and *B*, travel in the same direction. Car *A* travels toward *B* at a rate of 60 kilometres per hour. Car *B* travels at a rate of 50 km/h and is now 200 km ahead of *A*.

1. How many hours will it take *A* to catch *B*?
2. How many km must *A* travel in order to catch *B*?

More practice, page S-9, Set 16

UTILIZATION

Do not require that students use any one method for finding the LCM in these exercises. If the numbers are small, the students will probably find it easiest to think about multiples of each number. However, if the numbers are large, they may find that thinking about prime factorizations of the numbers is a more practical way of finding the LCM. For example, the LCM of 63 and 35 may be found as follows:

$63 = 3^2 \cdot 7$ $\quad$ $35 = 5 \cdot 7$

From these two equations, the students can reason that the smallest nonzero number that is a multiple of both 63 and 35 must contain all the factors of the larger number, plus any factors of the smaller number which are not factors of the larger. Hence, the LCM of 63 and 35 is $3^2 \cdot 7 \cdot 5$, or 315.

EXTENSION

Further practice may be assigned from Supplementary Exercise Set 16 on page S-9, *Workbook* page 22, *Duplicator Masters* page 18, and *Arithmetic Skill Card* W-12.

Enrichment: Challenge your better students with these problems.

1. What is the smallest-size square that can be covered both by 9-cm and by 12-cm squares if we are not permitted to cut any of the squares? (36-cm square)
2. Repeat the first exercise for a 12-cm square and 15-cm square. (60-cm square)

Think Solution

1. With car *A* travelling at 60 km/h and car *B* travelling at 50 km/h, car *A* will gain 10 km for each hour of travel (60 km/h − 50 km/h = 10 km/h). Since car *B* has a 200 km lead, car *A* will require 200 ÷ 10 = 20 h to catch car *B*.
2. If car *A* must travel for 20 h at 60 km/h to catch car *B*, then car *A* will travel 20 · 60 = 1200 km.

Assignments

Minimum 1, oral; 2, 3A–C.
Average 1, oral; 2; 3; 4A–D.
Maximum 1, oral; 2–4.

Objective

The student can use divisibility rules for 2, 3, 5, and 10.

PREPARATION

Although this and the following lesson can be regarded as supplementary or enrichment material, they contain enough useful mathematical ideas to make them worthwhile for most students. However, it is not intended that students memorize or become proficient at working with the divisibility rules.

INVESTIGATION

Have the students read the expository material in the Investigation and then try the Investigation question. Some of them may already be familiar with the divisibility rule for 9. The rule itself will be discussed in the next lesson, but if the point comes up it could be discussed here as well.

DISCUSSION

A formal definition of divisibility can be stated as follows:

If a and b are nonzero whole numbers, and $a \cdot b = c$, then c is divisible by a, and c is divisible by b.

Most students will already know divisibility rules for 2, 5, and 10, and some may know the divisibility rule for 3 as well.

A proof that, if the sum of the digits of a number is divisible by 3, then the number is divisible by 3 is given for the case of a three-digit number abc:

We have $abc = 100a + 10b + c$

$= 99a + a + 9b + b + c$

$= 99a + 9b + (a + b + c)$

Clearly, 3 is a factor of the first two terms. Therefore the number will be divisible by 3 if 3 is a factor of $(a + b + c)$, that is, if 3 is a factor of the sum of the digits a, b, and c.

The same reasoning also shows that a number is divisible by 9 if the sum of the digits is divisible by 9. The proof can be extended to any number of digits.

Some Divisibility Rules

Investigating the Ideas

It is useful to know when one number is divisible by another number.

36 is **divisible** by 4 because $4 \cdot 9 = 36$.

42 is **divisible** by 3 because $3 \cdot 14 = 42$.

However, 15 is **not divisible** by 4 because

$$4\overline{)15}\quad 3,\ 12,\ 3$$

Also, 27 is **not divisible** by 6 because

$$6\overline{)27}\quad 4,\ 24,\ 3$$

Can you decide which of these numbers are divisible by 9?

A 126 **B** 2483 **C** 7696 **D** 828 135

Discussing the Ideas

1. Using the equation $7 \cdot 23 = 161$, you can say that "161 **is divisible by** 7," or "7 **is a divisor of** 161," or "7 **is a factor of** 161."
   What similar statements can you make regarding the numbers 23 and 161?

2. There are some simple divisibility rules that you may already know.
   - **A** How can you tell whether or not a number is divisible by 2?
   - **B** How can you tell whether or not a number is divisible by 5?
   - **C** How can you tell whether or not a number is divisible by 10?
   - **D** How can you tell whether or not a number is divisible by 25?
   - **E** How can you tell whether or not a number is divisible by 100?

3. Study the table at the right. Then complete this statement:
   If a number is divisible by 3, then the sum of the digits of the number is ?.

$3 \cdot 4 =$	$12 \rightarrow 1 + 2 = 3$
$3 \cdot 8 =$	$24 \rightarrow 2 + 4 = 6$
$3 \cdot 102 =$	$306 \rightarrow 3 + 0 + 6 = 9$
$3 \cdot 727 =$	$2181 \rightarrow 2 + 1 + 8 + 1 = 12$
$3 \cdot 1254 =$	$3762 \rightarrow 3 + 7 + 6 + 2 = 18$

4. Which of these numbers are divisible by 3?
   A 421 **B** 7114 **C** 705 **D** 121 212

Using the Ideas

1. In the set {12, 15, 17, 20, 24, 27, 30, 35, 48, 51, 70, 87}, which numbers are

A divisible by 2? **B** divisible by 5? **C** divisible by 10?

2. Tell whether each statement is true or false.

A Every number which divisible by 10 is also divisible by 2.

B Every number which is divisible by 2 is also divisible by 10.

C Every number which is divisible by 10 is also divisible by 5.

D Every number which is divisible by 10 is also divisible by both 2 and 5.

3. Solve the equation, then answer the question.

A $1 + 2 + 3 = n$: is 123 divisible by 3?

B $2 + 1 + 1 = n$: is 211 divisible by 3?

C $7 + 0 + 5 = n$: is 705 divisible by 3?

D $7 + 5 + 6 = n$: is 756 divisible by 3?

E $2 + 0 + 0 + 1 = n$: is 2001 divisible by 3?

F $5 + 5 + 4 + 2 = n$: is 5542 divisible by 3?

G $7 + 8 + 9 + 2 + 1 = n$: is 78 921 divisible by 3?

4. Without actually dividing, tell which numbers are divisible by 3.

A 231	**C** 519	**E** 1968	**G** 20 001	**I** 1 010 100
B 426	**D** 733	**F** 4440	**H** 72 641	**J** 23 456

5. A number is divisible by 6 if and only if it is even **and** also divisible by 3. Which of the numbers are divisible by 6?

A 132	**C** 765	**E** 842	**G** 1260	**I** 8088	**K** 3234
B 321	**D** 576	**F** 702	**H** 888	**J** 8808	**L** 1968

6. Divisibility rules can help you find the prime factorization of a number. Study the example, then give the prime factorization of each number.

EXAMPLE: Give the prime factorization of 402.

SOLUTION: Since $4 + 0 + 2 = 6$, 3 is a factor of 402.
Also, 402 is an even number; therefore, 2 is a factor.
$402 = 2 \cdot 3 \cdot 67$

A 111 **B** 114 **C** 267 **D** 195 **E** 11 010

UTILIZATION

If your class is well prepared, you may elect to do many of the exercises orally and proceed to the next lesson, which extends the work on divisibility.

EXTENSION

Page 23 of the *Workbook* presents exercises that might be assigned to provide suitable additional practice with the ideas of this lesson.

Enrichment: Here is a challenge problem: A number has 1 for all of its digits. The number is divisible by both 7 and 13. What is the smallest number it could be?

Solution: Since 7 and 13 are both factors of the number, $7 \cdot 13$ or 91 is also a factor of the number. Carry out the division problem.

$$91\overline{)111\ldots}$$

When the dividend is 111111, it will be found that the remainder is 0 and the quotient is 1221.

Assignments

Minimum 1–2, oral; 3, 4.
Average 1–2, oral; 3–5.
Maximum 1–2, oral; 3–6.

Objective

The student can use divisibility rules for 4, 9, and 11.

PREPARATION

This lesson, like the previous one, is optional and can be considered as enrichment. Since many students may find the rules for divisibility interesting and motivational, the time spent in covering this material may be time well spent.

INVESTIGATION

The examples in the Investigation should enable students to observe that a number is divisible by 11 if and only if (units' digit + hundreds' digit + ten thousands' digit + . . .) − (tens' digit + thousands' digit + hundred thousands' digit + . . .) is divisible by 11. Although the proof that follows is quite straightforward, it is unlikely that its presentation would be worthwhile for any but the very able students.

If $abcd$ represents a four-digit number, then

$$\begin{aligned} abcd &= 1000a + 100b + 10c + d \\ &= 1001a - a + 99b + b + 11c - c + d \\ &= (1001a + 99b + 11c) + (b + d) - (a + c) \end{aligned}$$

Since 11 is a factor of $1001a + 99b + 11c$, the number $abcd$ will be divisible by 11 if and only if 11 is a factor of $(b + d) - (a + c)$.

The proof can be extended to numbers represented by any number of digits.

DISCUSSION

In testing for divisibility by 11, the difference may turn out to be a negative number. To avoid having students work with negative integers, suggest that they simply subtract the smaller sum from the larger.

Discussion Exercise 2 provides examples that suggest a divisibility rule for 4: A number is divisible by 4 if the number formed by the tens' and units' digits is divisible by 4.

The proof for the divisibility rule for 9 (Exercise 3) was given in the Discussion section on page E-76.

Other Divisibility Rules

Investigating the Ideas

Study the examples below to discover a rule for divisibility by 11.

253 → 5 − 5 = 0 → 253 is divisible by 11. → 253 = 11 · 23
924 → 13 − 2 = 11 → 924 is divisible by 11. → 924 = 11 · 84
942 → 11 − 4 = 7 → 942 is **not** divisible by 11.
4928 → 17 − 6 = 11 → 4928 is divisible by 11. → 4928 = 11 · 448
25 674 → 12 − 12 = 0 → 25 674 is divisible by 11. → 25 674 = 11 · 2334
342 715 → 16 − 6 = 10 → 342 715 is **not** divisible by 11.

Can you decide which of these numbers are divisible by 11?

A 748 **B** 407 **C** 4721 **D** 98 626 **E** 804 716

Discussing the Ideas

1. What is the divisibility rule for 11 illustrated by the examples in the Investigation?

2. If a number is less than 100, you can quickly decide by using direct division whether or not this number is divisible by 4.

100 is divisible by 4 because $4 \cdot 25 = 100$.

Every multiple of 100 is also divisible by 4.

Study these examples: 104 = 100 + 4 → 104 is divisible by 4.
114 = 100 + 14 → 114 is not divisible by 4.
224 = 200 + 24 → 224 is divisible by 4.
3936 = 3900 + 36 → 3936 is divisible by 4.

A What divisibility rule for 4 do the examples suggest?

B Is 728 divisible by 4? Is 2650 divisible by 4?

3. The rule for divisibility by 9 is much the same as the rule for divisibility by 3. Study the table of information.

$2 \cdot 9 = \blacksquare = 9$	$111 \cdot 9 = 999 \rightarrow 9 + 9 + 9 = 27$
$5 \cdot 9 = \blacksquare = 9$	$217 \cdot 9 = 1953 \rightarrow 1 + 9 + 5 + 3 = 18$
$11 \cdot 9 = \blacksquare = 18$	$834 \cdot 9 = \blacksquare = 18$
$26 \cdot 9 = \blacksquare = 9$	$2541 \cdot 9 = \blacksquare = 27$

A State a rule for divisibility by 9.

B Is 279 divisible by 9? Is 1663 divisible by 9?

Using the Ideas

1. Which of the following numbers are divisible by 11? Verify by actual division.

A	9394	**C**	8251	**E**	8250	**G**	71 577	**I**	437 404
B	8371	**D**	8252	**F**	3465	**H**	395 824	**J**	437 415

2. Which of the numbers are divisible by 4?

A	204	**C**	626	**E**	996	**G**	332	**I**	344	**K**	7112
B	740	**D**	1148	**F**	116	**H**	443	**J**	3008	**L**	6152

3. **A** If a number is divisible by 12, is it also divisible by 4?
B If a number is divisible by 12, is it also divisible by 3?
C If a number is divisible by 3 and also divisible by 4, is it necessarily divisible by 12?

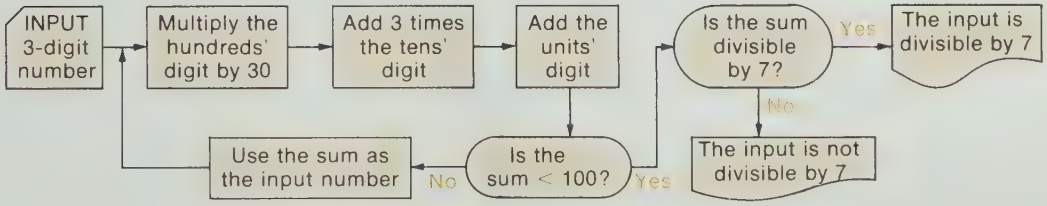
4. Which numbers are divisible by 12?

A	324	**B**	216	**C**	448	**D**	720	**E**	468	**F**	678

5. Which numbers are divisible by 9?

A	414	**C**	7001	**E**	16 164	**G**	441	**I**	3 244 617
B	3153	**D**	8001	**F**	809	**H**	123 456	**J**	38 496

★**6.** Most divisibility rules for 7 are complicated. The flow chart shows the steps for one such rule for 3-digit numbers only. Study the flow chart and the example.


INPUT 3-digit number
Multiply the hundreds' digit by 30
Add 3 times the tens' digit
Add the units' digit
Is the sum divisible by 7?
Yes
The input is divisible by 7
No
The input is not divisible by 7
Is the sum < 100?
Yes
No
Use the sum as the input number

EXAMPLE: 1 2 6 → 6 = 6

30 · 1 = 30
3 · 2 = 6
6
42

Since 42 is divisible by 7, the number 126 is divisible by 7.

Which of these numbers are divisible by 7?

A	246	**D**	647	**G**	973
B	222	**E**	595	**H**	861
C	378	**F**	196	**I**	241

Think

What is the smallest number, larger than zero, which is divisible by all whole numbers 1 through 9?

UTILIZATION

Since this lesson is primarily enrichment, you should not expect students to attain great proficiency in the use of the divisibility rules or to memorize them.

In Exercise 6, students will probably find it easier to use direct division to test for divisibility by 7. However, the relative complexity of rules for divisibility by 7 holds a certain amount of exotic appeal for many students. The rule presented in the text can be extended to four-digit numbers by multiplying the thousands' digit by 300. Thus, to test 2933, we compute

$$\begin{aligned} 300 \cdot 2 &= 600 \\ 30 \cdot 9 &= 270 \\ 3 \cdot 3 &= 9 \\ 3 &= +3 \\ \hline & 882 \end{aligned}$$

Now, using the rule for 3-digit numbers, we get $30 \cdot 8 + 3 \cdot 8 + 2 = 266$. Repeating for 266, we get $30 \cdot 2 + 6 \cdot 3 + 6 = 84$. Since 84 is divisible by 7, the original number, 2933, is divisible by 7.

EXTENSION

Think Solution

The smallest number must have factors of 9, 8, 7, and 5. All other smaller factors than these are contained in the product of this factor. $9 \cdot 8 \cdot 7 \cdot 5 = 2520$ is the smallest number with the required property.

Assignments

Minimum 1A–F, 2A–F, 3, 5A–F.
Average 1–5. Maximum 1–6.

Objective

Given several terms of a sequence and the general rule for the sequence, the student can find other terms in the sequence.

PREPARATION

Students have had at least some experience with number patterns in previous mathematics classes. This lesson presents such patterns somewhat more formally. However, no formal definition of a sequence need be given; it is sufficient for present purposes that the student think of a sequence simply as a number pattern. This lesson is intended as optional enrichment material.

INVESTIGATION

Allow time for students to try to find the patterns for the sequences given in parts A to E. The pattern for A, in which each succeeding term is 3 larger than the previous term, should be obvious to most students. When the difference of successive terms is fairly large, as in sequence E, the pattern is not so evident.

DISCUSSION

For Discussion Exercises 1 and 2, intuitive descriptions of the sequences should be accepted. Answers might be somewhat as follows:
A. Numbers increase by 3 (nth term is $3n - 2$).
B. Successive differences are 3, 4, 5, 6, . . . $\left(n\text{th term is } \frac{n(n+3)}{2}\right)$.
C. Terms increase by the odd numbers starting with 3 (nth term is $n^2 + 1$).
D. Each term is 3 times the one preceding it (nth term is 3^{n-1}), *or* Terms are powers of 3 (nth term is 3^{n-1}).
E. Terms increase by 138 (nth term is $23 + (n - 1) \cdot 138$).

The idea of a *general term* of a sequence is important to the extent that if we want to know the 19th term of the sequence 1, 3, 5, 7, . . . , we do not have to find each previous number of the sequence in order to find it. This should be discussed carefully. Additional examples such as those presented in Exercises 3 and 4 should be given as needed.

Sequences

Investigating the Ideas

Study the sequences at the right. Try to discover a pattern for each of them.

Can you find the next three numbers for any of these sequences?

A 1, 4, 7, 10, 13, . . .

B 2, 5, 9, 14, 20, . . .

C 2, 5, 10, 17, 26, . . .

D 1, 3, 9, 27, 81, . . .

E 23, 161, 299, 437, 575, . . .

Discussing the Ideas

1. What pattern did you find for sequence **A**?

2. Explain the patterns for sequences **B** through **E**.

3. The terms of the sequence are the numbers of the sequence. We speak of the **first** term, the **second** term, the **third** term, and, in general, the n^{th} term of the sequence.

Term:	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	. . .	n^{th}	. . .
	↓	↓	↓	↓	↓		↓	
Sequence:	1	3	5	7	9	. . .	$2n - 1$	. . .

In the sequence above, the rule for the n^{th} term is $2n - 1$.
Thus, if $n = 4$, then $2 \cdot 4 - 1 = 7$. (The 4^{th} term is 7.)
if $n = 6$, then $2 \cdot 6 - 1 = 11$. (The 6^{th} term is 11.)

A What is 7^{th} term of the sequence?
B What is the 10^{th} term of the sequence?
C What is the 11^{th} term of the sequence?

4. The rule for the n^{th} term of a sequence is $5n - 4$. Use the rule to give three more terms of the sequence.

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	. . .
↓	↓	↓	↓	↓	↓	
$5 \cdot 1 - 4$	$5 \cdot 2 - 4$	$5 \cdot 3 - 4$				
↓	↓	↓				
1 ,	6 ,	11 ,	,	,	,	. . .

Using the Ideas

1. Give the next three numbers for each sequence.

A 1, 3, 5, 7, 9, . . .

B 1, 10, 100, 1000, . . .

C 1, 2, 4, 8, 16, . . .

D 3, 7, 11, 15, 19, . . .

2. Give the 4th, 5th, and 6th terms of each sequence.
Use the rule for the ***n*th term.**

	TERM:							
		↓	↓	↓	↓	↓	↓	↓
SEQUENCES	**A**	4 ,	7 ,	10 ,	▒▒ ,	▒▒ ,	▒▒ , . . .	$3n + 1$ . . .
	B	1 ,	3 ,	6 ,	▒▒ ,	▒▒ ,	▒▒ , . . .	$\frac{n(n+1)}{2}$ . . .
	C	1 ,	4 ,	9 ,	▒▒ ,	▒▒ ,	▒▒ , . . .	n^2 . . .
	D	2 ,	3 ,	6 ,	▒▒ ,	▒▒ ,	▒▒ , . . .	$(n-1)^2 + 2$ . . .

3. Write the first five terms of the sequence whose ***n*th** term is $3n - 1$.

4. Write the first five terms of the sequence whose ***n*th** term is 0 if ***n*** is even and 1 if ***n*** is odd.

★ **5.** What is the rule for the ***n*th** term of each sequence?

A 3, 6, 9, 12, . . .

B 2, 3, 4, 5, . . .

C $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \ldots$

D 1, 4, 9, 16, 25, . . .

★ **6.** The **Fibonacci Sequence**, 1, 1, 2, 3, 5, 8, 13, 21, . . . , is a famous sequence that has many interesting properties.

A Give the next four numbers of the sequence.

B Can you explain the rule for the ***n*th** term of the sequence?

think

The ***n*th** term of a sequence is given by the rule $\frac{n^3 - 5n + 6}{6}$.
Find the first six terms of the sequence.

UTILIZATION

In Exercise 6B, more able students should observe that each term of the sequence after the first two terms is the sum of the two previous terms. The rule for the formation of the Fibonacci sequence is best stated recursively as follows: Let a_n denote the nth term of the Fibonacci sequence. Then

$a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$.

EXTENSION

Enrichment: Ask students to solve these problems concerning the Fibonacci sequence.

1. Find the sum of the first 4 terms of the Fibonacci sequence. Find the sum of the first 5 terms, the first 6 terms, etc. What do you notice? (Answer: The sum of the first n terms is one less than the $(n + 2)$ term.)
2. Divide the terms of the sequence by 4. Form a new sequence of the remainders. Is there a pattern to the remainder? (Answer: 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, . . .)
3. Continue the pattern below. What do you notice?

$1^2 + 1^2 = 2$ $\qquad$ $2^2 + 3^2 = 13$

$1^2 + 2^2 = 5$ $\qquad$ $3^2 + 5^2 = 34$

4. Here is another pattern. Can you continue it?

$2^3 + 1^3 - 1^3 = 8$

$3^3 + 2^3 - 1^3 = 34$

$5^3 + 3^3 - 2^3 =$ ▒▒

$8^3 + 5^3 - 3^3 =$ ▒▒

$13^3 + 8^3 - 5^3 =$ ▒▒

Think Solution

Simply substitute 1 through 6 for n to find 2, 4, 8, 15, 26, and 42.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–6.

Objective

The student can solve problems relating to the sequence of the square or the triangular numbers.

PREPARATION

The two lessons on this page represent optional material on square and triangular numbers. You might have the students work through these two pages either individually or in small groups, according to your students' ability.

UTILIZATION

Since students' previous work in number theory should have provided sound preparation for this lesson, have them begin the exercise set immediately and work through as much of it as they can independently. Of course, your less able students may need some help with the more complicated exercises. When the students have finished the assignment, some interesting discussion may be induced by having several students present their solutions to the class. Be particularly ready to commend any original ideas your students might contribute concerning the patterns displayed in the exercises.

Some generalizations that might be formulated from the exercises are listed below.

A. Each square number is the sum of consecutive odd whole numbers. (Exercises 1–3, page E-82)

B. $1 + 2 + 3 + \ldots + (n - 1) + n + (n - 1) + \ldots + 3 + 2 + 1 = n^2$ (Exercises 4 and 5, page E-82)

C. Every triangular number is a sum of consecutive whole numbers beginning with 1. (Exercises 1–3, page E-83)

D. Each square number greater than 1 is the sum of two consecutive triangular numbers. (Exercise 4, page E-83)

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–5.

SQUARE NUMBERS

1. The sequence 1, 4, 9, . . . , n^2, . . . , is a sequence of **square numbers**. The first three square numbers are represented by the arrays of dots. Give the next four square numbers and draw arrays of dots for them.

$1^2 = 1$ $2^2 = 4$ $3^2 = 9$

2. Study Figures **A** through **C**.

A **B** **C**

$1 = 1^2$ $1 + 3 = 2^2$ $1 + 3 + 5 = 3^2$

Draw square arrays of dots for the next three square numbers and write addition statements for each square.

3. Solve the equations. Try to avoid computation by using the pattern established in Exercise 2.

A $1 + 3 + 5 = n$

B $1 + 3 + 5 + 7 = n$

C $1 + 3 + 5 + 7 + 9 = n$

D $1 + 3 + 5 + 7 + 9 + 11 = n$

E $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = n$

F $1 + 3 + 5 + 7 + \ldots + 95 + 97 + 99 = n$

4. Figures **A** through **E** show that we may think of the square numbers as the following sums:

A **B** **C** **D** **E**

$1 = 1^2$ $1 + 2 + 1 = 2^2$ $1 + 2 + 3 + 2 + 1 = 3^2$

A Write similar addition statements for the numbers pictures in Figures **D** and **E**.

B Write 64 as a sum.

C Write 100 as a sum.

5. Find the square number for each sum.

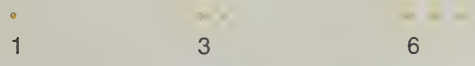
A $1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 + 3 + 2 + 1 =$ ▨.

B $1 + 2 + 3 + 4 + 5 + 6 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ ▨.

C $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ ▨.

Triangular Numbers

1. The sequence of **triangular numbers** is the sequence whose n^{th} term is $\frac{n(n+1)}{2}$. The first three triangular numbers are represented by these arrays of dots.



1 3 6

Give the next four triangular numbers and draw arrays of dots for these numbers.

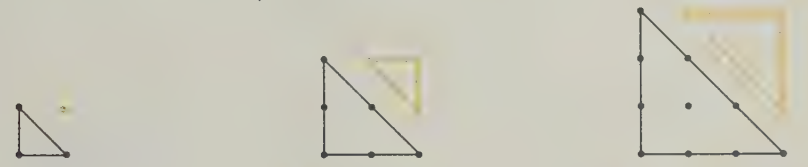
2. Give the next ten triangular numbers for the set $\{1, 3, 6, 10, \ldots\}$.

3. Each triangular number greater than 1 is the sum of consecutive whole numbers beginning with 1.

$$1 \quad , \quad \overbrace{3}^{(1+2)} \quad , \quad \overbrace{6}^{(1+2+3)} \quad , \quad \overbrace{10}^{(1+2+3+4)} \quad , \quad \ldots$$

Represent the next four larger triangular numbers as the sum of consecutive whole numbers.

4. Every square number greater than 1 is the sum of two consecutive triangular numbers. The examples below illustrate this.



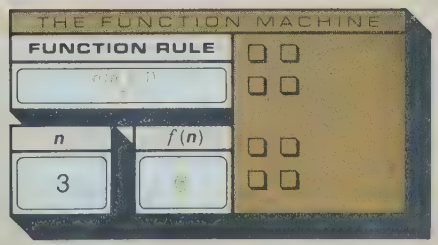
$1 + 3 = 4 = 2^2$ $3 + 6 = 9 = 3^2$ $6 + 10 = 16 = 4^2$

A Express each number 25, 36, and 49 as the sum of two triangular numbers.

B Express 100 as the sum of two triangular numbers.

5. This function machine produces triangular numbers. Find $f(n)$ for each number in the table.

n	3	7	10	20	100	1000
$f(n)$	6					



Answers, Exercises 1 and 2, Page E-82

1. $4^2 = 16$ $5^2 = 25$ $6^2 = 36$ $7^2 = 49$

2. $1 + 3 + 5 + 7 = 4^2$
$1 + 3 + 5 + 7 + 9 = 5^2$
$1 + 3 + 5 + 7 + 9 + 11 = 6^2$

Answers, Exercise 1, Page E-83

10 15 21 28

EXTENSION

Enrichment: Present the following problems for your more able students.

1. Give the next three numbers in this sequence of cubes.
$1^3, 2^3, 3^3, 4^3, 5^3, 6^3, \ldots, n^3, \ldots$
$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
1, 8, 27, ▒, ▒, ▒, . . . (64, 125, 216)
2. Study the following examples and try to describe a pattern:
$1 = 1^3$; $3 + 5 = 8 = 2^3$; $7 + 9 + 11 = 27 = 3^3$
Express the next three cubes as sums of consecutive odd numbers.
($13 + 15 + 17 + 19 = 64 = 4^3$; $21 + 23 + 25 + 27 + 29 = 125 = 5^3$; $31 + 33 + 35 + 37 + 39 + 41 = 216 = 6^3$)
3. Study examples (a), (b), and (c) and then complete the equations.
(a) $1^3 = 1^2$ (b) $1^3 + 2^3 = 9 = 3^2$
(c) $1^3 + 2^3 + 3^3 = 36 = 6^2$
A. $1^3 + 2^3 + 3^3 + 4^3 =$ ▒ = ▒^{2} (100, 10)
B. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 =$ ▒ = ▒^{2} (225, 15)
C. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 =$ ▒ = ▒^{2} (441, 21)

This would also be an appropriate time to suggest that some students undertake Research Project E on page E-87.

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–5.

Objective

The student can solve problems about the sequence whose general term is $3n - 2$.

PREPARATION

This optional lesson relates some ideas of number theory in the set of whole numbers to similar ideas using a proper subset of the whole numbers. If the lesson on sequences on pages E-80 and E-81 was covered, students should be adequately prepared for this lesson.

INVESTIGATION

The Investigation should not be particularly difficult for the students to complete. Students will find that the list of numbers they obtain in the Investigation will be referred to during the Discussion and Utilization stages.

The numbers in the sequence are:
1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58, 61, 64, 67, 70, 73, 76, 79, 82, 85, 88, 91, 94, 97, 100, 103, 106, 109, 112, 115, 118, 121, 124, 127, 130, 133, 136, 139, 142, 145, 148.

DISCUSSION

Discussion Exercise 1 gives the student a method of finding any term in the sequence, while Exercise 2 presents a method of determining whether a given number belongs in the sequence.

Exercise 3 brings out the fact that the product of any two numbers in the sequence is always another number in the sequence. If n_1 and n_2 are any two whole numbers, then $(3n_1 - 2)$ and $(3n_2 - 2)$ are two numbers in the sequence. Then:

$$\begin{aligned}(3n_1 - 2)(3n_2 - 2) &= 3 \cdot 3n_1n_2 - 6n_1 - 6n_2 + 4 \\ &= (9n_1n_2 - 6n_1 - 6n_2 + 6) \\ &\quad - 2 \\ &= 3(3n_1n_2 - 2n_1 - 2n_2 + 3) \\ &\quad - 2\end{aligned}$$

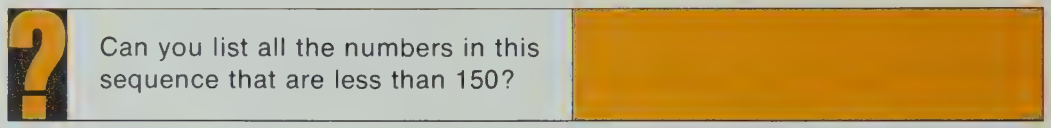
Since the expression in the parentheses is always a whole number, the product has the form $3n - 2$; hence the product of two numbers in S is always another number in S.

A Particular Sequence

Investigating the Ideas

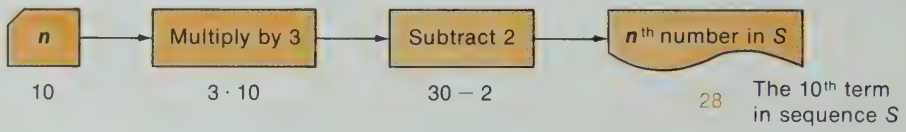
The sequence S given below has some interesting properties that you will study in this lesson.

S: 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, . . . , $3n - 2$, . . .



Discussing the Ideas

1. The flow chart below can help you find any term in sequence S.

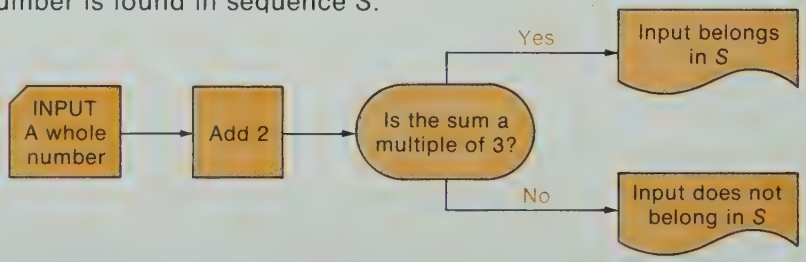


A What is the 16th term in S?

B What is the 20th term in S?

C What is the 100th term in S?

2. This flow chart can help you decide whether a given whole number is found in sequence S.



Use the flow chart to decide which of these numbers are terms in S.

A 70 **B** 116 **C** 213 **D** 997 **E** 4132

3. If you choose any two numbers in S, do you think their product is also a number in S?

Using the Ideas

1. Find the designated term in sequence *S*.

A 8^{th} term **C** 50^{th} term **E** 105^{th} term
B 26^{th} term **D** 82^{nd} term **F** 1000^{th} term

2. Is the given number a term of sequence *S*? Use the flow chart in Discussion Exercise 2 to help you decide.

A 127 **C** 250 **E** 306 **G** 888 **I** 1002
B 168 **D** 481 **F** 762 **H** 1001 **J** 1003

3. Each number below is a number from sequence *S*. List all the factors of the number which are also numbers in *S*.

EXAMPLE: **40** Factors in *S*: {1, 4, 10, 40}

A 28 **C** 10 **E** 70 **G** 25 **I** 52
B 16 **D** 34 **F** 76 **H** 100 **J** 91

★**4.** Suppose we call a number from *S* which is greater than 1 and which has exactly two factors in *S* an **S-prime**. Then 4 is an *S*-prime because 1 and 4 are the only numbers in *S* that are factors of 4.

A Is 10 an *S*-prime? **D** Is 16 an *S*-prime?
B Is 13 an *S*-prime? **E** Is 88 an *S*-prime?
C Is 25 an *S*-prime? **F** Is 46 an *S*-prime?

★**5.** What is the smallest number in *S*, greater than 1, that is an *S*-prime?

★**6.** If a number greater than 1 in *S* is not an *S*-prime, then it can be expressed as a product of *S*-primes.

EXAMPLE: $40 = 4 \cdot 10$ (4 and 10 are *S*-primes.)
$208 = 4 \cdot 4 \cdot 13$ (4 and 13 are *S*-primes.)

Express each number from *S* as a product of primes.

A 15 **B** 88 **C** 70 **D** 52 **E** 49

★**7.** Show that 100 can be expressed as a product of two different sets of *S*-primes.

UTILIZATION

Note in the example given in Exercise 3 that numbers such as 2, 5, and 8 are not listed because they are not numbers in *S*.

Exercise 7 illustrates that in the sequence *S*, there is not a unique factorization of a number into *S*-primes.

$100 = 10 \cdot 10$, and 10 is an *S*-prime.
$100 = 4 \cdot 25$, and both 4 and 25 are *S*-primes.

EXTENSION

Enrichment: Challenge some of your students with the problems below.

A. Use the set of numbers in the sequence *S*. Make and use a sieve similar to the Sieve of Eratosthenes to find the *S*-primes in this sequence.

B. How many square numbers are in the sequence *S*? (*Answer:* All squares which do not have 3 as a factor. {1, 4, 16, 25, 49, 64, 100, 121, 169, . . .})

Assignments

Minimum 1–2. Average 1–3. Maximum 1–7.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

Since the latter part of this module consisted of optional material, you may want to restrict the review to those exercises that are related to the essential material. If so, assign only Exercises 1 through 11.

Additional review exercises are provided on page 24 of the *Workbook*.

Answers, Exercises 1 and 12

1A. {1, 2, 3, 6, 9, 18}
B. {1, 2, 3, 5, 6, 10, 15, 30}
C. {1, 2, 11, 22}
D. {1, 2, 5, 10, 25, 50}
E. {1, 2, 4, 8, 16}
F. {1, 2, 4, 7, 14, 28}
G. {1, 3, 5, 15, 25, 75}
H. {1, 2, 4, 8, 16, 32, 64, 128}

12A. 54, 66, 70, 222
B. 54, 66, 111, 222
C. 54, 66, 222
D. 65, 70

REVIEWING THE IDEAS

1. Give the set of all factors of the following numbers:

EXAMPLE: 24
ANSWER: {1, 2, 3, 4, 6, 8, 12, 24}

A 18 C 22 E 16 G 75
B 30 D 50 F 28 H 128

2. What is the smallest number that has exactly three factors?

3. Only one of the numbers in the set is prime. Which is it?

{33, 27, 49, 87, 51, 91, 78, 97, 57, 39}

4. One of the numbers in the set is a composite number. Which is it?

{71, 37, 17, 67, 47, 69, 23, 31, 41}

5. Give the prime factorization of each number.

A 54 D 81 G 284
B 40 E 210 H 500
C 20 F 220 I 900

6. Which of the following prime numbers have twin primes? If the number has a twin, give both numbers.

A 17 C 43 E 31 G 97
B 23 D 89 F 59 H 71

7. What is the GCF of each pair?

A 16, 24 D 14, 35 G 6, 35
B 20, 25 E 66, 88 H 13, 52
C 9, 21 F 60, 75 I 48, 120

8. Find the GCF of each pair using the Euclidean Algorithm.

A 168, 315 C 1763, 2021
B 629, 851 D 2397, 3337

9. A List the first 10 nonzero multiples of 6.
B List the first 10 nonzero multiples of 10.
C List the first four common multiples of 6 and 10.
D What is the LCM of 6 and 10?

10. What is the LCM of each pair?

A 6, 8 E 12, 15 I $3^2 \cdot 5, 3 \cdot 5^2$
B 5, 8 F 4, 20 J $5^2 \cdot 7, 5 \cdot 7^2$
C 9, 12 G 7, 11 K $2 \cdot 3 \cdot 5, 2^2 \cdot 3$
D 14, 21 H 4, 18 L $7^2 \cdot 11^2, 7^2 \cdot 17$

11. What is the LCM in each group?

A 3, 4, 6 D 3, 5, 7 G 10, 20, 30
B 3, 6, 8 E 4, 6, 9 H 6, 9, 27
C 6, 8, 10 F 3, 4, 5 I 12, 18, 20

12. Which numbers in the set {54, 65, 66, 111, 70, 222} are divisible by

A 2? B 3? C 6? D 5?

13. For each number, tell if it is divisible by 4.

A 316 B 702 C 328 D 114

14. What is a divisibility rule for 9?

15. Which numbers are divisible by 9?

A 727 B 1512 C 6381 D 23 760

16. Give the next four numbers in each sequence.

A 1, 6, 11, 16, 21, . . . B 1, 3, 9, 27, . . .

17. Write the first four numbers of a sequence whose n^{th} term is $3n + 2$.

18. For each number 6, 7, 8, 9, and 10, tell whether it is **square**, **triangular**, or **prime**.

TEST YOURSELF

1. **A** Is 8 a factor of 144?
   B Is 6 a factor of 266?
2. List all the factors of 20.
3. Is 87 a prime number or a composite number?
4. Give the prime factorization of 140.
5. Complete the prime factorization of this number.
   $483 = 23 \cdot \blacksquare \cdot \blacksquare$
6. What is the greatest common factor of 42 and 77?
7. What is the GCF of two numbers that are relatively prime?
8. What is the LCM of 12 and 15?
9. Is 324 divisible by
   A 4? **B** 3? **C** 9? **D** 11?
10. Give the next three terms for each sequence.
    A 2, 5, 8, . . . , $3n - 1$, . . .
    B 2, 5, 10, 17, . . . , $n^2 + 1$, . . .

RESEARCH PROJECTS

A A famous unsolved problem in mathematics, called **Goldbach's Conjecture**, is concerned with whether or not every even number greater than 2 can be expressed as the sum of two primes. For example, $24 = 11 + 13$ and $44 = 31 + 13 = 37 + 7$. Can you show that each even number less than 100 can be expressed as the sum of two prime numbers?

B In reply to a question about the definition of a "friend," Pythagoras said: A friend is "one who is the other I, such are 220 and 284." Mathematically 220 and 284 are examples of **amicable** numbers. The sum of the proper divisors of each number equals the other number. Read about the history of amicable numbers and see how many you can find. (See *Number—The Language of Science* by Tobias Dantzig; New York: Doubleday and Company, 1956.)

C Another famous conjecture about primes is attributed to Edward Waring (1734–1793). He asserted that every odd number greater than 1 which is not a prime can be expressed as the sum of three primes. For example, $9 = 2 + 2 + 5$ and $27 = 3 + 7 + 17$. Show that Waring's conjecture is true for all the odd non-prime numbers less than 100 but greater than 1.

D Many formulas have been developed that give a finite number of primes. Read about some of these formulas and report on them to your class. (See *One, Two, Three, . . . Infinity*, by George Gamow; New York: The Viking Press, 1961; Macmillan.)

E Make a chart of the whole numbers from zero to 100, showing how each number can be classified as odd, even, prime, composite, triangular, square, perfect, cubic, and so forth. Make your chart suitable for bulletin board display.

TEST YOURSELF

Students who covered only the non-optional material should be expected to complete only Exercises 1 through 8.

Note that the correlations between the test items and the module objectives are indicated by annotations on the student book page.

RESEARCH PROJECTS

Students who undertake Project A should observe that there is not a unique set of prime numbers for each sum. Some students may be interested in trying to show all the possible ways an even number can be expressed as the sum of 2 primes.

For students who are interested in Research Project B, you may need to define the term *proper divisors* as the exact divisors of the number excluding the number itself. In the *Journal of Recreational Mathematics* (Vol. 5, Nos. 2, 3, 4, 1972), Elvin T. Lee and Joseph S. Madachy list 1095 known pairs of *amicable numbers*.

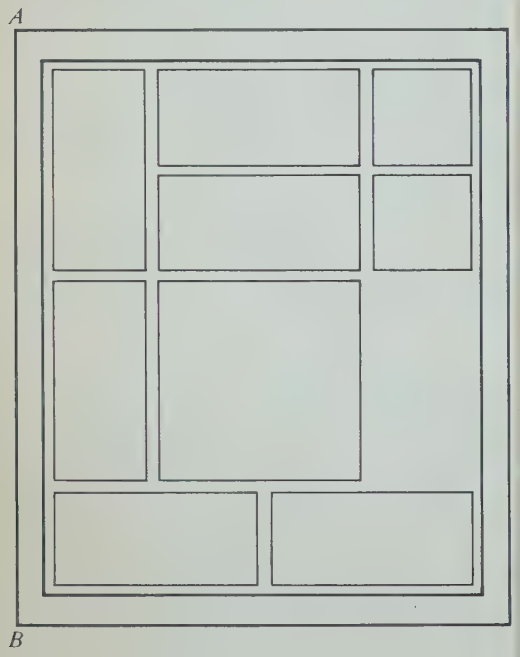
Students who try Project C will discover that there is more than one way to express some odd nonprime numbers as the sum of three prime numbers. For example, $21 = 3 + 5 + 13 = 3 + 7 + 11 = 7 + 7 + 7$.

In reference to Project E, you might want to recommend Philip J. Davis' *The Lore of Large Numbers* (New Haven: Yale University Press, 1961), which contains an interesting classification of the first 100 whole numbers (pages 92 and 93.)

MATHEMATICAL RECREATION

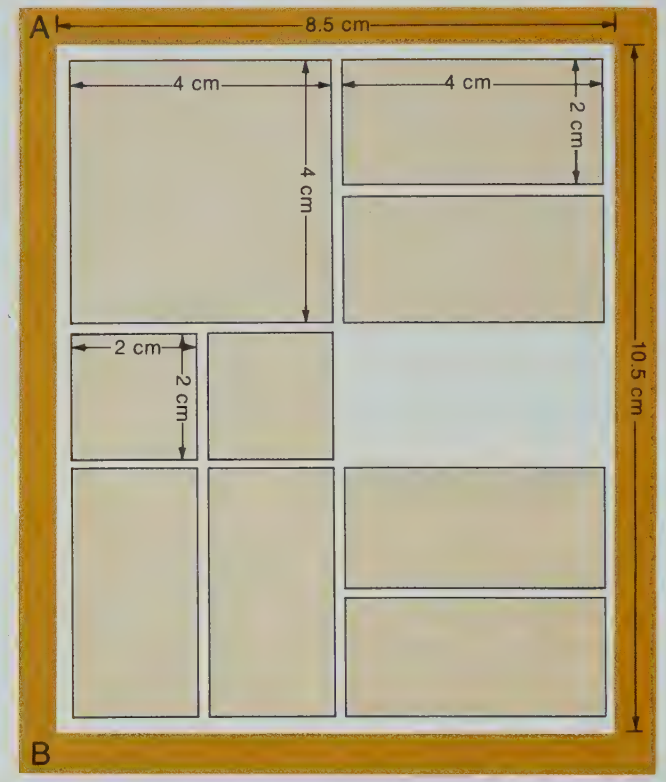
Some students might be permitted to use time in an industrial arts class to make a wooden model of the puzzle. If so, they too should first make a pattern from centimetre graph paper (*Duplicator Masters*, page 83).

This puzzle requires many steps in order to move the large square from corner A to corner B, and will be solved only after considerable trial and error. The key to the solution lies in moving the two small squares around the right side of the large square two times, which will ultimately allow the large square to move to corner B. The diagram below shows the pieces in a position wherefrom a few more moves will place the large square in corner B.



MATHEMATICAL RECREATIONS

A TRANSLATION PUZZLE



Cut out the nine pieces from centimetre graph paper and arrange them inside a cardboard frame as shown. Now try to move the large square from corner A to corner B by sliding the pieces. You cannot jump over a piece, turn a piece, or cover a piece.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A yes; B no **2.** {1, 2, 4, 5, 10, 20} **3.** composite **4.** $2^2 \cdot 5$
5. $483 = 23 \cdot 3 \cdot 7$ **6.** 7 **7.** 1 **8.** 60
9. A yes; B yes; C yes; D no **10.** A 11, 14, 17; B 26, 37, 50

Module 4: Integers

General Objectives

To develop skills in computation using integers.
To solve equations and inequalities using integer replacement sets.
To extend students' experiences with graphing of functions and equations to the set of integers.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
28	**RED** Given an integer, the student can give the opposite of the integer.	E-90, 91		**PD** d-1	
29	**RED** Given two integers, the student can find the sum.	E-92, 93 E-94, 95	**WB** 25 **DM** 19	**PD** d-2, 3 **DS** c-1, 2	**SWM 3** 70-79
30	**RED** Given two integers, the student can find the difference.	E-96, 97	**WB** 26, 27 **DM** 20	**PD** d-4 **DS** c-1, 2	**SWM 3** 80, 81
31	**RED** Given two integers, the student can find the product.	E-100, 101 E-102, 103	**WB** 28 **DM** 21	**DS** c-3, 4	**SWM 3** 82, 83
32	**RED** Given two integers, the student can find the quotient.	E-104, 105	**WB** 29 **DM** 22	**DS** c-3, 4	
33	**RED** Given two integers a and b, the student can state whether $a < b$ or $a > b$.	E-108, 109	**WB** 30	**PD** d-1	
34	**YELLOW** Given an inequality involving integers, the student can find its solution set.	E-110, 111		**PD** d-1	
35	**YELLOW** Given any integer or an equation involving integers, the student can find the absolute value of it.	E-112, 113	**WB** 31 **DM** 24		
36	**YELLOW** Given a simple equation such as $2x + 5 = {}^{-}9$, the student can find its solution.	E-116, 117	**WB** 32		
37	**YELLOW** Given a set of coordinate pairs or a rule for finding them, the student can graph them in the co-ordinate plane.	E-118, 119		**DS** d-1, 2, 3	**SWM 3** 152, 153
38	***GREEN*** *Given a function rule and a set of input numbers, the student can find corresponding output numbers and graph the function.*	E-120, 121	**WB** 33 **DM** 25	**DS** d-1	
39	***GREEN*** *Given a linear equation such as $x + y = 3$ and a set of values for x, the student can find corresponding values for y and graph the equation.*	E-122, 123		**DS** d-1, 2	
	Reviewing the Ideas	E-124	**WB** 34		

MATHEMATICS

There are two ideas involved in the development of the integers as a mathematical system in this module: (1) the integers are necessary in order to describe certain situations in the physical world, and (2) the integers can be developed mathematically apart from any physical considerations.

The physical approach is vital to the development of student understanding of applications of integers. Mathematically, the set of integers is formed by creating the *opposite* of each positive integer and adjoining the set of opposites or negatives to the set of positive integers and zero.

The basic principles for addition and multiplication are assumed: associativity, commutativity, the identity principles, and the distributive principle. Using these principles, the usual algorithms for computing with integers are developed. Although considerable emphasis is placed on computation with integers, complete mastery is not necessary but should be a continuing goal throughout the school year.

The latter part of the module is devoted to work with equations, inequalities, absolute value, functions, and linear equations and their graphs. The material presented here will give students an excellent background for a formal course in algebra. You will note that the approach to such topics as equation solving emphasizes understanding of the problem rather than manipulative skill in algebraic processes. Solving equations and inequalities with finite replacement sets gives students the opportunity to use intuitive methods. When the replacement set is the complete set of integers, emphasis is placed on the role of the basic principles and inverse operations in finding solution sets.

TEACHING THE MODULE

Materials

Graph paper, co-ordinate grid paper.

Vocabulary

absolute value
associative principle
closure
commutative principle
co-ordinate axes
co-ordinates
distributive principle
equation
function
graph
inequality
integer
negative integer
one principle
opposites principle
ordered pair
origin
positive integer
quadrant
replacement set
solution set
zero principle

For students who have studied integers before (Unit D, Module 1), much of the material on addition, subtraction, multiplication, and division of integers will be a review. Such students should be able to improve their computational skills considerably in this module and to cover the remaining topics in the module in greater depth than those students who lack this background.

In teaching computation with integers, both physical models and the basic principles should be employed. Students with weak backgrounds will definitely need to rely heavily on physical models as they work with integers. More able students can be expected to follow the logical arguments involved in the use of the basic principles to justify steps in computation.

The work with equations and graphing can be the most interesting and valuable experience for students. Success in the work with integers will increase the probability of success in the next learning unit, which deals with rational numbers.

Lesson Schedule

The amount of time devoted to this module will vary, depending on how much previous experience your students have had with integers and the amount of material covered in the module. In general, you should plan to allow about 15 to 18 days for adequate coverage.

Evaluation

When evaluating student progress in this module, keep in mind that computational skills with integers should be a continuing objective throughout the year. Complete mastery by all students would be an unrealistic goal. Since the student has been exposed to a large number of ideas in this module, you will need to give careful consideration to the relative importance of the concepts and skills that you are evaluating.

The module review and the Test Yourself quiz should aid the student in preparing for an achievement test for this module.

You should note that, since this is the last module in Unit E, a cumulative review of all modules in the unit is provided at the end of this module. A cumulative achievement test covering the learning unit is included in the test package that is available from the publisher.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 4, pp. 31–35, Creative Publications
Developmental Math Cards, L16–17, Addison-Wesley
Madison Project: *Discovery*, Chapters 2, 4–23, 25–29, 47; *Explorations*, "Variables, Graphs, and Signed Numbers," pp. 18–115 (TT), "Identities, Functions and Derivations," pp. 169–324, Addison-Wesley
Making and Using Graphs and Nomographs, pp. 39–48, Rand McNally
Mathematics: Man's Key to Progress, Book B, pp. 34–40, Rand McNally
Nuffield Project: *Computation and Structure 4* (integers), pp. 22–49; *Graphs Leading to Algebra*, "Co-ordinates" and "Open sentences and graphs," pp. 37–53, Wiley
Sourcebook for Substitutes, pp. 71–72, 80, 94–96, Addison-Wesley

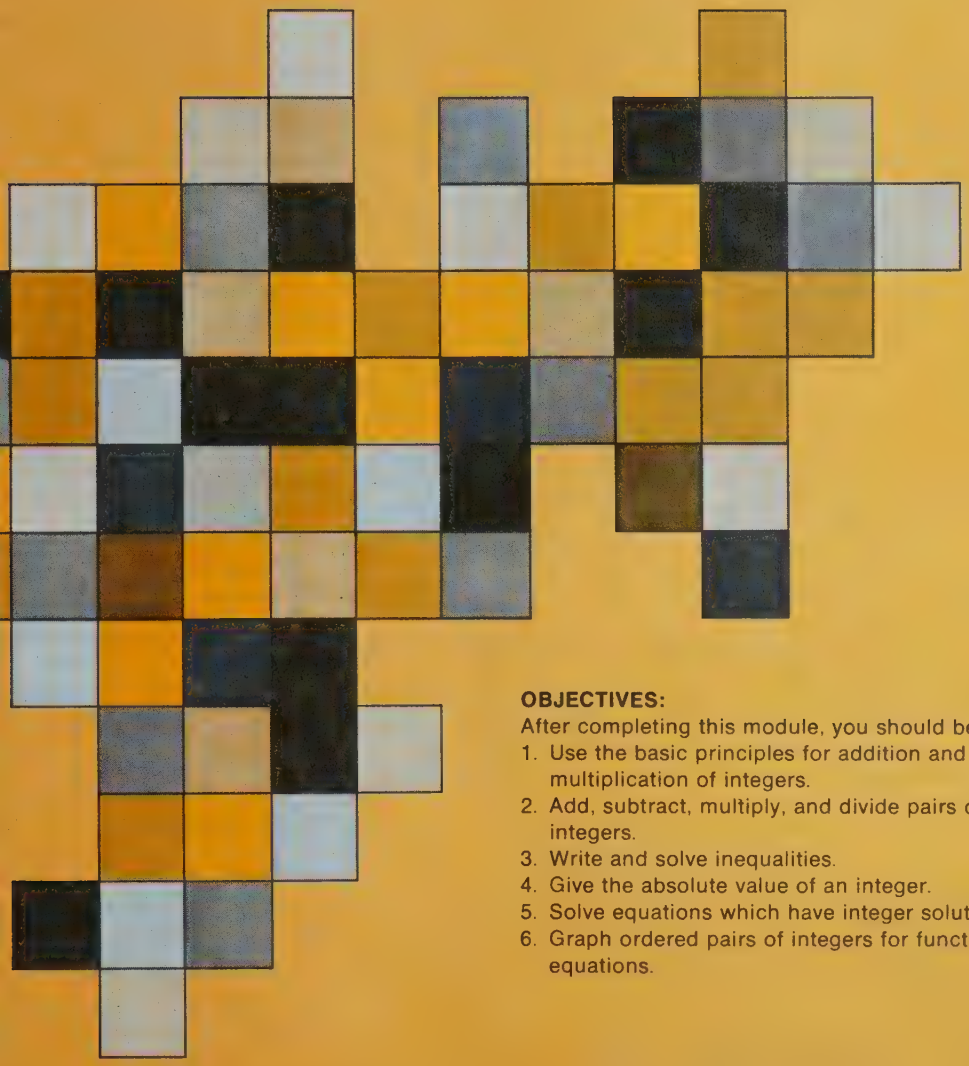
Manipulative Devices

Integers Kit (General Learning Corp.)
Polar Co-ordinate Graph Chart (Sargent-Welch; Yoder Instruments)
Rectangular Graph Chart (Sargent-Welch; Yoder Instruments)

Games and Puzzles

Battleship (retail market)
Counter-Trek (Sigma)
Vectors (Selective Educational Equipment; Sigma)

MODULE 4: *Integers*



OBJECTIVES:

After completing this module, you should be able to:

1. Use the basic principles for addition and multiplication of integers.
2. Add, subtract, multiply, and divide pairs of integers.
3. Write and solve inequalities.
4. Give the absolute value of an integer.
5. Solve equations which have integer solutions.
6. Graph ordered pairs of integers for functions and equations.

Objective

The student can state and represent the opposite of any integer.

PREPARATION

This lesson introduces the set of integers. Since many students will have studied integers in previous mathematics classes, this lesson may be considered primarily a review.

INVESTIGATION

You might wish to use the Investigation as a diagnostic tool to determine how many students remember negative integers and the symbols for them. You might prefer to have the students simply mark the points to the left of the number line rather than punch pin holes.

You might augment the Investigation by presenting a class demonstration. On a strip of paper about 3 cm wide by 50 cm long, draw a line. Mark 0 in the middle of it and then mark points for the positive integers 1 through 9 about 3 cm apart on the line. Now fold the line at 0 and with a hand paper punch, punch holes through the folded paper at each position number. Display the unfolded strip and have the students name the integers for each "hole" to the left of zero.

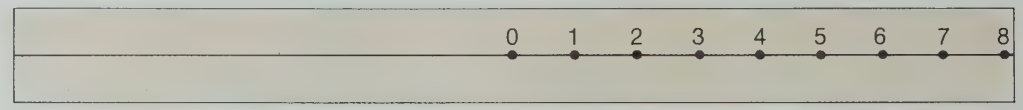
DISCUSSION

While the main goal of the lesson is to introduce the set of integers and their symbols, a secondary goal is to show that integers are useful in describing many quantitative situations in the real world. Exercise 3 suggests a few of these situations. The students may suggest such others as "going set" in a game, countdowns for missile launchings, distances above and below sea level, assets and liabilities, etc.

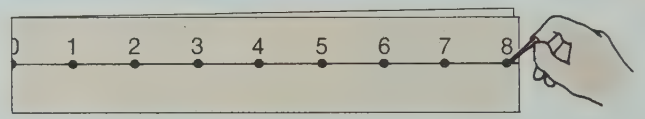
The Set of Integers

Investigating the Ideas

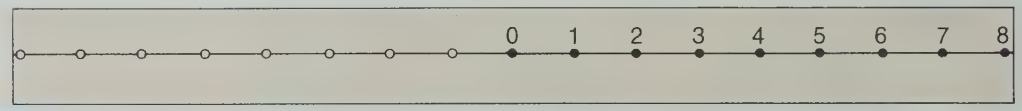
A Draw a number line on a strip of paper.
Start with 0 in the middle of the strip.



B Fold the strip at 0.
Punch pin holes through each point starting at 1.



C Open up your strip.



What numbers go with the points to the left of zero on your number line?

Label all the points you can on the number line.

Discussing the Ideas

1. The **negative integers** are paired with the points opposite each of the **positive integers** 1, 2, 3, 4, 5, ...
The symbol $^{-}1$ (negative 1) denotes the negative integer, which is opposite from 1 on the number line.
What are some other negative integers that you could name on your number line?

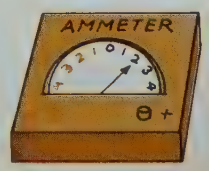
2. The integers 1 and $^{-}1$ are called **opposites**.

   "1 is the opposite of $^{-}1$" and "$^{-}1$ is the opposite of 1."

   What are some other pairs of integers that are opposites of each other?

3. Can you think of some practical uses of negative integers?
The figures may suggest some uses.

Stock	Sales in 100's	Open	High	Low	Close	Net change
Gropex	12	32	35	31	34	$^{+}2$
IS&M	57	49	53	46	47	$^{-}2$

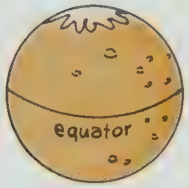


Using the Ideas

1. Give the opposite of each integer.

A	3	**C**	10	**E**	$^{-}4$	**G**	$^{-}11$	**I**	$^{-}89\ 376$	**K**	$^{-}1\ 000\ 000$
B	7	**D**	100	**F**	$^{-}8$	**H**	$^{-}99$	**J**	10 000	**L**	10 000 001

2. The temperature at the equator of the planet Mars in the daytime may be as high as 29°C. At night, the temperature may be as low as 87°C below zero. What integer represents the nighttime temperature?

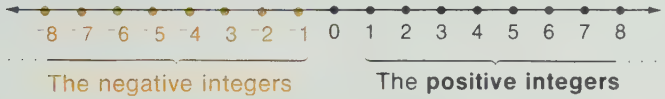


Mars

3. Death Valley, California, is the lowest point in the North American continent. It is 86 metres below sea level. What integer could be used to describe the altitude of Death Valley?

4. For each number given, indicate whether it is a positive integer, a negative integer, or zero. The number line may help you.

The set of integers



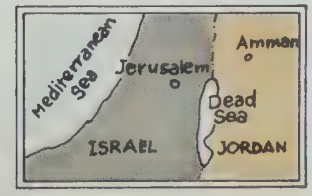
A	7	**B**	$^{-}6$	**C**	$^{-}1$	**D**	$^{-}99$	**E**	27

F The opposite of 4.
G The opposite of 10.
H The opposite of the opposite of 3.
I The opposite of the opposite of $^{-}5$.
J The only integer that is the opposite of itself.

5. Complete the sentences.

A The opposite of a positive integer is a ? integer.
B The opposite of a negative integer is a ? integer.
C If the opposite of a number is a positive integer, the number must be a ? integer.
D If the opposite of a number is a negative integer, the number must be a ? integer.
E ? is the only integer that is neither positive nor negative.

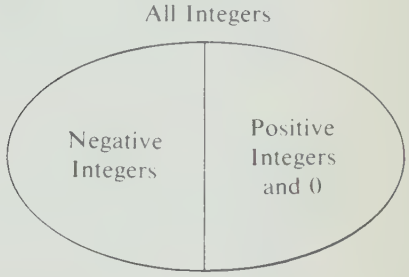
★ **6.** The lowest land point in the world surrounds the Dead Sea. It is 306 metres lower than Death Valley (Exercise 3). What integer gives the altitude of the Dead Sea?



UTILIZATION

The exercise set can be assigned as written work, but most of the exercises are also suitable for oral discussion.

When discussing Exercise 4, point out that the set of integers is the union of the set of whole numbers (non-negative integers) and the set of negative integers (the opposites of the positive integers). A Venn diagram showing how the sets are related may be helpful.



Assignments

Minimum 1–5, oral. Average 1–5, oral. Maximum 1–6, oral.

Objective

The student can find the sum of two integers by thinking about physical examples such as thermometers or the number line.

PREPARATION

This lesson focuses on interpreting addition of integers in terms of physical models.

This concrete approach in addition of integers is especially valuable for students who have had little previous experience with integers. In the following lesson a more formal approach to addition, based on the basic principles, will be presented.

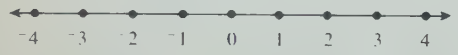
INVESTIGATION

Have the students write the sum for each of the equations given, then discuss each example. Substitute some other numbers for those of the given examples if time permits.

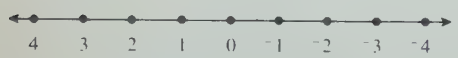
DISCUSSION

When integers are used to describe physical situations, agreements must be made on the use of positive and negative integers. Quite often in problems this agreement is not specified but left for the student. For example, we could label a horizontal number line with integers in either of the two ways shown below.

A.



B.

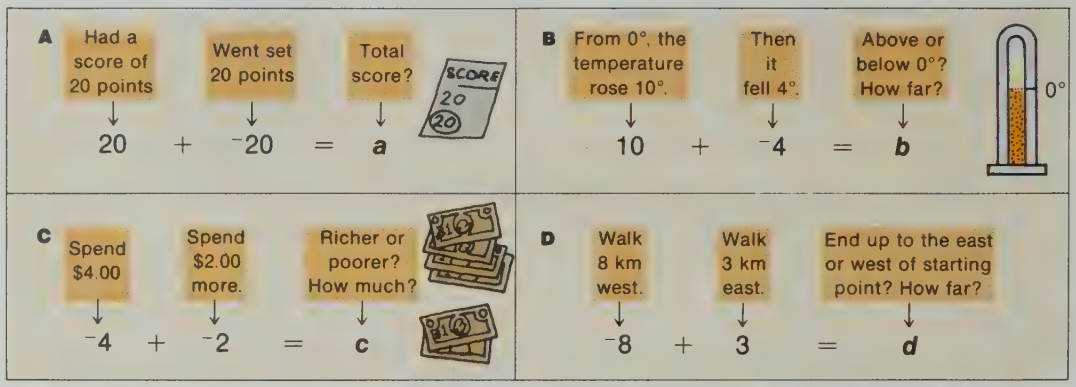


When discussing Exercise 4, point out the agreements made for the use of integers and direct the students in the use of the form you prefer in writing equations for such simple word problems.

Adding Integers

Investigating the Ideas

The examples below may suggest some ways for finding the sum of two integers.

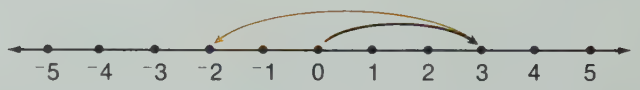


? Can you find the sum for each of the equations above?

Discussing the Ideas

1. When using integers to represent numbers in physical situations, we must agree upon the interpretation of positive and negative integers.
   - **A** What agreement must be made in keeping score in Example **A**?
   - **B** What interpretation is given to positive and negative integers in Example **B**? Example **C**? Example **D**?

2. What addition problem does this number-line picture show?

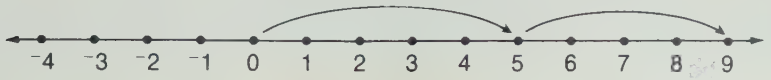


3. What are these sums? Use a number line to help you.
   A $5 + ⁻4$ **B** $⁻3 + 6$ **C** $⁻4 + 4$ **D** $⁻2 + ⁻3$

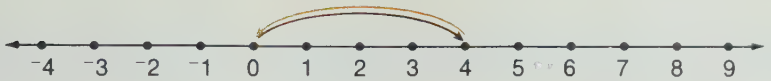
4. Write and solve an addition equation using integers for this problem. Then explain your solution.
   One morning the temperature was 9°C below zero. During the day, the temperature rose 14°C. What was the high temperature?

Using the Ideas

1. On the number line, jumps to the **right** show addition of positive integers. Jumps in the opposite direction (to the left) show addition of negative integers. Study the number lines and solve the equations.

A 

$5 + 4 = n$

B 

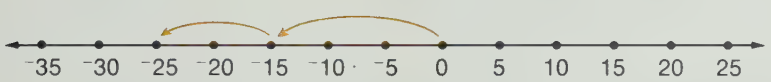
$4 + {}^{-}4 = n$

C

$8 + {}^{-}5 = n$

D

$^{-}50 + 20 = n$

E 

$^{-}15 + {}^{-}10 = n$

2. Draw number lines and label them so that you can show the jumps for each sum. Write an equation for each number line.

A $6 + {}^{-}3$ **B** $^{-}9 + 7$ **C** $^{-}30 + {}^{-}40$ **D** $^{-}15 + 35$ **E** $^{-}800 + 200$

3. Find the sums. Think about number-line jumps.

A $^{-}4 + 1$ **C** $^{-}7 + 4$ **E** $^{-}12 + 10$ **G** $7 + {}^{-}8$ **I** $9 + {}^{-}17$
B $^{-}6 + 5$ **D** $^{-}10 + 7$ **F** $^{-}15 + 8$ **H** $6 + {}^{-}12$ **J** $^{-}21 + 18$

4. Find the sums. Use any method you choose.

A $27 + {}^{-}9$ **D** $24 + (37 + {}^{-}24)$ **G** $(5 + {}^{-}3) + ({}^{-}5 + 3)$
B $^{-}18 + 11$ **E** $(9 + {}^{-}14) + 9$ **H** $(4 + {}^{-}5) + (7 + {}^{-}6)$
C $^{-}16 + {}^{-}17$ **F** $({}^{-}7 + {}^{-}11) + 20$ **I** $(4 + {}^{-}11) + (12 + {}^{-}3)$

5. Write an **addition equation** using integers for each part. Then solve the equation.

A Bill had a score of 50 points in Canasta. The next game he lost 60 points. What was his score then?

B The temperature was 2°C below zero. Then the temperature fell 8° more. What was the new temperature?

C Kathy has $14. She spent $9 for a pair of jeans. How much money did she have then?

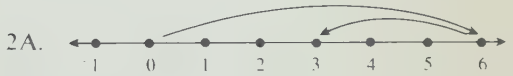
D Jan drove 40 km north. Then she drove back on the same road 62 km south. How far and in what direction was she from her starting point?

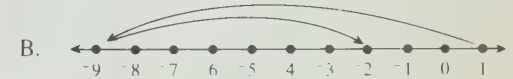
UTILIZATION

Since students will be relying on physical models for the exercises, you should not expect them to be highly efficient in finding sums. At this stage interpretation and understanding of addition is most important. Improvement of computational skills with integers is a long-range, continuing objective.

Plan some time to discuss the interpretation given to the use of integers in the short story problems in Exercise 5.

Answers, Exercise 2

2A. 

B. 

C.

D.

E.

EXTENSION

Workbook page 25 may be assigned for students who would benefit from further practice.

Enrichment: A challenging puzzle involving addition of integers can be made by labelling the sides of 6 wooden cubes with integers as follows:

A. $^{-}2, 7, 8, {}^{-}12, 3, 13$ D. $7, {}^{-}7, 17, {}^{-}3, 2, 12$
B. $8, 13, 18, 3, {}^{-}7, {}^{-}2$ E. $4, {}^{-}11, 14, {}^{-}6, 10, {}^{-}1$
C. $^{-}4, 16, {}^{-}9, 7, 1, 11$ F. $^{-}1, 10, 19, {}^{-}6, 4, 14$

Many problems can be constructed using the cubes. For example, a challenging problem is to arrange the six cubes so that the sum of the integers on the six faces is 1.

Assignments

Minimum 1, 3, 5A, B.
Average 1, 2, 4, 5. Maximum 1–5.

Objective

The student can find the sum of two integers by applying the basic principles for addition of integers.

PREPARATION

Review addition of integers using the number line or other physical models. In this lesson the emphasis is on the role of the basic principles in addition.

DISCUSSION

Have your students read the table of basic principles, noting the examples accompanying each principle. Then work through the Discussion Exercises as a class activity. You need not place much emphasis on the use of the associative and commutative principles as long as your students understand that these principles, working together, make it possible to rearrange addends in any way that is convenient.

Students should realize that the basic principles for addition of integers are the same as those for addition of whole numbers, with the single exception that there is now a new principle, called the "opposites principle." Once students accept this set of basic principles, they can derive methods for finding the sum of any two integers.

Exercises 2, 3, and 4 illustrate the basic cases of addition of integers except for two positive integers.

Keep in mind that students do not need to master the steps illustrated by these problems. Once they understand the role of the basic principles, they should be encouraged to develop their own shortcuts in finding sums of integers.

Answers, Discussion Exercise 2

A. $8 + {}^{-}3 = (5 + 3) + {}^{-}3$ Addition fact
$= 5 + (3 + {}^{-}3)$ Associative principle
$= 5 + 0$ Opposites principle
$= 5$ Zero principle

B. ${}^{-}6 + 9 = {}^{-}6 + (6 + 3)$ Addition fact
$= ({}^{-}6 + 6) + 3$ Associative principle
$= 0 + 3$ Opposites principle
$= 3$ Zero principle

Basic Principles for Addition of Integers

Discussing the Ideas

Since the set of whole numbers is a subset of the integers, the basic principles for addition of integers are much the same as for whole numbers. If we accept these basic principles, we can justify the rules used to add integers.

Name of principle	Statement of principle	Examples
Opposites	The sum of each integer and its opposite is zero.	$12 + {}^{-}12 = 0$ ${}^{-}7 + 7 = 0$
Zero	For each integer a, $a + 0 = a$	$8 + 0 = 8$ ${}^{-}9 + 0 = {}^{-}9$
Commutative	For each pair of integers a and b, $a + b = b + a$	${}^{-}6 + 4 = 4 + {}^{-}6$ ${}^{-}9 + {}^{-}3 = {}^{-}3 + {}^{-}9$
Associative	For each three integers a, b, and c, $(a + b) + c = a + (b + c)$.	$(5 + {}^{-}3) + 3 = 5 + (3 + {}^{-}3)$ ${}^{-}8 + ({}^{-}4 + 4) = ({}^{-}8 + {}^{-}4) + 4$

1. If the sum of two integers is zero, each integer is the **additive inverse** of the other.

${}^{-}7$ is the additive inverse of 7 because $7 + {}^{-}7 = 0$.
7 is also the additive inverse of ${}^{-}7$.

A What is the additive inverse of ${}^{-}11$?

B What is the additive inverse of 0?

C Does every integer have a unique additive inverse?

2. The example shows how the basic principles can be used to find the sum of 10 and ${}^{-}4$. How could you use the principles to show that

A $8 + {}^{-}3 = 5$? **B** ${}^{-}6 + 9 = 3$?

$10 + {}^{-}4 = (6 + 4) + {}^{-}4$ Addition fact
$= 6 + (4 + {}^{-}4)$ Associative (+)
$= 6 + 0$ Opposites
$= 6$ Zero principle

3. Give the missing number in each step of the example. Which basic principles are used?

$(3 + {}^{-}3) + (5 + {}^{-}5) =$ ▒▒ Why?
$(3 + 5) + ({}^{-}3 + {}^{-}5) =$ ▒▒ Why?
$8 + ({}^{-}3 + {}^{-}5) =$ ▒▒ Addition fact
$({}^{-}3 + {}^{-}5) =$ ▒▒ Why?

4. Give a reason for each step of the example.

${}^{-}7 + 5 = ({}^{-}2 + {}^{-}5) + 5$
$= {}^{-}2 + ({}^{-}5 + 5)$
$= {}^{-}2 + 0$
$= {}^{-}2$

Using the Ideas

1. Find the number for ***a***. Then find the sum ***b***.

A $^{-}16 + 16 = a \rightarrow 16 + {}^{-}16 = b$

B $9 + 0 = a \rightarrow 0 + 9 = b$

C $^{-}76 + 76 = a \rightarrow 76 + {}^{-}76 = b$

D $^{-}947 + 947 = a \rightarrow 947 + {}^{-}947 = b$

2. Find the number for ***a***. Then find the sum ***b***.

A $4 + (2 + {}^{-}2) = a \rightarrow 6 + {}^{-}2 = b$

B $5 + 3 + {}^{-}3 = a \rightarrow 8 + {}^{-}3 = b$

C $^{-}4 + {}^{-}2 + 2 = a \rightarrow {}^{-}6 + 2 = b$

D $^{-}5 + {}^{-}4 + 4 = a \rightarrow {}^{-}9 + 4 = b$

E $^{-}2 + ({}^{-}3 + 3) = a \rightarrow {}^{-}5 + 3 = b$

F $^{-}7 + ({}^{-}4 + 4) = a \rightarrow {}^{-}11 + 4 = b$

G $({}^{-}8 + 8) + 7 = a \rightarrow {}^{-}8 + 15 = b$

H $^{-}6 + 6 + 9 = a \rightarrow {}^{-}6 + 15 = b$

I $4 + {}^{-}4 + {}^{-}10 = a \rightarrow 4 + {}^{-}14 = b$

J $7 + {}^{-}7 + {}^{-}5 = a \rightarrow 7 + {}^{-}12 = b$

K $^{-}8 + {}^{-}5 + 5 = a \rightarrow {}^{-}13 + 5 = b$

L $^{-}6 + {}^{-}9 + 7 = a \rightarrow {}^{-}15 + 7 = b$

3. Find the sums.

A $2 + {}^{-}1$	**G** $8 + {}^{-}5$	**M** $^{-}15 + 8$	**S** $12 + {}^{-}25$
B $4 + {}^{-}2$	**H** $10 + {}^{-}6$	**N** $7 + {}^{-}8$	**T** $^{-}53 + 17$
C $^{-}5 + {}^{-}7$	**I** $^{-}9 + {}^{-}8$	**O** $12 + {}^{-}5$	**U** $^{-}64 + 38$
D $^{-}4 + 1$	**J** $^{-}7 + 4$	**P** $^{-}7 + 15$	**V** $^{-}99 + {}^{-}101$
E $^{-}6 + 5$	**K** $^{-}10 + 7$	**Q** $20 + {}^{-}9$	**W** $250 + {}^{-}89$
F $9 + {}^{-}4$	**L** $^{-}12 + 10$	**R** $9 + {}^{-}17$	**X** $^{-}138 + {}^{-}965$

4. Find the sums. (Use any method you choose.)

A $24 + (37 + {}^{-}24)$

B $(9 + {}^{-}14) + {}^{-}9$

C $({}^{-}7 + {}^{-}11) + 20$

D $(5 + {}^{-}3) + ({}^{-}5 + 3)$

E $(120 + {}^{-}11) + (14 + {}^{-}120)$

F $({}^{-}1 + {}^{-}1) + ({}^{-}1 + {}^{-}1) + ({}^{-}1 + 4)$

5. Find the sums.

A 6	**B** $^{-}8$	**C** $^{-}18$	**D** 12
$^{-}7$	$^{-}3$	9	$^{-}21$
E $^{-}3$	**F** 24	**G** 6	**H** $^{-}9$
$^{-}8$	12	15	$^{-}6$
$^{-}2$	$^{-}7$	30	$^{-}1$
9	$^{-}9$	$^{-}20$	$^{-}4$

think

The matches form six small squares. Remove only two of the matches in order to form four squares of the same size.

★ **6.** Solve the equations.

A $^{-}7 + n = 0$

B $5 + (3 + y) = 0$

C $z + {}^{-}4 = {}^{-}4$

D $^{-}3 + (8 + t) = {}^{-}3$

E $12 + (r + {}^{-}12) = 9$

F $({}^{-}2 + {}^{-}3) + 3 = s$

G $(r + 3) + 5 = 0$

H $9 + ({}^{-}3 + w) = {}^{-}1$

I $^{-}6 + ({}^{-}4 + c) = {}^{-}4$

More practice, page S-10, Set 17

UTILIZATION

Exercises 1 and 2 may be discussed orally, if you prefer, and the remainder assigned as written exercises.

Encourage students to develop shortcuts to increase their efficiency in finding sums. The role of the opposites principle should be emphasized. Statements such as "$^{-}3$ is the only integer that adds to 3 to give 0" will help underscore the importance of this principle.

The equations in Exercise 6 should be solved intuitively by finding the missing addends. Students are not expected to use subtraction at this time.

EXTENSION

Extra practice assignments may be made from Supplementary Exercise Set 17 on page S-10 and *Duplicator Masters* page 19.

Think Solution

If, in Figure A, we remove the matches indicated by the arrows labelled *a* and *b*, we will have remaining four squares the same size as those in Figure B.

Figure A

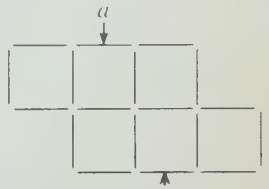
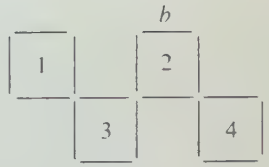


Figure B



Assignments

Minimum 1 and 2, oral; 3.
Average 1 and 2, oral; 3–5.
Maximum 1 and 2, oral; 3–6.

Objective

The student can find the difference of two integers by finding a missing addend.

PREPARATION

Give some oral drill in finding sums of two integers. Then ask a few questions like these:

"What number added to 5 gives a sum of 3?"

"What number added to $^-1$ gives a sum of $^-4$?"

"What number added to 4 gives a sum of $^-2$?"

INVESTIGATION

Have the students write the solution to the 5 equations in the Investigation. Then have them check the answers they have found.

DISCUSSION

Subtraction for the set of integers is defined the same way as subtraction for whole numbers:

If a, b, and c are integers and $a + b = c$, then $a = c - b$.

The integer a is said to be the difference of the integers c and b. Subtraction will not be confusing to students if they think in terms of addition. Thus suggest that the student, given the subtraction problem $5 - {}^-2$, ask himself the question, "What integer will add to $^-2$ to give a sum of 5?"

In Exercise 3 the important mathematical concept of "closure" is introduced. To further illustrate this concept, you might ask whether or not the set of even whole numbers is closed under addition (yes) and under subtraction (no: $4 - 8$ does not result in an even whole number). You might also ask if the set of odd whole numbers is closed under addition (no: $3 + 5$ does not result in an odd whole number) or under subtraction (no: $3 - 15$ does not result in an odd whole number).

Subtraction of Integers

Investigating the Ideas

In each equation, the variable ***n*** represents a missing addend.

	A	A	S
1	$n + 6 = 9$		
2	$n + 6 = 2$		
3	$n + 6 = {}^-2$		
4	$n + {}^-6 = 2$		
5	$n + {}^-6 = {}^-2$		

Can you find the missing addend in these equations?

Discussing the Ideas

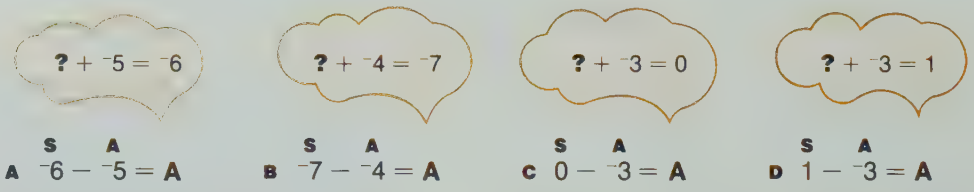
1. When you find a missing addend in an addition problem, you have found the difference for an equivalent subtraction problem.

$$\overset{A\ \ A\ \ S}{n + 6 = 9} \qquad \overset{S\ \ A\ \ A}{9 - 6 = n}$$

   A What subtraction equation can you write for each equation 2 through 5 in the Investigation?

   B What are the solutions to the subtraction equations?

2. Find the differences. The addition equation in the "think cloud" may help you.



A $\overset{S\ \ A}{{}^-6 - {}^-5 = A}$ **B** $\overset{S\ \ A}{{}^-7 - {}^-4 = A}$ **C** $\overset{S\ \ A}{0 - {}^-3 = A}$ **D** $\overset{S\ \ A}{1 - {}^-3 = A}$

3. The set of whole numbers is **closed** under addition because any two whole numbers can be added to give a whole number sum. The set of whole numbers is **not closed** under subtraction because if we select a first whole number and a second whole number, we cannot always subtract the second from the first to get a whole number.

   A Is the set of **integers** closed under addition; that is, is the sum of any two integers always an integer?

   B Is the set of integers closed under subtraction?

Using the Ideas

1. Find the differences by finding the missing addends.

? + 4 = 0	? + ⁻4 = 0	? + 9 = 0	? + ⁻12 = 0
A $0 - 4 = A$	F $0 - {}^{-}4 = A$	K $0 - 9 = A$	P $0 - {}^{-}12 = A$
B $2 - {}^{-}3 = A$	G $0 - 6 = A$	L $1 - 6 = A$	Q $2 - 6 = A$
C $0 - 9 = A$	H $^{-}1 - 9 = A$	M $^{-}2 - 9 = A$	R $^{-}3 - 9 = A$
D $0 - {}^{-}7 = A$	I $^{-}1 - {}^{-}7 = A$	N $^{-}2 - {}^{-}7 = A$	S $^{-}3 - {}^{-}7 = A$
E $^{-}9 - 0 = A$	J $^{-}9 - 1 = A$	O $^{-}9 - {}^{-}1 = A$	T $^{-}9 - {}^{-}9 = A$

2. Find the integer for ***a***. Then find the integer for ***b***.

A $0 - {}^{-}5 = a \rightarrow 1 - {}^{-}5 = b$
B $0 - {}^{-}5 = a \rightarrow 2 - {}^{-}5 = b$
C $0 - {}^{-}10 = a \rightarrow 1 - {}^{-}10 = b$
D $0 - 6 = a \rightarrow 2 - 6 = b$
E $0 - 6 = a \rightarrow 3 - 6 = b$
F $0 - {}^{-}8 = a \rightarrow 3 - {}^{-}8 = b$

3. Think about what the missing addend would be if the sum were zero. Then find the correct missing addend.

A $3 - {}^{-}8 = n$	D $6 - {}^{-}4 = n$	G $^{-}5 - 2 = n$	J $^{-}4 - {}^{-}3 = n$
B $4 - {}^{-}5 = n$	E $^{-}2 - 3 = n$	H $^{-}8 - 3 = n$	K $^{-}9 - {}^{-}8 = n$
C $9 - {}^{-}3 = n$	F $^{-}4 - 6 = n$	I $^{-}6 - {}^{-}2 = n$	L $^{-}10 - {}^{-}4 = n$

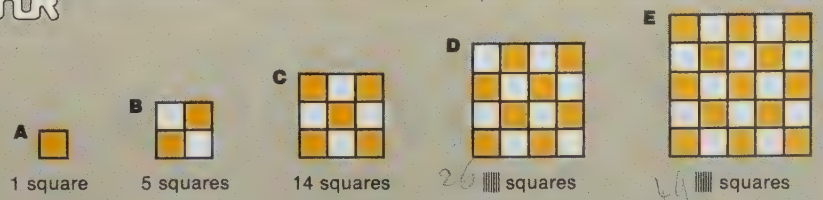
4. Find the differences. (Use any method you choose.)

A $18 - 3$	D $^{-}13 - 8$	G $10 - {}^{-}5$	J $^{-}15 - {}^{-}2$
B $3 - 18$	E $10 - {}^{-}4$	H $^{-}4 - 20$	K $57 - 120$
C $18 - {}^{-}3$	F $^{-}4 - 10$	I $^{-}23 - {}^{-}5$	L $27 - {}^{-}12$

★ **5.** Give the integer for **x**.

A $(9 - {}^{-}4) + {}^{-}3 = x$
B $9 - ({}^{-}4 + {}^{-}3) = x$
C $9 + (4 + {}^{-}3) = x$

think



1. The **total** number of squares in each of the Figures A through C is given. What is the total number of squares in Figures D and E?

2. What is the total number of squares in an 8 by 8-square checkerboard?

More practice, page S-10, Set 18

UTILIZATION

Exercise 1 and 2 might be treated orally and the remainder of the exercise set assigned as written work.

EXTENSION

Supplementary Exercise Set 18 on page S-10 and *Duplicator Masters* page 20 are recommended for students who would benefit from further practice.

Enrichment: Conduct a discussion centered on the following questions:

1. Do you think the commutative principle holds true for subtraction of integers? If not, give an example to prove that you are correct. (*No:* $^{-}2 - {}^{-}6 \neq {}^{-}6 - {}^{-}2$, $4 \neq {}^{-}4$)
2. Do you think the associative principle holds true for subtraction of integers? If not, give an example to prove that you are correct. (*No:* $(6 - 4) - 2 \neq 6 - (4 - 2)$)

Think Solution

The students will find the solving simpler if they make a table and, from the information given in the examples, form their own solution.

Sides	*Area*	*Area and previous total*	*Total squares*
1	1	1	1
2	4	1 + 4	5
3	9	5 + 9	14
4	16	14 + 16	30
5	25	30 + 25	55
6	36	55 + 36	91
7	49	91 + 49	140
8	64	140 + 64	204

The total number of squares T in each example can be computed using the formula

$$T = \frac{n(n+1)(2n+1)}{6}$$

where n represents the number of smallest squares along each edge of the largest square.

Assignments

Minimum 1 and 2, oral; 3; 4.
Average 1 and 2, oral; 3; 4.
Maximum 1 and 2 oral; 3–5.

Objective

The student can find the difference of two integers by adding the opposite of the integer subtracted.

PREPARATION

Conduct an oral review of both addition and subtraction of integers.

This lesson introduces another method of finding differences. It is, of course, mathematically equivalent to the method used in the previous lesson.

INVESTIGATION

The Investigation gives the student the opportunity to discover that each difference can be written as a sum. Students should use any technique they wish in carrying out the Investigation. Have some student or students write two completed equations on the chalkboard to facilitate discussion of the patterns.

DISCUSSION

The proof that for all integers a and b

$$a - b = a + {}^{-}b$$

is not presented in the text. Instead, sufficient examples are presented to convince students of its truth. The proof can be made as follows:

$(a + {}^{-}b) + b =$

$a + ({}^{-}b + b)$ Associative principle

$= a + 0$ Opposites principle

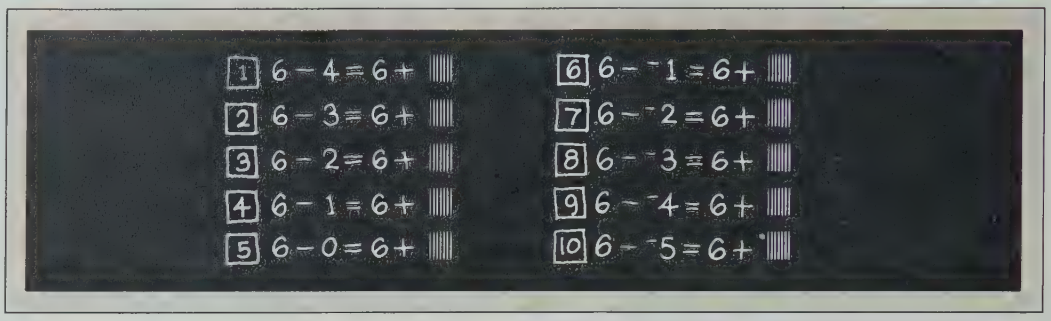
$= a$ Zero principle

Since $(a + {}^{-}b) + a = b$, it follows by the definition of subtraction that $a + {}^{-}b = a - b$.

Adding and Subtracting Integers

Investigating the Ideas

Can you find the missing integer in each equation below?



Discussing the Ideas

1. Did you discover a pattern in the problems above? What is the pattern?

2. Do you agree with the statement below? Give some examples to support your answer.

   "To subtract one integer from another, just add the opposite of the integer that is to be subtracted."

3. You can write every integer subtraction problem as an addition problem.

   EXAMPLE: $^{-}9 - 3 = {}^{-}9 + {}^{-}3 = {}^{-}12$

   How could you write these subtraction problems as addition problems?

   A $3 - 7$ B $8 - {}^{-}1$ C $^{-}9 - {}^{-}3$ D $^{-}7 - 6$

4. Explain or show how to solve each equation. Think about finding missing sums or addends. The replacement set is the set of integers.

   EXAMPLE: $x - {}^{-}2 = {}^{-}3$

   SOLUTION: $^{-}3 + {}^{-}2 = x$, $x = {}^{-}5$

   A $y - 3 = {}^{-}7$ B $r - {}^{-}2 = 6$ C $7 - k = 10$

Using the Ideas

1. Solve the equations. The replacement set is the set of integers.

A $4 - {}^{-}3 = 4 + n$	**E** ${}^{-}2 - 7 = {}^{-}2 + n$	**I** ${}^{-}12 - {}^{-}4 = {}^{-}12 + n$
B $5 - {}^{-}2 = 5 + n$	**F** ${}^{-}7 - 4 = {}^{-}7 + n$	**J** ${}^{-}15 - 6 = n + {}^{-}6$
C $8 - {}^{-}5 = 8 + n$	**G** ${}^{-}1 - 3 = n + {}^{-}3$	**K** $10 - {}^{-}6 = 10 + n$
D ${}^{-}3 - 4 = n + {}^{-}4$	**H** ${}^{-}4 - {}^{-}6 = {}^{-}4 + n$	**L** ${}^{-}20 - {}^{-}4 = {}^{-}20 + n$

2. Rewrite each subtraction problem as an addition problem. Then find the sum.

A $6 - 3$	**D** ${}^{-}10 - 1$	**G** $4 - 14$	**J** ${}^{-}70 - {}^{-}70$
B ${}^{-}2 - 8$	**E** $17 - {}^{-}3$	**H** $0 - 5$	**K** ${}^{-}100 - 43$
C ${}^{-}4 - {}^{-}9$	**F** ${}^{-}8 - 8$	**I** $0 - {}^{-}6$	**L** ${}^{-}201 - {}^{-}179$

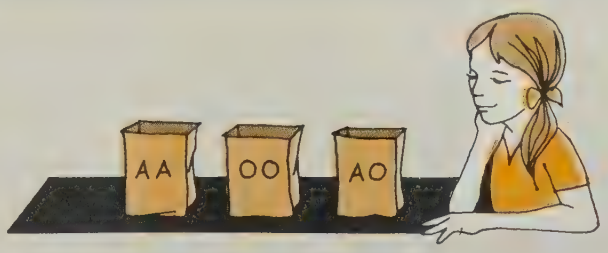
3. Solve the equations. You may find it helpful to replace all subtraction operations with addition.

A $(3 - 7) + {}^{-}1 = r$	**E** $(1 - 2) - 3 = x$	**I** ${}^{-}6 + ({}^{-}3 - 7) = h$
B $({}^{-}2 - {}^{-}1) + 6 = t$	**F** $({}^{-}4 - 4) + 4 = n$	**J** $({}^{-}1 - {}^{-}9) + {}^{-}5 = j$
C $({}^{-}8 + 4) - 2 = s$	**G** $10 - (6 - 4) = b$	**K** $(4 - {}^{-}2) + ({}^{-}3 - 1) = v$
D $3 - ({}^{-}8 + 9) = p$	**H** $({}^{-}1 - {}^{-}1) - {}^{-}1 = q$	**L** $({}^{-}2 - {}^{-}3) - ({}^{-}7 - 4) = m$

4. Solve the equations. The replacement set is the set of integers.

A $n - {}^{-}3 = 7$	**D** $x - {}^{-}7 = 1$	**G** $y - 2 = {}^{-}3$
B $z - 4 = 9$	**E** $r - {}^{-}2 = {}^{-}2$	**H** $t - 10 = 4$
C $k - {}^{-}10 = 13$	**F** ${}^{-}3 - s = 5$	**I** ${}^{-}19 - p = {}^{-}2$

think



In one sack, there are two apples; in another sack, there are two oranges. In a third sack, there are an apple and an orange. **Each sack is incorrectly labelled** *AA*, *OO*, or *AO*. By looking at just one piece of fruit from one of the sacks, you are to tell what is in each sack. How can you do this?

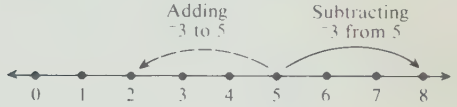
UTILIZATION

Assign the exercises according to the needs of your students. There are 45 responses required for the exercises on this page.

EXTENSION

Additional practice exercises are provided on page 27 of the *Workbook*.

Remedial: If some students have difficulty with subtraction of integers, you might suggest a number line interpretation of subtraction. You can use number line pictures such as the ones below:



Point out that when subtracting ⁻3 from 5, we jump in the direction opposite to the direction we jump when adding ⁻3 to 5.

When subtracting 5 from ⁻2, we jump in the direction opposite to the direction we jump when adding 5 to ⁻2.

Think Solution

Pick a fruit from the sack labelled *AO*. If the fruit is an orange, then the correct label for the sack will be *OO*. That is, since *AO* is an incorrect label, the correct label must be *OO*. (If an apple were picked from the sack, instead of an orange, then the correct label would be *AA*.) The remaining two sacks are also labelled incorrectly; thus the correct label for *OO* is *AA*, and the correct label for *AA* is *AO*.

Assignments

Minimum 1A–D, 2A–F, 3A–D.
Average 1A–H, 2A–I, 3A–H.
Maximum 1–4.

Objective

The student can find the product of two integers by interpreting physical models for multiplication.

PREPARATION

Multiplication of integers is treated in a manner similar to that used for addition: First, multiplication is developed from a few carefully chosen physical examples that should be familiar to students. Then, in the following lesson, a more formal approach, which relies on the basic principles for multiplication of integers, is used.

INVESTIGATION

Students will need only a short time to complete the Investigation. It is essential, however, that they properly interpret the graphic presentation before they try the exercises.

DISCUSSION

Display a table on the overhead projector or chalkboard like the one below. Fill out the table using examples A, B, and C from the Investigation. Then give other examples, such as D and E in the chart below, for students to interpret and solve.

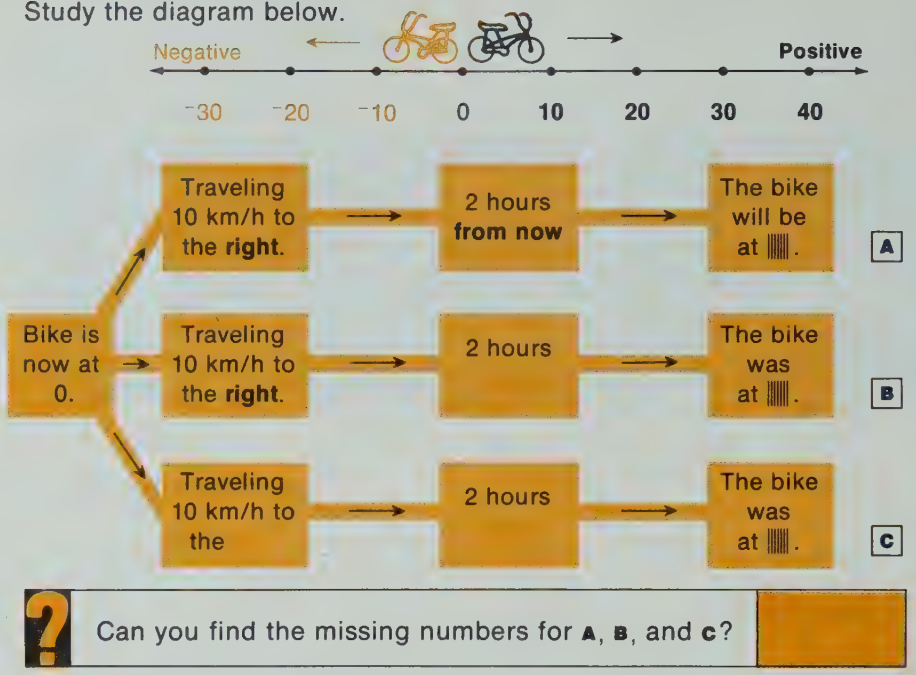
	Speed (km/h)		Time (h)		Distance (km)	
	Right (+)	Left (−)	Hours from Now (+)	Hours Ago (−)	Right (+)	Left (−)
A	10		2		20	
B	10			−2		−20
C		−10		−2	20	
D		−6	5			
E		−5		−4		

Some students may quickly discover that there are shortcuts for multiplying two integers. You need not discourage them from using such methods, but there is little need for formal presentation of the shortcuts at this time.

Multiplication of Integers

Investigating the Ideas

Study the diagram below.



? Can you find the missing numbers for A, B, and C?

Discussing the Ideas

1. The examples above suggest a model for multiplication of integers. Some agreements must be made in the use of the diagram above.

   1. Speeds to the **right** are represented by positive integers.
   2. Speeds to the left are represented by negative integers.
   3. Hours **from now** are represented by positive integers.
   4. Hours ago are represented by negative integers.

   Example A could be illustrated by the multiplication equation $2 \cdot 10 = 20$.

   A What multiplication equation is represented by B?

   B What is the multiplication equation suggested by C?

2. Suppose the bike is now at 0 and traveling 10 km/h to the left.

   A Where will it be 2 hours from now?

   B What multiplication equation does this suggest?

Using the Ideas

1.

Think about trips along the number line starting at zero. Complete the table below. Part **A** has been completed for you as an example.

	SPEED	DIRECTION	TIME	FINAL POINT	MULTIPLICATION EQUATION
A	5 km/h	left	3 hours from now	⁻15	⁻5 · 3 = ⁻15
B	5 km/h	right	3 hours ago		
C	5 km/h	left	3 hours ago		
D	10 km/h	left	4 hours from now		
E	10 km/h	right	4 hours ago		
F	10 km/h	left	4 hours ago		
G	15 km/h	right	2 hours from now		
H	15 km/h	left	4 hours ago		
I	12 km/h	right	5 hours ago		

2. Find the products. Use the number line model if necessary.

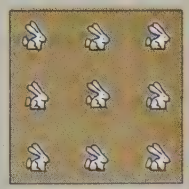
A $3 \cdot {}^{-}4$	**D** ${}^{-}5 \cdot {}^{-}2$	**G** $10 \cdot {}^{-}7$	**J** $9 \cdot {}^{-}10$
B $2 \cdot {}^{-}8$	**E** $6 \cdot {}^{-}4$	**H** ${}^{-}2 \cdot {}^{-}9$	**K** ${}^{-}8 \cdot 5$
C ${}^{-}4 \cdot 2$	**F** ${}^{-}8 \cdot {}^{-}2$	**I** $7 \cdot {}^{-}7$	**L** ${}^{-}1 \cdot 10$

3. Find the products. Look for patterns.

A $4 \cdot 3 = P$	**H** ${}^{-}4 \cdot 3 = P$
B $4 \cdot 2 = P$	**I** ${}^{-}4 \cdot 2 = P$
C $4 \cdot 1 = P$	**J** ${}^{-}4 \cdot 1 = P$
D $4 \cdot 0 = P$	**K** ${}^{-}4 \cdot 0 = P$
E $4 \cdot {}^{-}1 = P$	**L** ${}^{-}4 \cdot {}^{-}1 = P$
F $4 \cdot {}^{-}2 = P$	**M** ${}^{-}4 \cdot {}^{-}2 = P$
G $4 \cdot {}^{-}3 = P$	**N** ${}^{-}4 \cdot {}^{-}3 = P$

think

Here is a picture of nine rabbits in a pen. Copy this picture and then draw two more squares so that each rabbit will be in a separate pen.



UTILIZATION

Plan to allow time for a discussion of Exercise 1. Give students the opportunity to describe the interpretations they have given to each integer used in the multiplication equation.

EXTENSION

Remedial: Ask the students to find the product P.

A. $3 \cdot {}^{-}6 = P$	G. ${}^{-}3 \cdot {}^{-}6 = P$
B. $2 \cdot {}^{-}6 = P$	H. ${}^{-}4 \cdot {}^{-}6 = P$
C. $1 \cdot {}^{-}6 = P$	I. ${}^{-}5 \cdot {}^{-}6 = P$
D. $0 \cdot {}^{-}6 = P$	J. ${}^{-}6 \cdot {}^{-}6 = P$
E. ${}^{-}1 \cdot {}^{-}6 = P$	K. ${}^{-}7 \cdot {}^{-}6 = P$
F. ${}^{-}2 \cdot {}^{-}6 = P$	L. ${}^{-}8 \cdot {}^{-}6 = P$

Enrichment: Provide equations like these for students to solve.

A. $8 \cdot {}^{-}3 = x$
B. $(2 + {}^{-}7) \cdot 4 = y$
C. $(2 \cdot {}^{-}3) \cdot {}^{-}2 = z$
D. $(3 - 5) \cdot 7 = t$
E. $(2 \cdot {}^{-}6) + {}^{-}7 = r$
F. $2 \cdot ({}^{-}6 + {}^{-}7) = s$
G. $({}^{-}1 \cdot {}^{-}1) \cdot {}^{-}1 + 1 = n$
H. $({}^{-}3 + 7) \cdot {}^{-}4 = k$
I. $(6 \cdot {}^{-}2) + (6 \cdot {}^{-}3) = j$
J. $({}^{-}5 \cdot {}^{-}4) + ({}^{-}5 \cdot 4) = a$
K. $({}^{-}8 \cdot {}^{-}8) + ({}^{-}8 \cdot 7) = b$
L. ${}^{-}2 \cdot {}^{-}2 \cdot {}^{-}2 \cdot {}^{-}2 = c$

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–3.

Objective

The student can identify and use the basic principles for multiplication.

PREPARATION

The previous lesson presented multiplication of integers in a fairly intuitive manner. This lesson uses a more formal mathematical approach based on acceptance of the axioms or basic principles for multiplication of integers.

DISCUSSION

Have the students read through the statements of the basic principles for multiplication. Emphasize that the only way these principles differ from those accepted for whole numbers is that we agree that they hold for negative integers as well. Once we accept these principles, the basic rules for multiplying integers are fixed.

As with the work on addition of integers, you need not stress the various steps for rearranging factors once you are sure that students understand that the commutative and associative principles, working together, allow them to rearrange factors in any way that is convenient.

Exercise 4 illustrates how the product of a negative integer and a positive integer can be found using the basic principles to justify each step. Exercise 5 illustrates the case of multiplication of two negative integers.

Students are not expected to show every step in finding products, as in Exercises 4 and 5. Rather, the examples should help students understand that their shortcut methods for multiplying two integers are justified.

Basic Principles for Multiplication of Integers

Discussing the Ideas

We accept the same principles for multiplication of integers as for multiplication of whole numbers. Using the principles, we can justify the products which were found by using physical models in the previous lesson.

THE BASIC PRINCIPLES

One Principle
For each integer a, $a \cdot 1 = a$

Commutative Principle
For each pair of integers a and b,
$a \cdot b = b \cdot a$

Associative Principle
For each three integers a, b, and c,
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive Principle
For each three integers a, b, and c,
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

1. A Which principle can be used to help you solve these equations?

$^-5 \cdot 1 = r$ $\quad$ $^-73 \cdot t = {}^-73$

B What are the solutions?

2. If $^-2 \cdot 8 = {}^-16$, what is $8 \cdot {}^-2$? What principle is involved?

3. A What is the integer for m?

B What is the integer for n?

C What principle do these two equations suggest?

$(^-12 \cdot 4) \cdot 3 = {}^-12 \cdot (4 \cdot m)$

$(^-6 \cdot n) \cdot {}^-7 = {}^-6 \cdot (3 \cdot {}^-7)$

4. The example at the right illustrates how the distributive principle can be used to find the product $^-2 \cdot 6$.

Give the missing number for each step.

$2 + {}^-2 = \blacksquare$

$6 \cdot (2 + {}^-2) = \blacksquare$

$(6 \cdot 2) + (6 \cdot {}^-2) = \blacksquare$

$12 + 6 \cdot {}^-2 = \blacksquare$

$6 \cdot {}^-2 = \blacksquare$

$^-2 \cdot 6 = \blacksquare$

This is the number that adds to 12 and gives zero.

5. Give the missing number for each step in finding the product $^-6 \cdot {}^-2$.

$6 + {}^-6 = \blacksquare$

$^-2 \cdot (6 + {}^-6) = \blacksquare$

$(^-2 \cdot 6) + (^-2 \cdot {}^-6) = \blacksquare$

$^-12 + {}^-2 \cdot {}^-6 = \blacksquare$

$^-2 \cdot {}^-6 = \blacksquare$

$^-6 \cdot {}^-2 = \blacksquare$

This is the number that adds to $^-12$ and gives zero.

Using the Ideas

1. Use the one principle and find the integer for ***a***. Use the commutative principle and find the integer for ***b***.

A $19 \cdot 1 = a \rightarrow 1 \cdot 19 = b$

B $^{-}8 \cdot 1 = a \rightarrow 1 \cdot {}^{-}8 = b$

C $^{-}24 \cdot 1 = a \rightarrow 1 \cdot {}^{-}24 = b$

D $86 \cdot 1 = a \rightarrow 1 \cdot 86 = b$

2. Give the integer for ***a***. Then give the integer for ***b***.

A $^{-}6 + {}^{-}6 = a \rightarrow 2 \cdot {}^{-}6 = b$

B $^{-}9 + {}^{-}9 = a \rightarrow 2 \cdot {}^{-}9 = b$

C $^{-}18 + {}^{-}18 = a \rightarrow 2 \cdot {}^{-}18 = b$

D $^{-}1 + {}^{-}1 = a \rightarrow 2 \cdot {}^{-}1 = b$

E $^{-}63 + {}^{-}63 = a \rightarrow 2 \cdot {}^{-}63 = b$

F $^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 = a \rightarrow 4 \cdot {}^{-}3 = b$

3. Give the integer for ***a***. Use the commutative principle and give the integer for ***b***.

A $2 \cdot {}^{-}8 = a \rightarrow {}^{-}8 \cdot 2 = b$

B $3 \cdot {}^{-}1 = a \rightarrow {}^{-}1 \cdot 3 = b$

C $2 \cdot {}^{-}42 = a \rightarrow {}^{-}42 \cdot 2 = b$

D $3 \cdot {}^{-}9 = a \rightarrow {}^{-}9 \cdot 3 = b$

E $4 \cdot {}^{-}96 = a \rightarrow {}^{-}96 \cdot 4 = b$

F $7 \cdot {}^{-}1 = a \rightarrow {}^{-}1 \cdot 7 = b$

G $6 \cdot {}^{-}8 = a \rightarrow {}^{-}8 \cdot 6 = b$

H $5 \cdot {}^{-}72 = a \rightarrow {}^{-}72 \cdot 5 = b$

4. Copy the equations, giving the correct number for each ▥. Each equation can be completed by thinking about the equation above it and a basic principle or multiplication fact.

A

$3 + {}^{-}3 = ▥$
$4 \cdot (3 + {}^{-}3) = ▥$
$(4 \cdot 3) + (4 \cdot {}^{-}3) = ▥$
$12 + 4 \cdot {}^{-}3 = ▥$
$4 \cdot {}^{-}3 = ▥$

This number adds to 12 to give zero. It must be what integer?

B

$5 + {}^{-}5 = ▥$
$2(5 + {}^{-}5) = ▥$
$(2 \cdot 5) + (2 \cdot {}^{-}5) = ▥$
$10 + 2 \cdot {}^{-}5 = ▥$
$2 \cdot {}^{-}5 = ▥$

C

$4 + {}^{-}4 = ▥$
$0 \cdot (4 + {}^{-}4) = ▥$
$(0 \cdot 4) + (0 \cdot {}^{-}4) = ▥$
$0 + 0 \cdot {}^{-}4 = ▥$
$0 \cdot {}^{-}4 = ▥$

5. Find the products.

A $8 \cdot {}^{-}9$

B $^{-}7 \cdot 6$

C $^{-}5 \cdot {}^{-}9$

D $^{-}17 \cdot 0$

E $^{-}23 \cdot 7$

F $8 \cdot {}^{-}56$

G $^{-}7 \cdot {}^{-}98$

H $^{-}30 \cdot {}^{-}40$

I $27 \cdot {}^{-}36$

J $^{-}42 \cdot 58$

K $^{-}29 \cdot {}^{-}97$

L $0 \cdot {}^{-}864$

M $^{-}92 \cdot 10$

N $100 \cdot {}^{-}48$

O $^{-}62 \cdot {}^{-}1000$

P $^{-}100 \cdot {}^{-}100$

Q $({}^{-}15 \cdot 13) \cdot ({}^{-}6 \cdot 0)$

R $^{-}5 \cdot {}^{-}5 \cdot {}^{-}5$

S $4 \cdot {}^{-}4 \cdot 4 \cdot {}^{-}4$

T $^{-}9 \cdot ({}^{-}4 \cdot {}^{-}10)$

6. **A** The product of a positive integer and a negative integer is a ? integer.

B The product of a negative integer and a negative integer is a ? integer.

C The product of any integer and zero is ? .

More practice, page S-11, Set 19

UTILIZATION

Exercises 1 through 4 illustrate the role of the basic principles in multiplication of integers. The emphasis placed on these exercises should depend on the ability of your students. You may prefer to treat some of them as oral discussion exercises. For very able students, you might want to show the role of the distributive principle in Exercise 2A:

$^{-}6 + {}^{-}6 = {}^{-}12$ Addition of integers

Also:

$^{-}6 + {}^{-}6 = 1 \cdot {}^{-}6 + 1 \cdot {}^{-}6$ One principle

$= (1 + 1) \cdot {}^{-}6$ Distributive principle

$= 2 \cdot {}^{-}6$ Addition fact

Therefore, $^{-}6 + {}^{-}6 = {}^{-}12 = 2 \cdot {}^{-}6$.

Exercise 6 summarizes the usual rules for multiplication of integers.

EXTENSION

Selective assignments from Supplementary Exercise Set 19 on page S-11, *Workbook* page 28, and *Duplicator Masters* page 21 would be beneficial for students needing further practice with the ideas presented in this lesson.

Assignments

Minimum 1–3, oral; 5A–J; 6.
Average 1–4, oral; 5A–O; 6.
Maximum 1–4, oral; 5, 6.

Objective

The student can find the quotient of two integers by finding a missing factor in the related multiplication problem.

PREPARATION

A brief oral review of multiplication of integers should prepare the student for the Investigation.

INVESTIGATION

The Investigation will take only a few minutes. If you wish, direct students to write a multiplication equation for each flow chart. Then use these equations in the Discussion phase of the lesson.

DISCUSSION

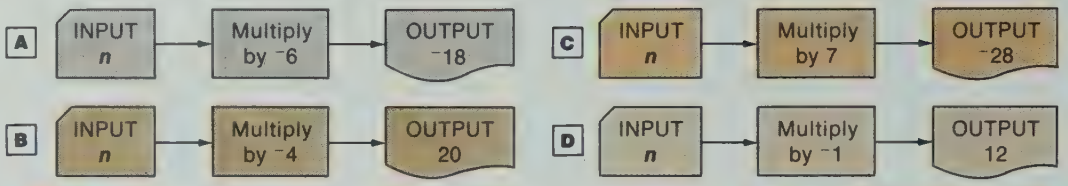
Division of integers causes little difficulty for students who understand multiplication of integers and relate division to multiplication in the manner suggested in Discussion Exercise 2.

Discussion Exercise 5B raises the question of dividing an integer by 0. As was the case with the whole numbers, division by 0 is not defined because either the quotient does not exist in cases such as $\frac{n}{0}$, where $n \neq 0$, or it is not unique for the case of $\frac{0}{0}$.

Division of Integers

Investigating the Ideas

Study the flow charts below.



? Can you find the number for n in each flow chart?

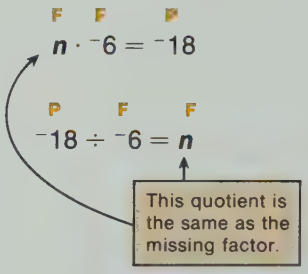
Discussing the Ideas

1. Each flow chart above suggests an equation.

   For flow chart A, you can write $n \cdot {}^-6 = {}^-18$. (F F P)
   What are the equations for flow charts B, C, and D?

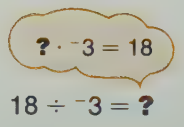
2. When you find the missing factor in a multiplication equation, you have found the **quotient** in an equivalent division equation.

   What is the quotient in the example at the right?



3. Write equivalent division equations for flow charts B, C, and D in the Investigation. Refer to the equations you found in Exercise 1.

4. What is $18 \div {}^-3$? Use the diagram to help you.



5. A What is $0 \div {}^-6$?
   B Can any integer be divided by zero? Explain your answer.

Using the Ideas

1. Find the missing factor, **F**, in each equation.

A $^{-}15 \div 5 = F$
B $^{-}15 \div {}^{-}5 = F$
C $^{-}15 \div {}^{-}15 = F$
D $15 \div {}^{-}1 = F$
E $0 \div {}^{-}15 = F$
F $^{-}20 \div 4 = F$
G $^{-}39 \div {}^{-}3 = F$
H $64 \div {}^{-}8 = F$
I $0 \div {}^{-}54 = F$
J $^{-}86 \div {}^{-}1 = F$
K $42 \div {}^{-}6 = F$
L $^{-}24 \div 8 = F$
M $^{-}24 \div ({}^{-}6 \div {}^{-}2) = F$
N $({}^{-}24 \div {}^{-}6) \div {}^{-}2 = F$
O $({}^{-}24 \div {}^{-}2) \div {}^{-}6 = F$

2. Write and solve an equivalent division equation which has n as the quotient. Use part **A** as an example.

A $^{-}8 \cdot n = 32$
$32 \div {}^{-}8 = n$
$n = {}^{-}4$
B $^{-}5 \cdot n = {}^{-}20$
C $9 \cdot n = {}^{-}54$
D $n \cdot {}^{-}6 = 6$
E $7 \cdot n = 21$
F $^{-}12 \cdot n = 60$
G $3 \cdot n = {}^{-}27$
H $n \cdot {}^{-}17 = 17$
I $^{-}8 \cdot n = 0$
J $n \cdot {}^{-}9 = 72$
K $^{-}1 \cdot n = 1$
L $^{-}15 \cdot n = 90$
M $6 \cdot n = {}^{-}144$

3. Instead of writing $^{-}12 \div 3$, we can write $\frac{^{-}12}{3}$.
Find the quotients for the following:

A $\frac{^{-}6}{2}$
B $\frac{18}{^{-}3}$
C $\frac{^{-}4}{^{-}2}$
D $\frac{^{-}28}{7}$
E $\frac{63}{^{-}9}$
F $\frac{175}{^{-}25}$
G $\frac{(15 + {}^{-}7)}{^{-}2}$
H $\frac{12}{(^{-}2 + {}^{-}1)}$
I $\frac{(9 - 3)}{(2 - 4)}$
J $\frac{(^{-}2 - {}^{-}1)}{(^{-}1 - {}^{-}2)}$

4. Complete each statement.

A The quotient of two positive integers is a ? integer.
B The quotient of a positive integer and a negative integer is a ? integer.
C The quotient of two negative integers is a ? integer.

think

The number 1729 is the smallest number that can be expressed in two different ways as the sum of two cubes.
One way is: $1729 = 12^3 + 1^3$
Find another pair of whole numbers for a and b so that $1729 = a^3 + b^3$.

More practice, page S-11, Set 20

UTILIZATION

Exercise 3 introduces another way of writing quotients. Since these quotients are all integers, there is little need to emphasize the fact that we are using fractional notation for quotients.

Upon completion of Exercise 4, it might prove desirable to have some students state in their own words rules for finding the quotient of two integers.

EXTENSION

Additional practice assignments may be selected from Supplementary Exercise Set 20 on page S-11, *Workbook* page 29, and *Duplicator Masters* page 22.

Think Solution

Both a and b must be between 1 and 12. Now, consider the following trial-and-error method. If a is cubed, then the difference, $1729 - a^3$, must be a cube also.

a	$1729 - a^3$	Is remainder a perfect cube?
6	1513	No
8	1217	No
10	729	Yes

Thus, 10^3 is 1000, and the remainder, 729, is 9^3. The solution is $1729 = 10^3 + 9^3$.

Assignments

Minimum 1A–J, 2, 4.
Average 1, 2, 4.
Maximum 1–4.

Objective

The student can perform several operations using integers by means of function machine rules and equations.

PREPARATION

Review the use of the function machine by means of simple rules such as $3 \cdot n$, $n + 4$, and $n \div 2$.

INVESTIGATION

The function machine was used for similar purposes with whole numbers in Module 2 of this learning unit. Thus, the only new idea here is the use of integers with the function machine. You may need to remind some students that the expression $3n + {}^{-}4$ means "multiply n by 3, then add $^{-}4$." You can use parentheses to emphasize this: $(3n) + {}^{-}4$, as in the function rule shown on the function machine.

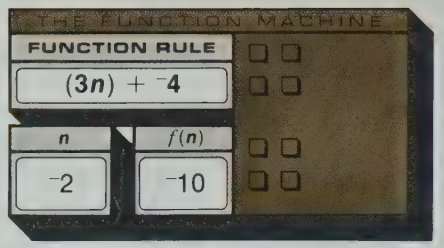
DISCUSSION

Although one of the prime objectives of this section is to provide practice in operations with integers, there are many other benefits to be derived. Note that the student is given a number of interesting function concepts to work with, and at the same time, he is required to compute with the set of integers. Also included are a number of equations which require integer computation while providing valuable experience in equation solving. Since work with both function concepts and equation solving is an extremely important part of mathematical training, a section of this type is doubly valuable.

Combining Integer Operations

Investigating the Ideas

The function machine, with the rule shown, combines the multiplication and addition operations. The function rule can be written as $f(n) = 3n + {}^{-}4$. When we use $^{-}2$ for n, the output number is $(3 \cdot {}^{-}2) + {}^{-}4$ or $^{-}10$. We could write, "$f(^{-}2) = {}^{-}10$."

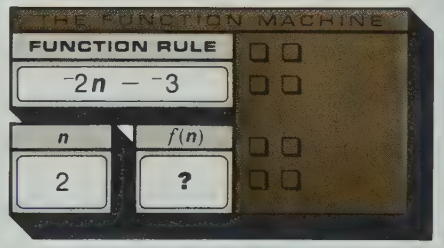


Can you find the output numbers for each of the input numbers in the set $\{^{-}5, ^{-}4, ^{-}3, ^{-}2, ^{-}1, 0, 1, 2, 3, 4, 5\}$?

Make a table to show your answers.

Discussing the Ideas

1. **A** What did you find for $f(^{-}5)$?
   B What is $f(3)$?



2. Suppose a function rule is $^{-}2n - {}^{-}3$.
   A What would you do to find $f(2)$?
   B What is $f(^{-}3)$?
   C What is $f(0)$?

3. What is $f(n)$ for each part of the function table?

	Function Rule $^{-}3n + 6$	
	n	$f(n)$
A	$^{-}4$	
B	6	
C	$^{-}7$	
D	$^{-}2$	
E	0	
F	$^{-}30$	

4. How would you find the number n in these equations?
   A $(^{-}14 - 4) \div 3 = n$
   B $(3 \cdot {}^{-}6) \div {}^{-}9 = n$
   C $(^{-}4 \cdot {}^{-}4) - {}^{-}6 = n$

Using the Ideas

Complete the function tables for Exercises 1 through 6.

1.

Function Rule: $^-2n + ^-3$

	n	$f(n)$
A	4	
B	$^-6$	
C	0	
D	$^-5$	
E	$^-9$	

2.

Function Rule: $(n \div ^-4) - 3$

	n	$f(n)$
A	28	
B	$^-36$	
C	100	
D	$^-44$	
E	$^-12$	

3.

Function Rule: $^-2n - 5$

	n	$f(n)$
A	6	
B	$^-5$	
C	4	
D	$^-10$	
E	$^-50$	

4.

Function Rule: $n^2 + 1$

	n	$f(n)$
A	2	
B	$^-2$	
C	5	
D	$^-5$	
E	1	

5.

Function Rule: n^3

	n	$f(n)$
A	2	
B	$^-1$	
C	4	
D	$^-3$	
E		$^-8$

6.

Function Rule: $(^-2)^n$

	n	$f(n)$
A	1	
B	2	
C	6	
D		$^-8$
E		$^-32$

7. Solve the equations. The replacement set is the set of integers.

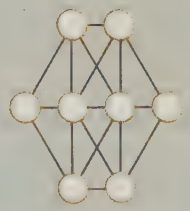
A $(^-5 + 8) - ^-2 = n$
B $(9 \cdot ^-6) + ^-6 = n$
C $(72 \div ^-8) \cdot ^-8 = n$
D $(^-16 + ^-8) - ^-8 = n$
E $(^-7 - ^-4) \cdot ^-9 = n$
F $(^-9 \cdot ^-6) \div ^-6 = n$
G $(25 - ^-4) + ^-4 = n$
H $(^-4 \cdot 9) - 10 = n$
I $(29 + ^-8) \div ^-7 = n$

★ **8.** Solve the equations. The replacement set is the set of integers.

A $(t + 8) + ^-10 = ^-7$
B $(n + 84) - 85 = ^-93$
C $(^-1b) - 4 = ^-10$
D $^-4x + ^-17 = ^-1$
E $(t + ^-7) + ^-7 = 0$
F $(^-5 + 5) + ^-1 = v$
G $2a + 3 = ^-5$

think

Copy the diagram and arrange the numbers 1 through 8 in the circles so that no two consecutive integers are in circles that are directly connected by lines.



UTILIZATION

In Exercises 7 and 8, no formal rules for solving the equations need be given. Students are expected to use intuitive methods for finding the solutions.

EXTENSION

Additional practice exercises are provided on page 23 of the *Duplicator Masters*.

Enrichment: Explain to the students that each function rule in the function tables below has the form $f(n) = (a \cdot n) + b$; that is, each function multiplies the input numbers by some integer a and then adds b. Ask students to find a and b for the function rule and then give the missing numbers in the table. (For your convenience, answers are shown in parentheses.)

1.

function rule: $(a \cdot n) + b$

n	$f(n)$	
2	6	
4	10	
10	22	
100		(202)
$^-50$		($^-98$)

($a = 2$, $b = 2$)

2.

function rule: $(a \cdot n) + b$

n	$f(n)$	
0	4	
5	29	
10	54	
$^-8$		($^-36$)
$^-1$		($^-1$)

($a = 5$, $b = 4$)

3.

function rule: $(a \cdot n) + b$

n	$f(n)$	
1	7	
3	19	
5	31	
$^-4$		($^-23$)
$^-12$		($^-71$)

($a = 6$, $b = 1$)

4.

function rule: $(a \cdot n) + b$

n	$f(n)$	
12	40	
25	79	
$^-10$		($^-26$)
$^-1$		(1)
0		(4)

($a = 3$, $b = 4$)

Assignments

Minimum 1–3, 7A–E.
Average 1–4, 7. Maximum 1–8.

Objective

Given two different integers, the student can decide which integer is the larger and write an inequality for the two integers.

PREPARATION

Comparison of integers is developed in this lesson in two ways:

(1) Through implicit use of the formal definition of "less than" or "greater than," which is as follows:

If a and b are integers, $a < b$ if and only if $a + p = b$, where p is a positive integer.

(2) Through an intuitive comparison based upon representation of the integers as points on the integer number line.

INVESTIGATION

In the equations in the Investigation, the student must determine the missing addend x and determine in which cases this addend is a positive integer. If it is positive, the other addend is less than the sum; if not, the other addend is greater than the sum.

DISCUSSION

Discussion Exercises 1 and 2 focus on the ideas of the Investigation. You may want to present the formal definition of "less than" given in the Preparation section to some classes.

Exercise 3 employs the more intuitive device of relative positions on the number line for comparing two integers. However, it should be noted that, mathematically, the number line comparison has some shortcomings. First, we must know the order of the integers before we can represent the integers as points on a line. Second, the number line must be depicted as a horizontal line with its right half reserved for the positive integers before we can say that the larger of two integers lies to the right of the smaller integer on the number line. Such prerequisites do not apply to the formal definition of the relation "less than" for the integers.

Inequalities

Investigating the Ideas

Study the equations below. Can you solve them?

A	$^{-}4 + x = 3$	D	$^{-}7 + x = {}^{-}7$	G	$^{-}1 + x = 1$
B	$^{-}2 + x = {}^{-}5$	E	$4 + x = 10$	H	$^{-}9 + x = {}^{-}12$
C	$6 + x = {}^{-}2$	F	$8 + x = 1$	I	$^{-}10 + x = {}^{-}3$

? In which equations does **x** represent a positive integer?

Discussing the Ideas

1. In equation **A**, you probably found that **x** was the *positive integer* 7. Because **x** is positive, we say that **$^{-}4$ is less than 3** and write this as $^{-}4 < 3$.

   A $^{-}4 + x = 3$

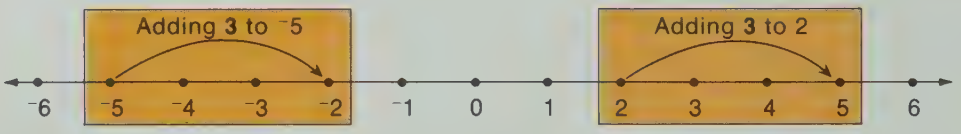
   $^{-}4 + 7 = 3$

   A **positive** integer

   Since $^{-}4 < 3$, we also say that **3 is greater than $^{-}4$** and write this as $3 > {}^{-}4$.

   Compare the numbers in the other equations above in this manner.

2. The number line can help you decide which of two integers is the greater. Study the examples.



   A What integer must be added to 2 to give 5?

   B Is $2 < 5$ or is $5 < 2$?

   C What integer must be added to $^{-}5$ to give $^{-}2$?

   D Is $^{-}5 < {}^{-}2$ or is $^{-}2 < {}^{-}5$?

3. Complete the statements.

   A If ***a*** and ***b*** are any integers and $a < b$, then ***a*** is to the ? (**left**, **right**) of ***b*** on the number line.

   B If $b > a$, then ***b*** is to the ? (**left**, **right**) of ***a*** on the number line.

Using the Ideas

1. Give the integer for each ▨.

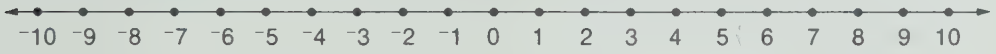
- **A** $4 > 2$ because the positive integer ▨ must be added to 2 to give 4.
- **B** $^{-}5 < ^{-}1$ because the positive integer ▨ must be added to $^{-}5$ to give $^{-}1$.
- **C** $^{-}7 < 0$ because the positive integer ▨ must be added to $^{-}7$ to give 0.
- **D** $^{-}1 > ^{-}99$ because the positive integer ▨ must be added to $^{-}99$ to give $^{-}1$.

2. Write the correct symbol ($<$ or $>$) for each ▨.

A $9 \square 0$	**D** $^{-}5 \square ^{-}1$	**G** $^{-}8 \square ^{-}10$	**J** $^{-}100 \square 100$
B $1 \square 0$	**E** $4 \square ^{-}4$	**H** $^{-}15 \square ^{-}20$	**K** $^{-}2 \square ^{-}98$
C $0 \square ^{-}1$	**F** $^{-}2 \square ^{-}1$	**I** $^{-}75 \square ^{-}72$	**L** $4 \square ^{-}84$

3. Answer the questions about the set S of integers shown on the number line:

$$S = \{0, 1, ^{-}1, 2, ^{-}2, \ldots, 9, ^{-}9, 10, ^{-}10\}$$



- **A** What is the smallest integer in S?
- **B** What is the largest integer in S?
- **C** List all integers in S that are less than 0.
- **D** List all integers between 3 and $^{-}3$.
- **E** List all integers less than $^{-}8$.
- **F** List the integers that are both greater than $^{-}4$ and less than 0.
- **G** What is the greatest integer that is less than $^{-}2$?
- **H** List the integers that belong to one or the other or both of these sets: the integers less than $^{-}7$ or the integers greater than 7.

4.

- **A** Any negative integer is ? than any positive integer.
- **B** Zero is ? than any negative integer.
- **C** One integer is either greater than, equal to, or ? than another integer.
- **D** If a first integer is greater than a second, and the second integer is greater than a third, then the first integer is ? than the third.
- **E** If a first integer is less than a second, and the second integer is less than a third, then the first integer is ? than the third.

think

This is an **integer magic square**. Complete the square so that the sum in each row, column, and diagonal is the same number.

6			3
	1	0	
$^{-}1$	$^{-}3$	$^{-}4$	
	4		$^{-}9$

UTILIZATION

If the ideas of inequalities are readily grasped by your students, you may find that an oral discussion of the exercises on this page suffices and you can proceed to the next lesson, which concerns solution sets for inequalities. For slower students, the lesson needs careful treatment at a more deliberate pace.

EXTENSION

Workbook page 30 provides additional practice exercises that would be appropriate for use with this lesson.

Enrichment: Challenge your students with the problems below. (Answers are given in parentheses.)

If x and y are integers, which integer is larger if you know that:

1. $x - y = 4$ (x)
2. $x + 5 = y$ (y)
3. $y - x = ^{-}3$ (x)
4. $x - y = ^{-}6$ (y)
5. $y + ^{-}3 = x$ (y)
6. $x - 4 = y$ (x)

Think Solution

The sum of the numbers in the completed diagonal is $^{-}6$. When this sum has been found, the number in the lower left corner can be found to be $^{-}6$. Using addition and subtraction, the numbers for the remaining squares can readily be determined.

Assignments

Minimum 1, 4, oral; 2.
Average 1, 3, 4, oral; 2.
Maximum 1–4, oral.

Objective

Given inequalities and a specific replacement set, the student can find the solution set.

PREPARATION

Review the concepts of "less than" and "greater than." Provide some oral practice by asking questions such as: "Which is greater, $^-7$ or $^-2$?" "Which is larger, 5 or $^-5$?" "What is the smallest integer greater than $^-99$?"

INVESTIGATION

Students have worked with inequalities and solution sets using positive integers in Module 2 of this learning unit. Thus, they should have little difficulty extending these ideas to solution sets involving negative integers.

In the Investigation, the student is to select the proper subset of S that is described by each of the given statements. The algebraic description of these sets is introduced in the Discussion phase of the lesson.

DISCUSSION

When discussing Exercises 2 and 3, you may want to use other examples to illustrate how *union* or *intersection of sets* can be used to determine solution sets of inequalities such as "$^-2 < x < 5$ ($^-2 < x$ and $x < 5$)" and "$x < ^-6$ or $x > 3$." However, you should expect students to be able to use the correct mathematical interpretation of "and" and "or" in the open sentences to find the solution sets for the exercises.

Solution Sets for Inequalities

Investigating the Ideas

Each statement below refers to a subset of the set of integers S shown on the number line below.

$$S = \{^-10, ^-9, ^-8, \ldots, 0, \ldots, 8, 9, 10\}$$

How many of the subsets of S described below can you find?

Record the elements in each set.

- **A** The integers that are less than $^-5$.
- **B** The integers that are greater than 4.
- **C** The integers between $^-4$ and 3.
- **D** The integers that are either less than $^-7$ or greater than 6.
- **E** The integers that when multiplied by $^-2$ give a product greater than 10.
- **F** The integers that add to 3 to give a sum that is less than $^-5$.

Discussing the Ideas

1. Sentence A can be shown by the *inequality* $x < ^-5$
   Using S as the replacement set, the **solution set** for the inequality is $\{^-10, ^-9, ^-8, ^-7, ^-6\}$
   - **A** What inequality could be written for B?
   - **B** What is the solution set?

2. For C, we can write $x > ^-4$ and $x < 3$ or $^-4 < x < 3$
   - **A** What is the solution set for $x > ^-4$?
   - **B** What is the solution set for $x < 3$?
   - **C** What is the intersection of the sets found in parts **A** and **B**?

3. For D, we can write $x < ^-7$ or $x > 6$
   - **A** What is the solution set for $x < ^-7$?
   - **B** What is the solution set for $x > 6$?
   - **C** Is the solution set for D the union or the intersection of these two sets?

4. **A** Describe how you found the solution set for E.
   B Describe how you found the solution set for F.

Using the Ideas

1. Find the solution set for each part in the table.
The replacement set is $\{^-4, ^-3, ^-2, ^-1, 0, 1, 2, 3, 4\}$.
Parts **C** and **D** are partially completed.

	INEQUALITY	SOLUTION SET
A	$x > ^-3$	
B	$x < 2$	
C	$x > ^-3$ and $x < 2$	$\{^-2, ^-1, 0, 1, 2, 3, 4\}$ $\{^-4, ^-3, ^-2, ^-1, 0, 1\} =$
D	$x < ^-2$ or $x > 3$	$\{^-3, ^-4\} \cup \{4\} =$
E	$x + 3 > 0$	
F	$x + ^-2 < 0$	

2. Find the solution set for the following sentences.
The replacement set is $T = \{^-3; ^-2, ^-1, 0, 1, 2, 3, 4, 5\}$.

A $x < 2$
B $x < ^-1$
C $x > ^-3$
D $x < 0$
E $x + ^-1 < 0$
F $x + 3 < 3$
G $x - 2 > ^-2$
H $^-2 \cdot x > 0$
I $x \div ^-1 < 2$
J $x < ^-1$ and $x > ^-3$
K $x > ^-1$ and $x < 3$
L $x > ^-1$ and $x < 2$
M $x < ^-2$ or $x > 2$
N $x < ^-1$ or $x > 1$
O $x < 0$ or $x > 0$
P $x < 1$ or $x > ^-1$
Q $(2 \cdot x) + 3 < 0$
R $^-4 - x > ^-1$

3. Find the solution set for each part.
The replacement set is
$S = \{0, 1, ^-1, 2, ^-2, 3, ^-3, \ldots, 10, ^-10\}$.

A $x > 7$
B $y < ^-7$
C $a + 2 < ^-7$
D $x + 5 > 0$
E $m - 2 < ^-5$
F $2 \cdot t < ^-4$
G $^-3 \cdot n > 10$
H $y \leq ^-6$
I $z \geq 5$
J $y^2 < 5$
K $t^3 < ^-7$
L $s^2 - s > 6$

Think

Three men stopped at a hotel and rented a suite of rooms for \$60. Each man paid \$20. On the way to the rooms, the clerk discovered he had erred and that the suite of rooms was only \$55. He found the men in their rooms and returned the \$5.00. Not knowing how to split the \$5.00, they gave the clerk \$2.00 and kept \$1.00 apiece. Thus, each man paid \$19 for his room or \$57 for all three rooms. The clerk got \$2.00, making a total of \$59. What happened to the other dollar?

UTILIZATION

When assigning the exercises on this page, ask students to note that a different replacement set is given for each exercise.

Presenting selected exercises on the chalkboard for student explanation should prove quite useful to your less able students.

EXTENSION

Enrichment: Ask students to find the solution set for each exercise, given the replacement set $R = \{0, 1, ^-1, 2, ^-2, 3, \ldots, ^-9, 10, ^-10\}$.

1. $t^2 > 5$ $\{3, ^-3, 4, ^-4 \ldots 9, ^-9, 10, ^-10\}$
2. $2x + 3x = 5x$ $\{0, 1, ^-1, \ldots 9, ^-9, 10, ^-10\}$
3. $y^2 < 5$ $\{^-2, ^-1, 0, 1, 2\}$
4. $s^2 - 7s + 12 = 0$ $\{3, 4\}$
5. $(n - 4) \cdot (n + 5) = 0$ $\{^-5, 4\}$
6. $b > 8$ or $b < ^-8$ $\{^-10, ^-9, 9, 10\}$
7. $c > ^-2$ and $c < 2$ $\{^-1, 0, 1\}$
8. $b \cdot (b - 4) = 0$ $\{0, 4\}$
9. $d \cdot (d + 5) = d^2 + 5d$ $\{0, 1, ^-1 \ldots 9, ^-9, 10, ^-10\}$
10. $p^2 - p - 6 = 0$ $\{^-2, 3\}$
11. $r^2 - 1 = 8$ $\{^-3, 3\}$
12. $x < 1$ and $x > ^-1$ $\{0\}$

Think Solution

Nothing actually happened to the extra dollar. This can be explained in the following ways.

(1) Consider the total amount for the three men. The three men paid \$55 for the room, plus a \$2 tip for the clerk. This \$57 added to the \$3 refund gives the original sum, \$60.

(2) It is incorrect to assume that each man paid \$19 for his room. The men actually paid $\frac{\$55}{3}$, or $\$18.33\frac{1}{3}$, apiece for their rooms. Each man contributed $67\frac{2}{3}¢$, tip to the clerk. Each man received \$1 in refund. The sum, $\$18.33\frac{1}{3} + \$.67\frac{2}{3} + 1 = \$20$, the amount each man paid originally.

Assignments

Minimum Alternate parts of 1–3.
Average 1, 2A–L, 3A–G. Maximum 1–3.

Objective

The student can give the absolute value of any integer and can solve simple equations involving absolute value.

PREPARATION

Play a "Guess the Function Rule" game for a few minutes. One student selects a rule. When other students give an input number, the student applies his function rule and tells them the output number. The students attempt to guess the function rule.

INVESTIGATION

Students should have little difficulty in completing the function table. However, it is quite likely that they will not be able to give a simple rule describing the function. Some of your most able students might invent a rule such as "n, if $n \geq 0$; ^-n, if $n < 0$." The goal of the Investigation is for students to discover that the output number is related to the input number and is nonnegative.

DISCUSSION

The concept of *absolute value* is used extensively throughout mathematics, and thus this section is of great importance for the students. The basic idea is quite simple and difficulties with absolute value generally arise only when complicated expressions are involved.

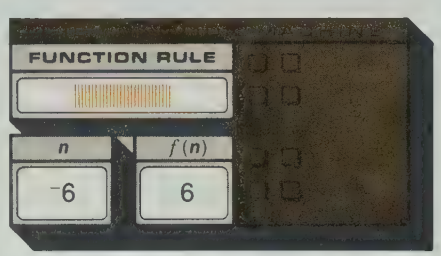
Emphasize that the absolute value of any integer x is either x or the opposite of x, whichever is nonnegative. Note that by this definition the absolute value of zero is zero.

Absolute Value

Investigating the Ideas

Function table A shows the input-output pairs for an interesting function rule.

Can you discover the function rule and complete function table B?



A

Function Rule	
n	$f(n)$
$^-6$	6
4	4
$^-5$	5
0	0
17	17
$^-12$	12

B

Function Rule	
n	$f(n)$
$^-4$	
5	
$^-7$	
13	
$^-10$	
$^-30$	

Discussing the Ideas

1. **A** If the input number n is a positive integer, what is $f(n)$?
   B If the input number n is a negative integer, what is $f(n)$?
   C If the input number is 0, what is $f(n)$?

2. The function rule illustrates the **absolute value** function. In this function, the output number is either the input number or its opposite, whichever is nonnegative. Instead of using $f(n)$ to denote the output number, it is standard practice to use the symbol $|n|$. Thus, $|^-7| = 7$.
   We say, "The absolute value of $^-7$ is 7."
   What is the absolute value of each of these numbers?
   A $^-4$ **B** 10 **C** 0 **D** $^-11$ **E** $^-1$ **F** 15

3. What number is represented by each of the following:
   A $|^-2|$ **B** $|8|$ **C** $|^-23|$ **D** $|^-14|$ **E** $|100|$

4. Explain how to find the integer for n in each part.
   A $|^-4 + 2| = n$ **B** $|2 \cdot {}^-3| = n$ **C** $|^-4 \cdot {}^-2| = n$ **D** $|6 - 8| = n$

Using the Ideas

1. Copy and complete the function tables.

A

Function Rule: $\lvert n \rvert$	
n	$f(n)$
$^-8$	
$^-2$	
20	
53	
$^-3$	
16	

B

Function Rule: n	
n	$f(n)$
$^-3 + {}^-1$	
$^-3 - {}^-1$	
$^-3 + 1$	
$3 \cdot {}^-1$	
$^-3 - 1$	
$^-3 \cdot 2$	

C

Function Rule: $\lvert n \rvert$	
n	$f(n)$
$2 + {}^-3$	
$^-2 + 3$	
$^-2 \cdot 3$	
$^-2 \div 1$	
$^-2 \div {}^-1$	
$^-2 \cdot {}^-3$	

2. What integer is represented by each of the following?

A $|^-5|$ **B** $|5|$ **C** $|^-18|$ **D** $|2|$ **E** $|^-47|$ **F** $|7 - 3|$ **G** $|3 - 7|$ **H** $|^-9 + 6|$ **I** $|^-4 + {}^-2|$ **J** $|^-5 \cdot 4|$

3. Solve the equations.

A $x = |^-5|$ **B** $|6| = a$ **C** $|^-2 + {}^-3| = b$ **D** $n = |0 - 5|$ **E** $|^-4 \cdot 2| = v$ **F** $s = |^-24 + 5|$ **G** $t = |15 - 6|$ **H** $|8 \div {}^-2| = y$

4. Give all solutions for each equation.

EXAMPLE: $|x| = 6$

SOLUTION: Since $|6| = 6$ and $|^-6| = 6$, then $x = 6$ or $x = {}^-6$.

A $|x| = 3$ **B** $|x| = 8$ **C** $|x| = 4$ **D** $|x| = 10$ **E** $|x^2| = 25$ **F** $|x + 1| = 4$ **G** $|x - 1| = 3$ **H** $|^-3 \cdot x| = 9$

★ **5.** Give the integer for each ▒.

A $|x| = 7$ and $x < 0 \rightarrow x =$ ▒

B $|x| = 5$ and $x > 0 \rightarrow x =$ ▒

C $|2x| = 6$ and $x < 0 \rightarrow x =$ ▒

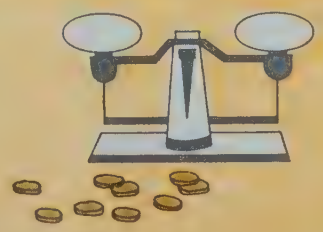
D $|x| = 2$ and $x > 0 \rightarrow x =$ ▒

E $|x| = 2$ and $x < 0 \rightarrow x =$ ▒

F $|x| = 13$ and $x < 0 \rightarrow x =$ ▒

Think

You have nine coins and a pan balance scale. The coins are alike in appearance, but you know that just one of them is counterfeit and lighter than or heavier than the others, but you do not know which. In three weighings, how could you find the bad coin and determine whether it is a heavier or a lighter coin?



UTILIZATION

Exercise 4 presents equations involving absolute value which have two solutions. You may want to discuss this type of problem before making an assignment. Questions such as this may be helpful: "I am thinking of a number whose absolute value is 4. What is the number?" (Answer: 4 or $^-4$.)

Solution, Exercise 4H

If $|^-3x| = 9$, then $^-3x = 9$ or $^-3x = {}^-9$. Thus, $x = {}^-3$ or $x = 3$.

EXTENSION

Appropriate additional practice exercises are provided on *Workbook* page 31 and *Duplicator Masters* page 24.

Think Solution

A. Group the nine coins into sets of three and call these A, B, and C. Begin by weighing set A against set B. If they balance, choose 2 coins from set C. Weigh one coin against another. If they balance, the third coin is counterfeit. If they do not balance, weigh one of the two coins against the third coin. You know that the third is a good coin. You can tell whether the counterfeit coin is heavier or lighter by observation when it is on the scale with a good coin. (This may occur in either the first or second weighing, depending on the choice of coin for the second weighing.)

B. If the first two sets, A and B, do not balance, you know that the remaining three coins (set C) are good. Now weigh one of the questionable sets, say A, against the three good coins (set C). If they balance, set B contains the counterfeit coin, which can be distinguished as discussed above in one more weighing. (You have already determined that set B is either lighter or heavier.) If this second weighing does not result in balancing, then you should have observed whether the set containing the bad coin was either lighter or heavier. Then as discussed before, the lighter or heavier coin can be detected in one more weighing.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

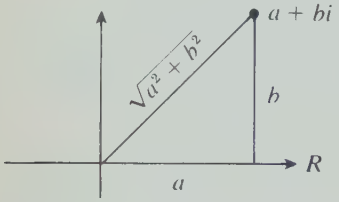
The student can solve problems involving absolute value by using the number line.

PREPARATION

Use examples from the previous lesson to review the concept of absolute value of an integer. In this lesson, absolute value is interpreted in terms of the positive distance of a point from the origin on the number line. This definition is also applicable to rational numbers and real numbers.

The same idea is used in defining absolute value of a complex number $(a + bi)$. Thus, $|a + bi|$ can be interpreted as the positive distance of the point for $a + bi$ from zero.

$$|a + bi| = \sqrt{a^2 + b^2}$$



DISCUSSION

Using the number-line interpretation for absolute value, students can solve some problems which otherwise would require rather sophisticated algebraic techniques. Discuss each of the exercises carefully, supplying other examples as needed by your students.

Absolute Value and the Number Line

Discussing the Ideas

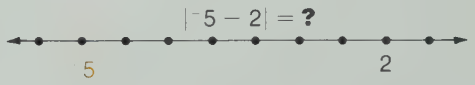
1. Using the number line, you can think of $|n|$ as asking the question, "How far is n from zero on the number line?"

For $|^-6|$, we think $^-6$ is 6 units from zero

A Using the number line, how far is 7 from zero?
B What is the absolute value of 7?
C How far is $^-20$ from zero?
D What is $|^-20|$?

2. The statement $|n| = 9$ means that a certain number n is 9 units from zero on the number line. Give two possible values for n.

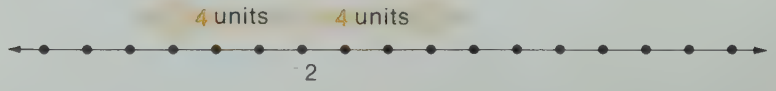
3. You can think of $|a - b|$ as asking the question, "How far is a from b on the number line?"



We think: $^-5$ is 7 units from 2.
We write: $|^-5 - 2| = 7$.
We say: "The absolute value of $^-5 - 2$ is 7."

A How far is $^-8$ from $^-3$ on the number line?
B What is $|^-8 - {}^-3|$?
C How far is 6 from 13 on the number line?
D What is $|6 - 13|$?

4. The equation $|^-2 - x| = 4$ can be solved by thinking, "What numbers are 4 units from $^-2$ on the number line?"



A How many solutions does the equation have?
B What are the solutions?

Using the Ideas

1. Copy the equations, giving the correct integer for each ▒. (It may help you to use the number line.)

A How far is it from $^-8$ to 0? → $|^-8| =$ ▒

B How far is it from 12 to 0? → $|12| =$ ▒

C How far is it between $^-3$ and 5? → $|^-3 - 5| =$ ▒

D How far is it between 5 and $^-7$? → $|5 - {}^-7| =$ ▒

E $|^-15| =$ ▒

F $|2| =$ ▒

G $|49| =$ ▒

H $|^-100| =$ ▒

I $|^-2 - 12| =$ ▒

J $|12 - {}^-2| =$ ▒

K $|^-15 - 0| =$ ▒

L $|9 - 20| =$ ▒

2. Think about the number line and give all the possible numbers for b in each equation.

A $|6 - b| = 6$

B $|4 - b| = 4$

C $|^-2 - b| = 2$

D $|^-9 - b| = 9$

E $|6 - b| = 5$

F $|4 - b| = 3$

G $|^-2 - b| = 1$

H $|^-9 - b| = 10$

3. Give the correct symbol ($<$, $>$, or $=$) for each ▒.

A $|6|$ ▒ $|^-6|$

B $|5 - (^-5)|$ ▒ 5

C $|^-6 - 0|$ ▒ $|^-6|$

D $|^-7|$ ▒ $^-7$

E $|6 - 1|$ ▒ $|6| - |1|$

F $|6 - {}^-1|$ ▒ $|6| - |^-1|$

4. Solve each inequality. The replacement set is $S = \{^-10, ^-9, ^-8, \ldots, 0, \ldots, 8, 9, 10\}$.

A $|x| > 3$ Solution set: $\{9, 10, ^-9, ^-10\}$

B $|x| < 4$

C $|x - 2| < 3$

D $|x - 3| > 5$

think

1. If you were offered a job and could choose between Salary Plans A and B, which would you choose?
2. What would your third-year salary be under Plan A? Under Plan B?
3. What would your total earnings for two years be under Plan A? Under Plan B?
4. If you were an employer choosing between two equally qualified men, would you choose the one who accepted Plan A or the one who accepted Plan B? Explain your choice.

A Start at $8000 per year with a $400 increase on the previous salary at the end of each year

B Start at $4000 per each half year with a $100 increase on the previous salary at the end of each half year

More practice, page S-12, Set 21

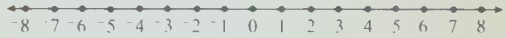
UTILIZATION

Adjust the assignment to the needs and capabilities of your students. Slower students may need additional help on the more difficult parts of the exercises.

EXTENSION

To provide further practice refer students to Supplementary Exercise Set 21 on page S-12.

Remedial: Students should think about the number line and give all the possible numbers for a in each equation below.



1. $|a| = 2$ (Answer: $a = 2$ or $a = ^-2$)
2. $|a| = 7$ (7, $^-7$)
3. $|a| = 0$ (0)
4. $|a| = 3$ (3, $^-3$)
5. $|a| = 4$ (4, $^-4$)
6. $|a - 0| = 2$ (2, $^-2$)
7. $|a - 0| = 7$ (7, $^-7$)
8. $|a - 0| = 0$ (0)
9. $|a - 0| = 4$ (4, $^-4$)
10. $|a - 0| = 8$ (8, $^-8$)
11. $|a - 1| = 2$ (3, $^-1$)
12. $|a - 1| = 7$ (8, $^-6$)
13. $|a - 1| = 0$ (1)
14. $|a - 1| = 6$ (7, $^-5$)
15. $|a - 1| = 4$ (5, $^-3$)

Enrichment: Ask students to find the solution set for each inequality if the replacement set is $\{^-10, ^-9, ^-8, \ldots 0, \ldots 8, 9, 10\}$.

1. $|x| > 4$ $\{^-10, ^-9, ^-8, ^-7, ^-6, ^-5, 5, 6, 7, 8, 9, 10\}$
2. $|x| < 3$ $\{^-2, ^-1, 0, 1, 2\}$
3. $|x - 1| > 5$ $\{7, 8, 9, 10, ^-5, ^-6, ^-7, ^-8, ^-9, ^-10\}$
4. $|x| + ^-3 < 0$ $\{0, 1, 2, ^-1, ^-2\}$
5. $|7 - x| > 5$ $\{1, 0, ^-1, ^-2, \ldots, ^-9, ^-1\}$

Think Solution

4. The man who chose Plan B would be the better choice because of his reasoning power and common sense. He would be worth the higher wage.

Year	Plan A	Plan B
1st	$8000	$8100 ($4000 + $4100)
2nd	$8400	$8500 ($4200 + $4300)
3rd	$8800	$8900 ($4400 + $4500)

Assignments

Minimum 1, 2A–D. Average 1–3.
Maximum 1–4.

Objective

The student can solve linear equations by using logical reasoning and inverse operations.

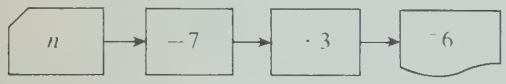
PREPARATION

Students have solved equations similar to the ones in this lesson in Module 2 of this learning unit. However, there the replacement set was restricted to whole numbers, while in this lesson the replacement set is the set of all integers. Plan to review solutions of equations with whole number solutions before starting the Investigation. Use examples similar to those on pages E-46 and 47.

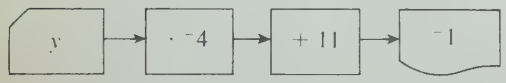
INVESTIGATION

Students are expected to use trial-and-error methods in the Investigation.

If you wish to extend the Investigation, present other flow charts for equations. Then have students try to write equations for the flow charts and solve the equations. Here are two examples.



$(n - 7)3 = -6$ Solution: $n = 5$



$-4y + 11 = -1$ Solution: $y = 3$

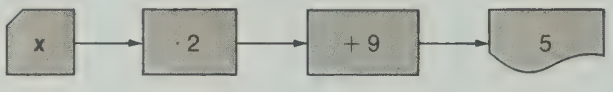
DISCUSSION

Exercise 3 introduces the use of *implication* arrows to indicate "if-then" reasoning. It is important for students to be able to use logical if-then reasoning as it applies to simple algebraic situations, but there is no need for them to become concerned with the more formal aspects of logic.

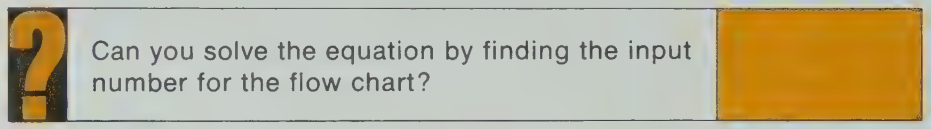
Integers in Equations

Investigating the Ideas

The flow chart below suggests an equation with the set of all integers for the replacement set.



x $2x$ $2x + 9$ $2x + 9 = 5$

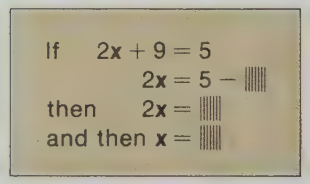


Discussing the Ideas

1. Tony said he thought about an inverse flow chart for the one in the Investigation.

   What would the rest of the flow chart be like?

2. Kitty said she thought about inverse operations to solve the equation. What are the missing numbers in her solution?



3. Instead of writing → If $3x + 2 = 17$,
   then $3x = 15$,
   and then $x = 5$,

   you can indicate the same reasoning by using arrows:

   $$3x + 2 = 17 \Rightarrow 3x = 15 \Rightarrow x = 5.$$

   The arrow $\Rightarrow$ can be read as "implies."

   What are the missing numbers in each step below?

   $$4x - 3 = -15 \Rightarrow 4x = \square \Rightarrow x = \square$$

4. How can you check the number you found for x in Exercise 3 to see if it is a solution for the equation?

Using the Ideas

The replacement set for all equations on this page is the set of integers.

1. Give the correct number for each ▒.

A $x + 4 = 1 \Rightarrow x =$ ▒

B $x + {}^-6 = 2 \Rightarrow x =$ ▒

C $3x = {}^-15 \Rightarrow x =$ ▒

D $x \div {}^-4 = 2 \Rightarrow x =$ ▒

E $x - {}^-7 = {}^-3 \Rightarrow x =$ ▒

F $^-7x = {}^-28 \Rightarrow x =$ ▒

2. Give the missing number in each step.

A If $3x + 2 = {}^-7$, then $3x =$ ▒, and $x =$ ▒.

B If $2x - 4 = {}^-10$, then $2x =$ ▒, and $x =$ ▒.

C If $4x + 5 = 1$, then $4x =$ ▒, and $x =$ ▒.

3. Give the missing integers in each equation.

A $2x + 3 = 13 \Rightarrow 2x =$ ▒ $\Rightarrow x =$ ▒

B $3x + 7 = 1 \Rightarrow 3x =$ ▒ $\Rightarrow x =$ ▒

C $^-2x + 1 = 9 \Rightarrow {}^-2x =$ ▒ $\Rightarrow x =$ ▒

D $2(x + 3) = {}^-6 \Rightarrow x + 3 =$ ▒ $\Rightarrow x =$ ▒

E $^-3(x + 4) = {}^-9 \Rightarrow x + 4 =$ ▒ $\Rightarrow x =$ ▒

4. Solve the equations.

A $6x + {}^-5 = 7$

B $^-4x + 1 = 17$

C $(x + 3) \div 2 = {}^-3$

D $(x \div {}^-5) + 2 = 0$

E $^-2x + {}^-9 = {}^-3$

F $(x + {}^-7) - 2 = {}^-4$

G $4(x + 5) = {}^-12$

H $^-5(x + {}^-7) = 10$

I $^-3(x + 6) = {}^-12$

★ **5.** Copy the equations, replacing each ▒ by the correct numbers. Part **A** is completed for you.

A If $|2x| = 6$, then $2x = 6$, or $2x = {}^-6$, and $x = 3$, or $x = {}^-3$.

B If $|3x| = 12$, then $3x =$ ▒, or $3x =$ ▒, and $x =$ ▒, or $x =$ ▒.

C If $|x + 3| = 5$, then $x + 3 =$ ▒, or $x + 3 =$ ▒, and $x =$ ▒, or $x =$ ▒.

Think

Using a certain code,

the word ▒

becomes ▒

Using the same code, what words do ▒ and ▒ represent?

More practice, page S-12, Set 22

UTILIZATION

The exercise set contains a large number of equations involving integer operations and absolute value of integers. The exercises are designed to give students careful guidance in arriving at logical solutions so that they will develop the ability to think about problems correctly when the work is less explicitly directed. A desirable outcome for the lesson, of course, is for students to gain confidence in this logical approach to equation solving.

EXTENSION

Additional practice assignments may be provided from Supplementary Exercise Set 22 on page S-12 and *Workbook* page 32.

Remedial: Give additional examples involving if-then reasoning in algebra like those in Exercises 1, 2, and 3. Both one-step and two-step examples should be exhibited; for instance:
1. $x + 9 = 5 \Rightarrow x = {}^-4$ 2. $^-2x = 6 \Rightarrow x = {}^-3$
3. $2x + {}^-7 = {}^-3 \Rightarrow 2x = 4$ or $x = 2$

Enrichment: Ask students to give the number for each ▒.
1. $|x| = 5 \Rightarrow |x + 3| = 8$, or $|x + 3| =$ ▒ (2)
2. $|x| = 4 \Rightarrow |2x| =$ ▒ (8)
3. $|x + 1| = 3 \Rightarrow |x + 3| = 5$, or $|x + 3| =$ ▒ (1)
4. $|x| = 5 \Rightarrow |{}^-x| =$ ▒ (5)
5. $|8 - x| = 4 \Rightarrow |x - 8| =$ ▒ (4)

Think Solution

The code is a simple translation which can be discovered by using the code letters for the word MATHEMATICS.

A B C D E F G H I J K L M
↓ ↓ ↓ ↓ ↓
L *M* N *O* P *Q* *R* *S* T *U* *V* *W* X

N O P Q R S T U V W X Y Z
↓ ↓
Y *Z* *A* *B* *C* D E *F* *G* *H* *I* *J* *K*

Thus, P is coded as A, Q as B, R as C, and so on. Using the completed code above, we find that TYEPRPCD represents INTEGERS and LOOTETZY represents ADDITION.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can graph points for ordered pairs of integers.

PREPARATION

Materials: Graph paper or co-ordinate grid paper (*Duplicator Masters*, page 83 or 87).

Your best preparation for this lesson is to review the graphing of whole numbers with your students. You might display on the chalkboard a pair of number rays drawn at right angles to each other, each labelled 0 through 10, and have the students graph several number pairs, such as (5, 4), (7, 2), (0, 3), (6, 0), and so forth. Most of the students should be quite familiar with the process.

INVESTIGATION

The responses of the students to the Investigation may help guide you in the amount of instruction needed graphing points for ordered pairs of integers. Many students will have studied this topic in Learning Unit D, pages D-22 through D-25.

DISCUSSION

Regardless of your students' background, you will want to cover this lesson thoroughly. The students should clearly understand the idea of drawing a pair of co-ordinate axes and finding a point for any integer pair. Once this idea is mastered, it is only a short step to the idea of graphing a function.

In discussing Exercise 2, allow for a wide variety of descriptions. For example, a student might say that point A $(^{-}4, 2)$ can be located by "going left 4 units, then up 2 units."

Exercise 4 introduces the word *quadrant*. It is convenient, but not vital, for the student to be able to identify a point as being located in a certain quadrant.

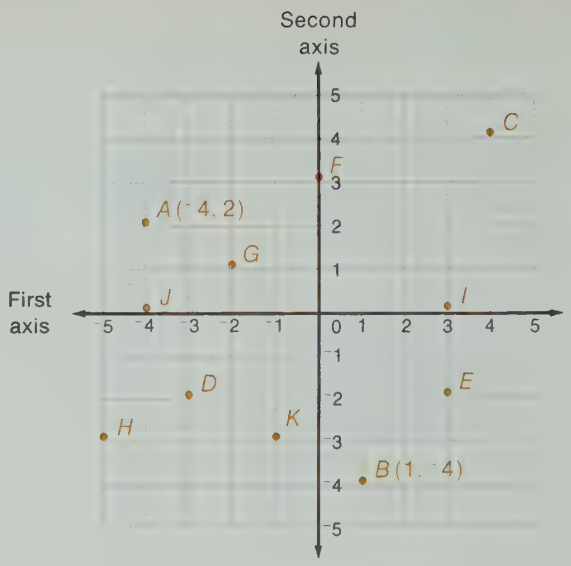
Integers and the Co-ordinate Plane

Investigating the Ideas

Point *A* can be associated with the **ordered pair** of integers $(^{-}4, 2)$.

The **co-ordinates** of point *A* are $(^{-}4, 2)$.

The co-ordinates of point *B* are $(1, ^{-}4)$.



Can you give the coordinates of the other lettered points?

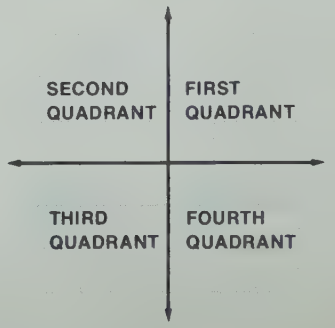
Discussing the Ideas

1. The two crossed number lines are called the **co-ordinate axes.** What are the co-ordinates of the point of intersection of the axes? (This point is called the **origin** of the co-ordinate axes.)

2. Explain how to locate the point for any ordered pairs of integers.

3. Jack said, "I am thinking about a point on the first axis." What can you say about the co-ordinates of his point?

4. The two axes divide the co-ordinate plane into four regions called **quadrants**, which are numbered as shown at the right.



   - **A** In which quadrant do all points have positive co-ordinates?
   - **B** In which quadrant is the first number in each pair negative and the second number positive?
   - **C** In which quadrant is the first number positive and the second number negative?
   - **D** In which quadrant do all points have negative numbers for both co-ordinates?

Using the Ideas

UTILIZATION

1. Graph these number pairs in the co-ordinate plane and label them with the letter indicated.

A $(7, 6)$, *A*	**D** $(6, -4)$, *D*	**G** $(6, 0)$, *G*	**J** $(3, 3)$, *J*
B $(-4, 5)$, *B*	**E** $(-7, 0)$, *E*	**H** $(0, 6)$, *H*	**K** $(3, -3)$, *K*
C $(-5, -6)$, *C*	**F** $(0, -7)$, *F*	**I** $(-3, 3)$, *I*	**L** $(-3, -3)$, *L*

For Exercises 2 through 9, draw the co-ordinate axes (indicating the integers between -10 and 10). Then graph the set of integer pairs indicated.

2. The set of integer pairs in which the first number is equal to the second number.

3. The set of integer pairs in which the first number is one greater than the second number.

4. The set of integer pairs in which the second number is one greater than the first number.

5. The set of integer pairs in which the sum of the first number and the second number is zero.

6. The set of integer pairs in which the second number is zero.

7. The set of integer pairs in which the first number is zero.

8. The set of integer pairs in which the second number is three times the first number.

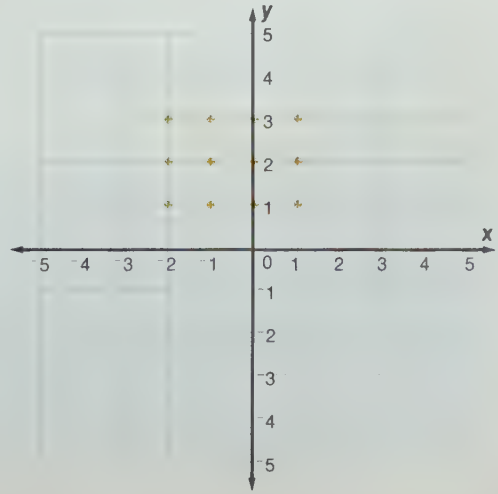
★ 9. The set of integer pairs in which the second number is the absolute value of the first number.

★ 10. If (x, y) represents the co-ordinates of the points in the co-ordinate plane, then the points graphed at the right are the set of points (x, y) such that $-2 \leq x \leq 1$ and $1 \leq y \leq 3$. Draw graphs of these inequalities.

A $-5 \leq x \leq -1$ and $1 \leq y \leq 4$

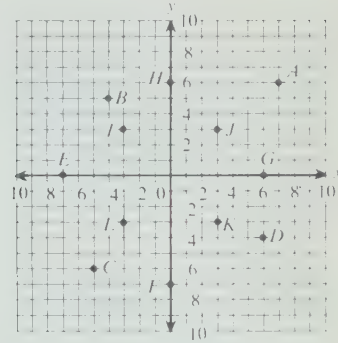
B $x = 3$ and $-3 \leq y \leq 3$

C $-2 \leq x \leq 2$ and $y = 2$

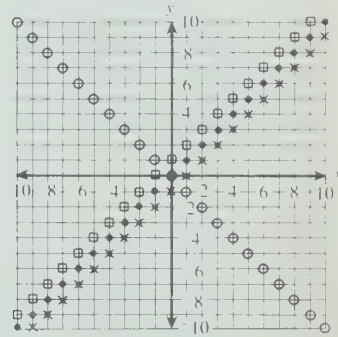


Solutions, Exercises 1–9

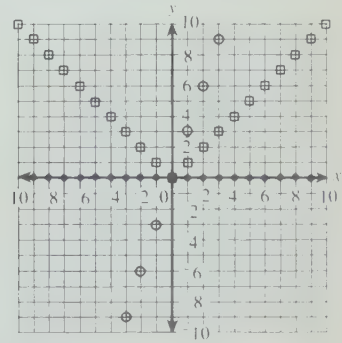
1.



2–5. Exercise 2 graph: • Exercise 4 graph: □
Exercise 3 graph: × Exercise 5 graph: ○



6–9. Exercise 6 graph: • Exercise 8 graph: ○
Exercise 7 graph: × Exercise 9 graph: □



Assignments

Minimum 1–4. Average 1–8. Maximum 1–10.

Objective

Given the input numbers and a function rule, the student can compute the output number and graph the pairs for the function.

PREPARATION

Materials: Graph paper or co-ordinate grid paper (*Duplicator Masters*, page 83 or 87).

Show a co-ordinate grid on the overhead screen or on the chalkboard. Review identification of co-ordinates of points in each of the four quadrants.

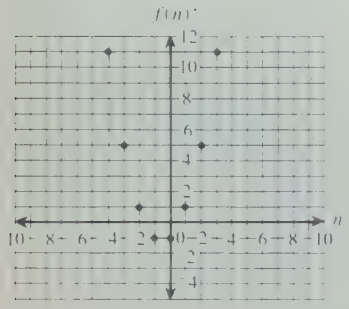
INVESTIGATION

Graphing functions is only a slight extension of the work of the previous lesson. If students have drawn graphs of functions involving whole numbers in Unit E, Module 2, they should have little difficulty with the Investigation. The function rule $f(n) = 9 - n^2$ produces ordered pairs which are points on a parabolic curve. The curve should not be drawn, however, since the domain is a finite set of integers and not all real numbers.

DISCUSSION

Work through Discussion Exercise 5 carefully. Your slower students may need extra help in evaluating functions such as this one. After the table has been completed, have some students show how to locate the ordered pairs for the function in the co-ordinate plane.

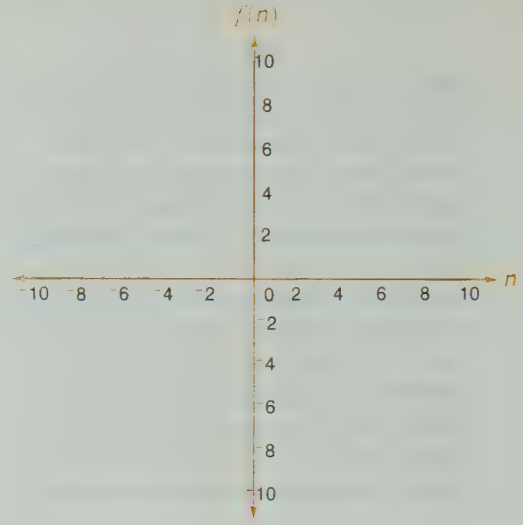
The graph for Exercise 6 is shown below.



Integer Functions

Investigating the Ideas

Given a function rule and a set of input numbers for the function, you can find the corresponding output numbers. When you graph the points for the input-output pairs, you have **graphed the function**.



Can you graph the function whose rule is $f(n) = 9 - n^2$, if the set of input numbers is $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$?

Discussing the Ideas

1. For which input number n is $f(n)$ the largest?

2. For which input number or numbers is $f(n)$ a negative integer?

3. For which integers n is $f(n) = 0$?

4. Do the points that you graphed for the function fall into a pattern? Describe the pattern.

5. A function rule is $f(n) = n^2 + n - 1$.
   If $n = -4$, then $f(-4) = (-4)^2 + -4 - 1$
   $= 16 + -4 - 1 = 11$.
   Give the other output numbers in the table below.

n	-4	-3	-2	-1	0	1	2	3
$f(n)$	11							

6. Draw a graph of the function in Exercise 5 using the number pairs in the table.

Using the Ideas

1. Complete each function table. Then graph the function for the ordered pairs in the table.

A

Function Rule	
$2n + {}^{-}3$	
n	$f(n)$
${}^{-}2$	${}^{-}7$
${}^{-}1$	${}^{-}5$
0	
1	
2	
3	
4	
5	

B

Function Rule	
$n^2 - 4n + 4$	
n	$f(n)$
5	9
4	
3	
2	
1	
0	
${}^{-}1$	
${}^{-}2$	

C

Function Rule	
$12 \div n$	
n	$f(n)$
12	1
6	
4	
3	
2	
1	
${}^{-}1$	
${}^{-}2$	
${}^{-}3$	
${}^{-}4$	
${}^{-}6$	

2. Draw the graph of a function whose rules are as follows:
$f(n) = 3$, if $n > 0$,
$f(n) = 0$, if $n = 0$,
$f(n) = {}^{-}2$, if $n < 0$.
Use the integers ${}^{-}5 \leq n \leq 5$ as input numbers.

★ **3.** Draw a graph of the absolute value function $f(n) = |n|$. Use the integers ${}^{-}5 \leq n \leq 5$ as input numbers.

★ **4.** **A** Draw graphs of $f(n) = n^2$ and $f(n) = n + 2$ on the same grid. Use integer inputs from ${}^{-}4$ to 4.

B What points do the two graphs have in common?

think

Can you complete the square to form a **multiplication magic square**? All the products in each column, row, and diagonal must be the same number.

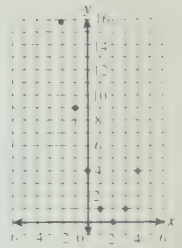
256		
	32	
16		4

UTILIZATION

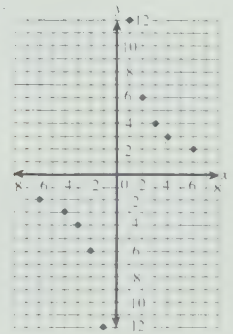
Solutions, Exercise 1, 2, 4

1A.

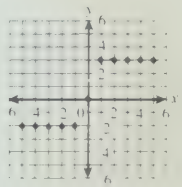
1B.



1C.



2.



4.

EXTENSION

Workbook page 33 and *Duplicator Masters* page 25 provide additional practice exercises.

Think Solution

The magic multiplication square contains only powers of 2.

2^8	2^1	2^6
2^3	2^5	2^7
2^4	2^9	2^2

The product in each row column or diagonal is $2^{15} = 32\,768$.

Assignments

Minimum 1. Average 1, 2. Maximum 1–4.

Objective

Given a linear equation $ax + by = c$, the student can find ordered pairs of integers satisfying the equation and draw the graph of the equation.

PREPARATION

Materials: Graph paper or co-ordinate grid paper (*Duplicator Masters*, page 83 or 87).

Graphs of equations in which the replacement set was ordered pairs of whole numbers were presented in Module 2 of this learning unit. Students who have covered that material should have the necessary background for this extension of graphing to ordered pairs of integers. Co-ordinate grid paper or graph paper should be available for the students.

INVESTIGATION

The equation presented in the Investigation is purposely simple in order that students can concentrate primarily on the ideas of graphing. The students should have little difficulty in finding several more ordered pairs of integers satisfying the equation and plotting the points for the pairs.

DISCUSSION

All of the equations in this lesson are linear equations. That is, the ordered pairs of integers satisfying an equation can be represented as points in the plane which lie along a line. The graphs of the equations, however, are not lines but sets of distinct points because we are restricting the ordered pairs to integers. Note also that we show only part of the graph of the equation, since there are infinitely many pairs satisfying each equation.

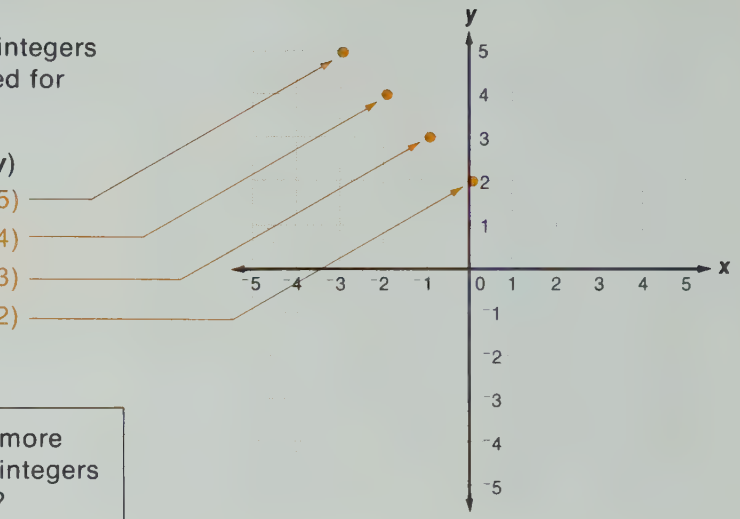
Exercise 2 should be discussed carefully inasmuch as some students may have difficulty in finding ordered pairs for the equations. You should emphasize the fact that several points should be graphed even though any two points determine a line. Since these are linear equations, all points should lie on one line. Thus the student can discover any error that might have been made in determining the pairs or plotting the points.

Graphing Integer Equations

Investigating the Ideas

Some ordered pairs of integers (x, y) have been graphed for the equation $x + y = 2$.

$x + y = 2$	(x, y)
$^-3 + 5 = 2$	$(^-3, 5)$
$^-2 + 4 = 2$	$(^-2, 4)$
$^-1 + 3 = 2$	$(^-1, 3)$
$0 + 2 = 2$	$(0, 2)$
⋮	⋮



Can you list five more ordered pairs of integers for the equation?

Discussing the Ideas

1. Locate the points for the ordered pairs that you found for the Investigation question. Do the points follow the pattern of the points already graphed?

2. To find the integer pairs (x, y) for an equation, you can try some integers for x in order to determine what y must be. Study the example at the right. Explain how you can find some other ordered pairs for the equation.

EQUATION: $3x - y = 2$

If $x = 0$, then $3 \cdot 0 - y = 2$

$0 - y = 2$

$y = ^-2$

$(0, ^-2)$ is one ordered pair of integers for the equation.

3. Complete the table to find some pairs (x, y) for the equation $3x - y = 2$.

$3x - y = 2$

x	0	1	2	3	$^-1$	$^-2$	$^-3$
y	2	1					

4. Graph the ordered pairs (x, y) in Exercise 3.

Using the Ideas

1. Complete the table of ordered pairs (x, y) for the equation $x + y = 0$.

x	$^-5$	5	$^-4$	4	$^-3$	3	$^-2$	$^-1$	1	0
y										

2. Graph the order pairs in the table of Exercise 1. This is part of the graph of $x + y = 0$.

3. Complete the table of ordered pairs (x, y) for the equation $x + 2y = 8$.

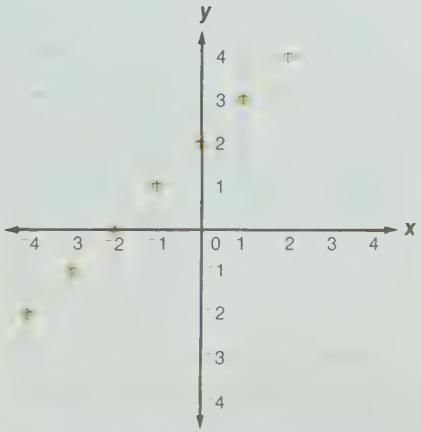
x	0	2	4	6	$^-2$	$^-4$	$^-6$
y							

4. Graph the ordered pairs of Exercise 3. This is part of the graph of $x + 2y = 8$.

5. Make a table of ordered pairs (x, y) for the equation $x - y = 0$. Use both positive and negative integers.

6. Draw the graph of the equation $x - y = 0$.

★ 7. Part of the graph of an equation is shown. What is the equation?



★ 8. Make a table of ordered pairs (x, y) for each equation. Then draw a graph for each equation.

- **A** $x + y = 6$
- **B** $x - y = 4$
- **C** $x = 2y$
- **D** $y = 2x$
- **E** $2x + 3y = 12$

think

Change LIKE into [illegible] in four steps by changing one letter at each step to get a new four-letter word each time.

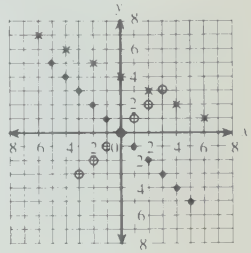
1
2
3
4

UTILIZATION

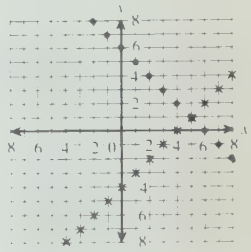
Graphing equations requires considerable time for many students. Therefore you should adjust your assignment to the needs and capabilities of your students.

Solutions, Exercises 2, 4, 6, 8

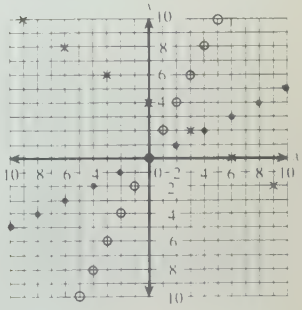
2, 4, 6. Graph for 2: •
Graph for 4: ×
Graph for 6: ○



8A, B. Graph for 8A: •
Graph for 8B: ×



8C–E. Graph for 8C: •
Graph for 8D: ○
Graph for 8E: ×



Think Solution

An alternate solution might be:

L I K E
M I K E (microphone)
M A K E
M A T E
M A T H

Assignments

Minimum 1–4. Average 1–6. Maximum 1–8.

Objective

The student can demonstrate the ability to work with the concepts developed in the module.

UTILIZATION

The review exercises emphasize the three major topics covered in this module:

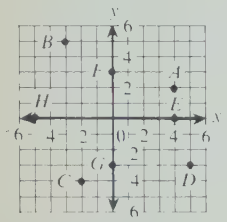
1. Computation with integers
2. Solutions of equations and inequalities
3. Graphs of ordered pairs of integers

Some of the review problems may be discussed orally. Others may involve written work. Solutions and methods used by the students should be discussed before an achievement test is administered.

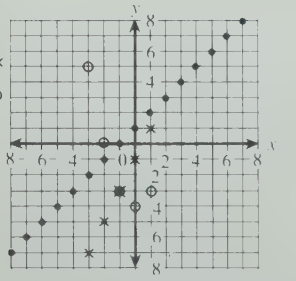
Additional review exercises are provided on page 34 of the *Workbook*.

Solutions, Exercises 12, 13, and 14

12.



13, 14. Graph for 13: •
Graph for 14A: ×
Graph for 14B: ○



REVIEWING THE IDEAS

1. Give the opposite of each integer.

A $^{-}5$ B 8 C $^{-}12$

2. The sum of any integer and its opposite is ?.

3. Find the sums.

A $18 + {}^{-}9$ B $6 + {}^{-}13$ C $^{-}8 + {}^{-}23$

4. Find the missing addends.

A $0 - {}^{-}2 = \mathbf{A}$ C $8 - {}^{-}2 = \mathbf{A}$

B $^{-}5 - {}^{-}3 = \mathbf{A}$ D $^{-}5 - {}^{-}5 = \mathbf{A}$

5. Solve each equation and give the name of the basic principle the equation illustrates.

A $16 + 0 = a$ B $^{-}9 \cdot 1 = b$

C $c + 8 = 0$ D $^{-}9 + d = 17 + {}^{-}9$

E $(8 + 4) + {}^{-}4 = e + (4 + {}^{-}4)$

F $^{-}5 \cdot (2 + 1) = (^{-}5 \cdot f) + (^{-}5 \cdot 1)$

G $(^{-}11 \cdot {}^{-}20) \cdot {}^{-}5 = {}^{-}11 \cdot (h \cdot {}^{-}5)$

6. Find the products.

A $7 \cdot 6$ B $^{-}7 \cdot 6$ C $^{-}7 \cdot {}^{-}6$

7. Find the missing factors.

A $^{-}54 \div 6 = \mathbf{F}$ C $^{-}72 \div {}^{-}9 = \mathbf{F}$

B $54 \div {}^{-}9 = \mathbf{F}$ D $0 \div {}^{-}8 = \mathbf{F}$

8. Give each solution set.
Replacement set:
$R = \{^{-}5, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, 4, 5\}$.

A $x < {}^{-}1$ C $x < 3$ and $x > {}^{-}2$

B $x > {}^{-}1$ D $x > 3$ or $x < {}^{-}4$

9. A What is the absolute value of 9?

B What is the absolute value of $^{-}9$?

10. Solve the equations.

A $|{}^{-}13| = a$ D $|{}^{-}6 - {}^{-}9| = a$

B $|27| = a$ E $|a - 2| = 1$

C $|5 - {}^{-}1| = a$ F $|4 - a| = 2$

11. Give the integer for each screen.

A $2(x + 5) = 14 \Rightarrow x + 5 =$ ▒
$\Rightarrow x =$ ▒

B $6 + (^{-}8 + y) = 0 \Rightarrow {}^{-}8 + y =$ ▒
$\Rightarrow y =$ ▒

C $6(x - 2) = 18 \Rightarrow x - 2 =$ ▒
$\Rightarrow x =$ ▒

D $^{-}2(5 + n) = {}^{-}2 \Rightarrow 5 + n =$ ▒
$\Rightarrow n =$ ▒

E $(-3 + a) \div 4 = {}^{-}2 \Rightarrow {}^{-}3 + a =$ ▒
$\Rightarrow a =$ ▒

12. Draw the co-ordinate axes (indicating points for the integers between $^{-}6$ and 6) and graph the integer pairs, using the indicated letters.

A $(4, 2)$, A E $(4, 0)$, E

B $(^{-}3, 5)$, B F $(0, 3)$, F

C $(^{-}2, {}^{-}4)$, C G $(0, {}^{-}3)$, G

D $(5, {}^{-}3)$, D H $(^{-}5, 0)$, H

13. Graph the set of integer pairs in which the second number is one more than the first number.

14. Give the missing integers. Then graph the functions.

A

Function Rule: $2n - 1$	
n	$f(n)$
$^{-}3$	$^{-}7$
$^{-}2$	▒
$^{-}1$	$^{-}3$
0	▒
1	▒

B

Function Rule: $n^2 - 4$	
n	$f(n)$
$^{-}3$	5
$^{-}2$	▒
$^{-}1$	▒
0	▒
1	▒

15. Which integer is larger, a or b, if

A $a + 4 = b$? C $a = b - 7$?

B $a - b = 6$? D $b - a = 5$?

TEST YOURSELF

1. Give the opposite of each integer.
   A $^{-}5$ B 16 C 0

2. Find the sums.
   A $^{-}7 + {}^{-}2$ C $9 + {}^{-}4$
   B $^{-}8 + 5$ D $^{-}3 + {}^{-}6$

3. Find the differences.
   A $8 - 10$ C $4 - {}^{-}2$
   B $^{-}5 - 3$ D $^{-}2 - {}^{-}7$

4. Find the products.
   A $6 \cdot {}^{-}3$ B $^{-}8 \cdot 4$ C $^{-}5 \cdot {}^{-}3$

5. Find the quotients.
   A $16 \div {}^{-}4$ C $^{-}30 \div 5$
   B $^{-}24 \div {}^{-}3$ D $^{-}18 \div {}^{-}6$

6. Write the correct symbol ($<$ or $>$) for each
   A 5 ▒ $^{-}3$ C $^{-}6$ ▒ $^{-}2$
   B $^{-}1$ ▒ 1 D $^{-}9$ ▒ $^{-}15$

7. What is the solution set for $x > {}^{-}4$ and $x < 3$?

8. If $|x - 2| = 5$, what are all the possible integers for x?

9. Solve the equation $3x + 8 = {}^{-}13$.

10. Complete the function rule table.

Function Rule

$4n - 3$

n	$f(n)$
2	5
1	1
0	A
$^{-}1$	B
$^{-}2$	C
$^{-}3$	D

RESEARCH PROJECTS

A Find how to make a **nomograph**. It is a device that can be used to find sums and differences of integers. Then construct your own nomograph. (See *Making and Using Graphs and Nomographs* by Richard A. Denholm; Pasadena, California: Franklin Publications, 1968, pp. 45-47; Grolier Ltd.)

B It is possible to represent any integer using a negative number base. For example, using base *negative three*, $21_{(^{-}3)}$ means $2 \cdot ({}^{-}3) + 1 = {}^{-}5$, $112_{(^{-}3)}$ means $1 \cdot ({}^{-}3)^2 + 1 \cdot ({}^{-}3) + 2 = 8$, and $2111_{(^{-}3)}$ means $2 \cdot ({}^{-}3)^3 + 1 \cdot ({}^{-}3)^2 + 1 \cdot ({}^{-}3) + 1 = {}^{-}47$. Choose some negative integer as a base and show how to represent both positive and negative integers in this base.

TEST YOURSELF

Students who do well on this self-evaluation test may be expected to perform well on an achievement test covering the concepts presented in this module. This self-test might serve as a model for a teacher-designed achievement test, or you might prefer to use the Unit E, Module 4, Achievement Test that is available from the publisher as part of the test package designed for use with this series.

Note that the correlation between Test Yourself items and the module objectives is indicated by annotations on the student book page.

RESEARCH PROJECTS

Students who are interested in Project A can find detailed descriptions and explanations of the construction and use of nomographs in *Multi-Sensory Aids in the Teaching of Mathematics*, 18th Yearbook of the NCTM, New York Bureau of Publications, Teachers College, Columbia University, 1945, pp. 164–181.

Research Project B may appeal to some of your better students. An article relating to this topic appears in *The Mathematical Log*, Vol. XVII, No. 1, Sept. 1972, Mu Alpha Theta, Dept. of Mathematics, The University of Oklahoma, Norman, Oklahoma.

Objective

The student can demonstrate the ability to work with the significant ideas presented in this learning unit.

UTILIZATION

Notice that the Cumulative Review requires about 120 responses from students. More than one class period might be needed if students are to cover all parts of these exercises. Hence, you might ask students to do only the first one or two parts of multi-part problems. As a general rule, students need only complete enough parts of a given problem to convince themselves that they can solve the type of problem given.

The table below indicates the correlation between the individual review items and the four modules of this learning unit.

Module Number	Related Review Items
1	1–10
2	11–18
3	19–24
4	25–36

After the cumulative review, an achievement test covering Modules 1, 2, 3, and 4 can be given. Packages of achievement and unit tests are available from the publisher.

CUMULATIVE REVIEW

1. A Write 1893 using Roman numerals.
B Write MDCXVII using Hindu-Arabic numerals.

2. Use exponents to express $7 \cdot 7 \cdot 7 \cdot 7$.

3. Solve the equations.
A $10^2 \cdot 10^4 = n$ C $10^4 \div 10^2 = n$
B $2^5 \cdot 2^2 = n$ D $10^3 \div 10^3 = n$

4. A Write the Hindu-Arabic numeral for $(5 \cdot 10^6) + (4 \cdot 10^3) + (3 \cdot 10^1) + (9)$.
B Write 78 463 097 using expanded notation.

5. Round 746 813 to the nearest
A thousand.
B ten thousand.
C hundred thousand.

6. Estimate the product.
$$\begin{array}{r} 421 \\ \times\ 77 \end{array}$$

7. Estimate the quotient. $394\overline{)287\,449}$

8. Compute.
A $87 + 287 + 778 + 19 + 5$
B $9000 - 576$
C $678 \cdot 932$
D $21\,544 \div 12$

9. Write each numeral in base two.
A $5_{(ten)}$ B $25_{(ten)}$ C $77_{(ten)}$

10. Compute as indicated and give the result as a binary numeral.

A
$$\begin{array}{r} 101_{(2)} \\ 11_{(2)} \\ +\,111_{(2)} \end{array}$$

B
$$\begin{array}{r} 1001_{(2)} \\ -\ \ 11_{(2)} \end{array}$$

C
$$\begin{array}{r} 101_{(2)} \\ \times\ 11_{(2)} \end{array}$$

11. $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
A Find a subset of X which contains all of the odd numbers in X.
B Find a subset of X which contains all of the even numbers in X.
C What is the union of the sets in parts A and B?

12. Sets R and S are disjoint. Which statement below is true?
A $R \subset S$ B $R \cap S = \phi$

13. Evaluate the expression $3x + 7$ when $x = 5$.

14. Find the solution set of each open sentence. The set $\{0, 1, 2, 3, 4, 5\}$ is the replacement set.
A $3 + n = 7$ B $n < 4$
C $n \cdot 2 > n + 2$ D $n^2 > 7$

15. Give the algebraic expression for each ▥.
A $5x + 2x =$ ▥ C $3x + 4x + 5x =$ ▥
B $9x + 6x =$ ▥ D $9x + x =$ ▥

16. Write the equation suggested by this flow chart.

17. What is the whole number that is the solution for the equation in Exercise 16?

18. Find the solution set for each inequality. The replacement set is the set of whole numbers.
A $x > 14$ and $x < 19$
B $x = 5$ *or* $x = 3$
C x is divisible by 3 and $x < 10$.
D $2x < 8$ and x is odd.

19. **A** What are the first 10 prime numbers?
B What are the first 10 composite numbers?

20. Give the prime factorization of each number.
A 98 **B** 102 **C** 625

21. Give the GCF of
A 16 and 30. **B** 100 and 30.

22. Give the LCM of
A 8 and 10. **B** 15 and 35.

23. Which of the following numbers are divisible by 2? By 3? By 5? By 9? By 10?
A 15 **B** 330 **C** 78 **D** 225

24. Write the first five terms of a sequence whose n^{th} term is $n^2 + 3$.

25. Give the opposite of each number.
A $^-46$ **B** 97 **C** $^-1000$

26. Find the sums.
A $23 + {}^-5$ **B** $^-17 + 9$

27. Find the differences.
A $13 - 19$ **C** $7 - {}^-5$
B $^-8 - 5$ **D** $^-10 - {}^-35$

28. Find the products.
A $^-4 \cdot 8$ **B** $^-5 \cdot 9$ **C** $^-7 \cdot 0$

29. Find the quotients.
A $45 \div {}^-9$ **C** $200 \div {}^-4$
B $^-27 \div 3$ **D** $(72 \div {}^-8) \div 3$

30. Solve the equations.
A $^-9 \cdot h = {}^-27$ **C** $k \div 5 = {}^-45$
B $w \cdot {}^-4 = 24$ **D** $^-8 = 32 \div y$

31. Complete each table.

A

Function Rule $n = 3$	
n	$f(n)$
2	
$^-5$	
	7
	$^-3$

B

Function Rule $n^2 - 3n$	
n	$f(n)$
1	
$^-2$	
	0
8	

32. Give the correct symbol ($<$ or $>$) for each ▨.
A 5 ▨ $^-7$ **C** $^-4$ ▨ 4
B $^-5$ ▨ 9 **D** 0 ▨ $^-17$

33. Solve the inequalities. The replacement set is $\{^-4, {}^-2, 0, 2, 4, 6\}$.
A $x + 1 > 3$ **C** $2x < {}^-1$
B $x - 3 < 6$ **D** $^-2x > 1$

34. Give the integer for each.
A $|{}^-4|$ **C** $|{}^-81 + 13|$ **E** $|8 - 8|$
B $|8 + 4|$ **D** $|{}^-13 + 81|$ **F** $|{}^-3 - {}^-2|$

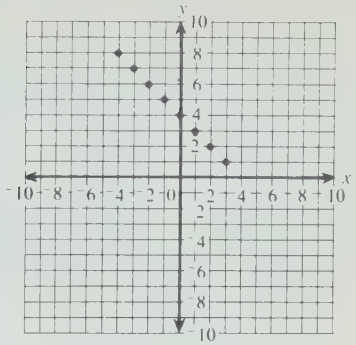
35. Give the whole numbef for each ▨.
A If $4(x - 7) = 0$, then $x - 7 =$ ▨ and then $x =$ ▨
B If $9(x - 4) = 9$, then $x - 4 =$ ▨ and then $x =$ ▨

36. **A** Complete the table for $x + y = 4$.

x	0	1	2	3	$^-1$	$^-2$	$^-3$	$^-4$
y	4	3	▨	▨	▨	▨	▨	▨

B Draw a graph of the ordered pairs in part **A**.

Solution, Exercise 36B



MATHEMATICAL RECREATION

To carry out the Mathematical Recreation, students can use actual toothpicks or draw segments to represent toothpicks.

The sequence for the stair-patterned square is

$$4, 10, 18, 28, \ldots, n^2 + 3n, \ldots$$

where n is the number of the toothpicks in the longest string of squares.

When $n = 20$, $n^2 + 3n = 460$.

When $n = 21$, $n^2 + 3n = 504$.

Therefore the longest row of squares in the stair-pattern is 20. The total number of such squares is

$$1 + 2 + 3 + \ldots + 19 + 20 = \frac{20 \cdot 21}{2} = 210$$

squares

There will be 40 toothpicks left over from a total of 500 toothpicks.

For the triangular pattern, it can be observed that the number of toothpicks at each stage is

$$3 \cdot [1 + 2 + 3 + \ldots + (n-1) + n] = \frac{3 \cdot n \cdot (n+1)}{2} = \frac{3}{2}(n^2 + n)$$

When $n = 17$, $\frac{3}{2}(n^2 + n) = 456$.

The number of triangles would be

$$(1 + 2 + 3 + \ldots + 16 + 17) = \frac{17 \cdot (18)}{2} = 153.$$

There will be 44 toothpicks left over.

MATHEMATICAL RECREATIONS

TOOTHPICK PATTERNS

Suppose that you have a box of 500 toothpicks.

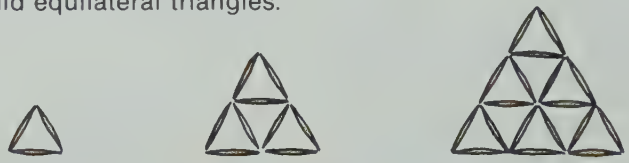
Using the toothpicks, you could build stair-patterned sets of squares like these:

4 toothpicks 10 toothpicks 18 toothpicks

A How many squares are in the largest complete stair-pattern that you could make using as many of the 500 toothpicks as possible?

B How many toothpicks would you have left?

Try the same problem, but this time build equilateral triangles.



3 toothpicks 9 toothpicks 18 toothpicks

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A 5; B $^{-}16$; C 0 **2.** A $^{-}9$; B $^{-}3$; C 5; D $^{-}9$ **3.** A $^{-}2$; B $^{-}8$; C 6; D 5

4. A $^{-}18$; B $^{-}32$; C 15 **5.** A $^{-}4$; B 8; C $^{-}6$; D 3 **6.** A $>$; B $<$; C $<$; D $>$

7. $\{^{-}3, ^{-}2, ^{-}1, 0, 1, 2\}$ **8.** $x = 7$, $x = ^{-}3$ **9.** $x = ^{-}7$

10. A $^{-}3$; B $^{-}7$; C $^{-}11$; D $^{-}15$

UNIT F: RATIONAL NUMBERS
Module 1: Rational Numbers

General Objectives

To introduce the complete set of rational numbers.
To emphasize addition, subtraction, multiplication, and division using both positive and negative rational numbers.
To provide problem-solving situations involving rational numbers.
To extend the student's experiences with graphing to include graphs of functions and equations.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
40	**RED** Given a fraction, the student can name an equivalent fraction.	F-2, 3 F-4, 5	**WB** 35 **DM** 26	**ASC** F-1, 3, 4 **SP** c-1, **PD** c-1
41	**RED** Given a fraction, the student can express it in lowest terms.	F-2, 3	**WB** 35 **DM** 26	**ASC** F-2 **SP** c-1
42	**RED** Given any rational number, the student can find its additive inverse.	F-6, 7 (F-8, 9)	**WB** 36, 37 **DM** 27	
43	**RED** Given two rational numbers, the student can find the sum of the numbers.	F-10, 11 F-12, 13	**WB** 38, 39 **DM** 28	**ASC** F-5, 7 **SP** c-2 **PD** c-1 (d-2, 3) **SWM 3** 112-113, 119, 210-211
44	**RED** Given two rational numbers, the student can find the difference of the numbers.	F-14, 15	**WB** 40 **DM** 29	**ASC** F-6, 8 **SP** c-3 **PD** c-2 (d-3)
45	**RED** Given two rational numbers, the student can find the product of the numbers.	F-16, 17 F-18, 19	**WB** 41	**ASC** F-9, 10, 11 **SP** d-1, 2 **PD** c-3 (DS c-3)
46	**RED** Given any nonzero rational number, the student can give the reciprocal of the number.	F-20, 21	**WB** 42	**SP** d-2 **PD** c-4
47	**RED** Given two rational numbers, the student can find the quotient of the numbers.	F-24, 25	**WB** 43	**ASC** F-12 **SP** d-3, 4 **PD** c-4 (DS c-3)
48	**YELLOW** Given two rational numbers, the student can name the larger.	F-26, 27	**WB** 44	(**PD** d-1)
49	**YELLOW** Given word problems involving rational numbers, the student can solve them.	F-22, 28, 29	**DM** 32	
50	***GREEN*** *Given a function with a set of rational numbers as the domain, the student can find the number pairs that satisfy the function and graph it.*	F-30, 31	**WB** 45	**DS** d-1
51	***GREEN*** *Given a simple linear equation such as $x + y = 1$, the student can find number pairs that satisfy the equation and graph it.*	F-32, 33	**DM** 33	**DS** d-2
	Reviewing the Ideas	F-34	**WB** 46	

MATHEMATICS

The major portion of this module is devoted to the development of the basic computational algorithms for the set of all rational numbers. The main task is to adjoin the negative rational numbers to the set of non-negative rational numbers together with zero and then show that the same computational rules hold for the complete set of rational numbers. The basic steps are these:

1. We assume that each positive rational number has an additive inverse or opposite such that the sum of each rational number and its opposite is zero.
2. We assume that the same basic principles for addition and multiplication hold for the set of rationals as hold for integers.
3. We then develop the usual rules of computation based on assumptions 1 and 2.

The work involving negative rational numbers involves demonstrating that if a and b are integers with $b \neq 0$, then $-\frac{a}{b}$, $\frac{-a}{b}$, and $\frac{a}{-b}$ are equivalent fractions. In particular, we show that the quotient of two integers can be expressed as a rational number in fractional form with a *positive* denominator.

The general algorithm for adding two rational numbers utilizes two ideas.

$$\text{If } \frac{a}{b} = \frac{a \cdot k}{b \cdot k},\ k \neq 0, \text{ and } \frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

$$\text{then } \frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{ad + bc}{bd}$$

Subtraction is defined in terms of additon:

$$\text{If } \frac{a}{b} + \frac{c}{d} = \frac{e}{f}, \text{ then } \frac{e}{f} - \frac{c}{d} = \frac{a}{b}.$$

The proof of the general algorithm for multiplying two rational numbers depends on the basic principles for multiplication and the two unit-fraction principles.

For any two rational numbers $\frac{1}{a}$ and $\frac{1}{b}$, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ and for any integer a, and any rational number $\frac{1}{b}$, $a \cdot \frac{1}{b} = \frac{a}{b}$.

$$\text{Then } \frac{a}{b} \cdot \frac{c}{d} = \left(a \cdot \frac{1}{b}\right) \cdot \left(c \cdot \frac{1}{d}\right) = a \cdot c\left(\frac{1}{b} \cdot \frac{1}{d}\right) = ac \cdot \frac{1}{bd} = \frac{ac}{bd}$$

Similarly, division is defined in terms of multiplication:

$$\text{If } \frac{a}{b} \cdot \frac{c}{d} = \frac{e}{f}, \text{ and } \frac{c}{d} \neq 0, \text{ then } \frac{e}{f} \div \frac{c}{d} = \frac{a}{b}.$$

Using this definition, we can prove the usual division algorithm:

$$\text{If } \frac{c}{d} \neq 0, \text{ then } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

TEACHING THE MODULE

Vocabulary

additive inverse
associative principle
axes
commutative principle
co-ordinates
denominator
distributive principle
domain
equivalent fractions
fraction
graph
improper fraction
least common denominator (LCD)
linear equation
lowest-terms fraction
means
negative rational number
numerator
multiplicative inverse
negative rational
numerator
one principle
opposites principle
positive rational number
proper fraction
range
rational number
reciprocal
unit fraction
zero principle

In teaching this module, you need to emphasize that if students can compute using integers, then they can compute using rational numbers, because the algorithms always reduce ultimately to a problem of computing with integers. Thus, if students can add $^-8 + 7$, they should be able to find the sum $\frac{-8}{9} + \frac{7}{9}$. If students can multiply $^-2 \cdot {}^-3$, they should be able to compute the product $\frac{-2}{5} \cdot \frac{-3}{7}$.

In addition to developing and strengthening computational skills, this module places considerable emphasis on problem solving and work with graphs. The material is a natural extension of similar material involving whole numbers and integers. The concepts are not new, but now the domain of discourse is the set of rational numbers.

Lesson Schedule

Although generally each lesson is designed to be covered in one class period, the time needed will vary considerably, depending on the ability of your students. With capable students, more than one lesson could be covered in a class period. This is especially true of the lessons at the beginning of the module. You probably should plan to spend about 12 to 15 days on the module.

Evaluation

Your day-to-day observation of students is particularly important in this module. Students who have difficulty with negative rational numbers are likely to be students who have also had difficulty with integers. Remedial work should be made available for such students.

The Reviewing the Ideas and Test Yourself pages should be helpful in making your evaluation of student progress.

RESOURCES FOR ACTIVE LEARNING

Activities

Cloudburst, Vol. 5 (fractions), Nos. 3118, 3318, Midwest Publications
Fraction Bars, Card Set II, Scott Resources
Let's Play Games in Mathematics, Vol. 7, Activities 11, 31, 44, 54, 57, National Textbook Co.
Nuffield Project: *Computation and Structure 5* (rationals), pp. 7–44, Wiley

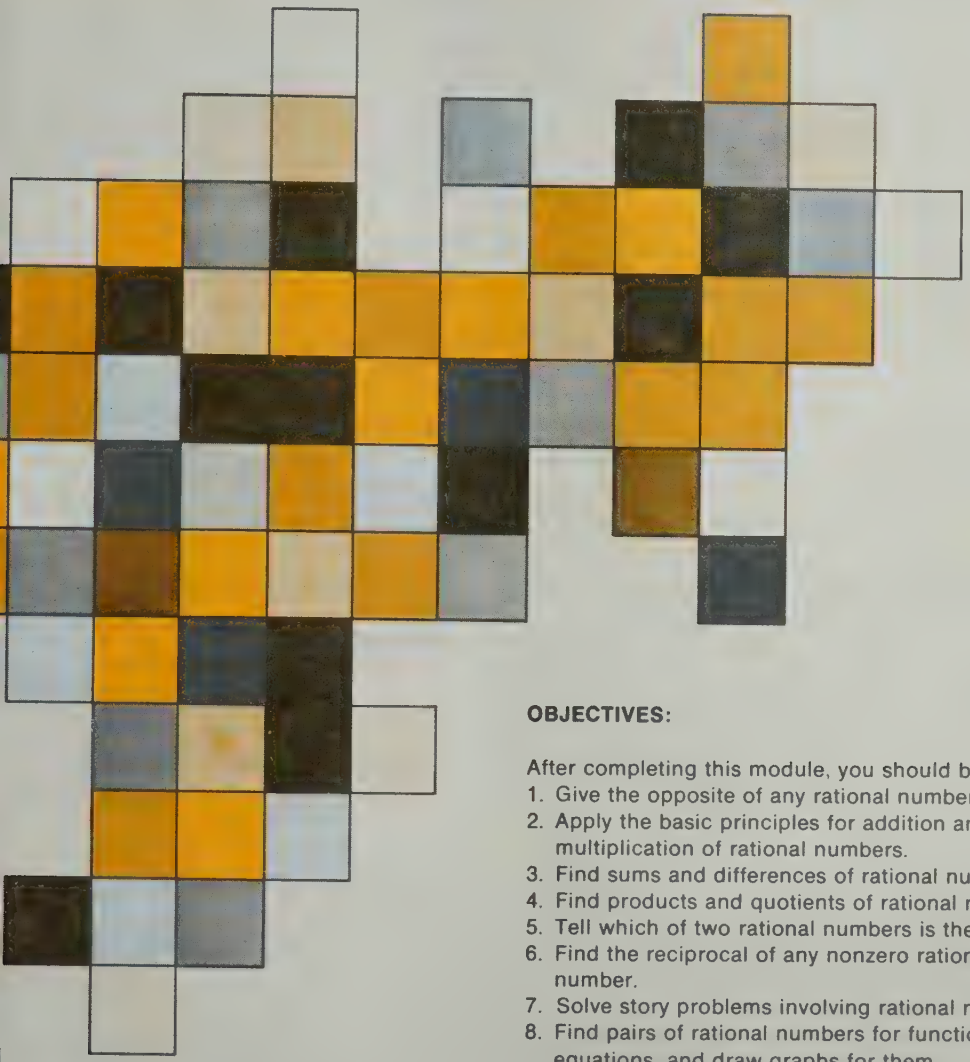
Manipulative Devices

Fraction Bars—manipulatives (Scott Resources)
Geoboards (Addison-Wesley)

Games and Puzzles

Come Out Even (Holt)
Here-to-There (Scott, Foresman)
In Order Game Kit (Midwest Publications)
Spinner Number Games (Heath)

MODULE 1: *Rational Numbers*



OBJECTIVES:

After completing this module, you should be able to:

1. Give the opposite of any rational number.
2. Apply the basic principles for addition and multiplication of rational numbers.
3. Find sums and differences of rational numbers.
4. Find products and quotients of rational numbers.
5. Tell which of two rational numbers is the greater.
6. Find the reciprocal of any nonzero rational number.
7. Solve story problems involving rational numbers.
8. Find pairs of rational numbers for functions and equations, and draw graphs for them.

Objectives

The student can (1) write a fraction that compares one set or region with another set or region; (2) write fractions equivalent to a given fraction; (3) find lowest-terms fractions; and (4) identify proper and improper fractions.

PREPARATION

Virtually all of the content of this lesson should represent review for most students. You might wish to review notation and terminology associated with fractions before beginning the Investigation, or you might begin the Investigation immediately.

INVESTIGATION

The Investigation is designed to stimulate students to recall some basic ideas associated with fractions, namely:

1. fraction notation
2. the use of fractions to compare two sets
3. the use of different fractions to describe a certain part of a region

The Investigation permits a wide range of responses from the student. Typical statements might be: "$\frac{2}{3}$ of the region is not shaded." "$\frac{8}{24}$ of the region is shaded." "$\frac{16}{24}$ of the region is not shaded." "The number of shaded squares is $\frac{1}{2}$ of the number of unshaded squares."

DISCUSSION

The example of an equivalent set of fractions in Exercise 4 can be generalized:

If k is any nonzero integer, $\frac{a}{b}$ and $\frac{ak}{bk}$ are equivalent fractions.

This concept is used when finding lowest-terms fractions and when choosing different denominators in problems of addition and subtraction of rational numbers.

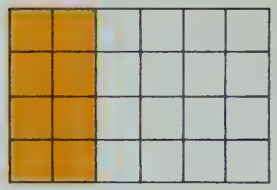
Exercise 5 reviews the idea of lowest-terms fractions. You should relate finding lowest-terms fractions to the ideas of prime factors and greatest common factor, which were developed in Unit E, Module 3.

A Review of Fractions

Investigating the Ideas

One third of this region is shaded.

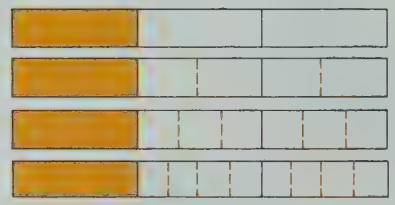
How many other statements (using fractions) can you write about this region?



Discussing the Ideas

1. **A** How many different fractions did you use to describe the shaded portion of the region above?
   B How many different fractions can you find for the unshaded portion of the region?

2. Fractions are used to compare one set with another set or to compare one region with another region. What fraction compares the shaded part of region R to the whole region?



3. The figures R, S, T, and U suggest that different fractions may be used to compare the same two regions. Give the fractions suggested by S, T, and U.

4. The sequence of fractions $\left\{\frac{2\cdot 1}{3\cdot 1}, \frac{2\cdot 2}{3\cdot 2}, \frac{2\cdot 3}{3\cdot 3}, \frac{2\cdot 4}{3\cdot 4}, \cdots, \frac{2\cdot n}{3\cdot n}, \cdots\right\}$ where n is a nonzero integer is a sequence of **equivalent fractions**.
   A What is the 5th fraction in the sequence?
   B What is the 10th fraction in the sequence?
   C What is the 100th fraction in the sequence?

5. A fraction is in **lowest terms** if the greatest common factor (GCF) of the numerator and denominator is 1.

$$\frac{3}{7} = \frac{3\cdot 1}{7\cdot 1} = \frac{3}{7} \qquad \frac{10}{15} = \frac{2\cdot 5}{3\cdot 5} = \frac{2}{3} \qquad \frac{16}{56} = \frac{2\cdot 8}{7\cdot 8} = \frac{2}{7}$$

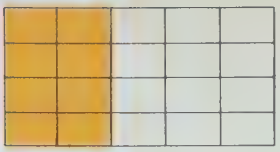
(GCF: 1, 5, 8; Lowest-terms fraction: $\frac{3}{7}$, $\frac{2}{3}$, $\frac{2}{7}$)

Explain how you would find the lowest-terms fraction for each of these fractions.

A $\frac{9}{15}$ **B** $\frac{10}{25}$ **C** $\frac{18}{36}$ **D** $\frac{90}{100}$ **E** $\frac{340}{1000}$

Using the Ideas

1. Write three fractions which compare the shaded portion of the rectangle to the whole rectangle.



2. Write the next three fractions for each set of equivalent fractions.

A $\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\right\}$ **C** $\left\{\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \ldots\right\}$ **E** $\left\{\frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \ldots\right\}$

B $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\right\}$ **D** $\left\{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \ldots\right\}$ **F** $\left\{\frac{9}{8}, \frac{18}{16}, \frac{27}{24}, \ldots\right\}$

3. Write two fractions equivalent to each given fraction.

A $\frac{1}{5}$ **C** $\frac{5}{8}$ **E** $\frac{5}{3}$ **G** $\frac{7}{8}$ **I** $\frac{5}{6}$ **K** $\frac{7}{16}$

B $\frac{4}{3}$ **D** $\frac{3}{10}$ **F** $\frac{1}{8}$ **H** $\frac{5}{12}$ **J** $\frac{11}{10}$ **L** $\frac{7}{20}$

4. For each fraction, give the GCF of the numerator and denominator. Then give the lowest-terms fraction.

A $\frac{12}{18}$ **D** $\frac{42}{49}$ **G** $\frac{20}{12}$ **J** $\frac{0}{10}$ **M** $\frac{48}{64}$

B $\frac{8}{12}$ **E** $\frac{10}{4}$ **H** $\frac{100}{150}$ **K** $\frac{63}{70}$ **N** $\frac{39}{52}$

C $\frac{24}{24}$ **F** $\frac{16}{24}$ **I** $\frac{25}{40}$ **L** $\frac{45}{60}$ **O** $\frac{12}{12}$

5. For each fraction, write an equivalent fraction with a denominator of 24.

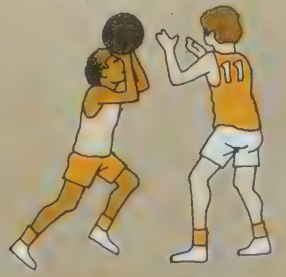
A $\frac{2}{3}$ **B** $\frac{1}{2}$ **C** $\frac{5}{12}$ **D** $\frac{3}{8}$ **E** $\frac{1}{4}$ **F** $\frac{5}{6}$ **G** $\frac{7}{8}$

6. A fraction whose numerator is smaller than its denominator is a **proper fraction**. Otherwise, it is an **improper fraction**. Which fractions below are proper and which are improper?

A $\frac{12}{5}$ **B** $\frac{7}{9}$ **C** $\frac{1}{10}$ **D** $\frac{7}{7}$ **E** $\frac{123}{719}$ **F** $\frac{100}{33}$

think

Suppose 623 teams are entered in a basketball tournament. A team is eliminated from the tournament as soon as it loses one game. How many tournament games must be played before the basketball championship is determined?



More practice, page S-13, Set 23

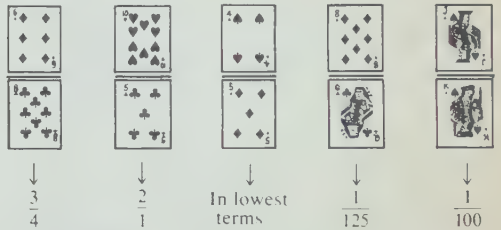
UTILIZATION

You may elect to do many of the exercises orally if your students seem to be well prepared from the Investigation and Discussion. Keep in mind that the objectives of this lesson are to review concepts studied many times before.

EXTENSION

To provide additional practice, make selective assignments from Supplementary Exercise Set 23 on page S-13, *Workbook* page 5, *Duplicator Masters* page 26, and *Arithmetic Skill Cards* W-1 through W-4.

Enrichment: To practice naming lowest-terms fractions, pairs of students can utilize a deck of playing cards to select fractions at random. One student deals two cards, one above the other, to represent fractions. Face cards can be assigned a value of powers of ten. For example, jacks could be 100, queens 1000, and kings 10 000. The lowest-terms fraction for each of the pairs is then named by the second student.



Think Solution

A total of 622 games will be played. All the teams entered in the tournament must lose one game except the winner. Thus, there must be 622 games.

There are several ways to structure such a tournament. Note that when an odd number of teams is entered, at least one team will draw a "bye" (advancing to the next round because the team has not drawn an opponent). You might ask a student to draw a chart for this tournament.

Assignments

Minimum 1–3, oral; 4. Average 1–3, oral; 4–6. Maximum 1–6, oral.

Objectives

The student can associate a set of equivalent fractions with one point on the number line.

The student can solve fractional equations such as $\frac{3}{5} = \frac{x}{30}$.

PREPARATION

Write several lowest-terms fractions on the chalkboard. Ask students to name or write several more fractions equivalent to each lowest-terms fraction.

INVESTIGATION

The Investigation is designed to help students review the fact that for each set of equivalent fractions there is a unique point on the number line associated with the set. Any fraction in the set can be used as a name for the co-ordinate of that point. Usually, the lowest-terms fraction is used.

Remind students that each point on the number line they draw is to be named with the letter denoting the set of equivalent fractions associated with that point. Thus, the letter *A* is associated with the sixth point to the right of zero since that point represents $\frac{1}{2}$ or $\frac{6}{12}$.

DISCUSSION

Exercise 2 reviews the "cross products" test for pairs of equivalent fractions. The examples illustrate the generalization that two fractional numbers, $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $ad = bc$. This generalization is used in Exercise 3 as an aid in solving fractional equations. This same concept will be extended later, when ratio and proportions are studied in Module 3 of this unit.

Fractional Numbers and the Number Line

Investigating the Ideas

For each set of equivalent fractions there is one fractional number and one point on the number line.

The number line below is divided into twelfths.
The letters *A* through *G* name sets of equivalent fractions.

$A = \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \ldots\}$

$B = \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\}$

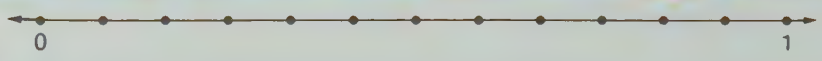
$C = \{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots\}$

$D = \{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots\}$

$E = \{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots\}$

$F = \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots\}$

$G = \{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots\}$



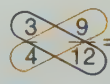
Can you find the point for each set of fractions on a copy of this number line?

Label each point that you find with the correct letter.

Discussing the Ideas

1. Any fraction from the set of equivalent fractions can be used to name the fractional number.
   - **A** Give two fractions to name the number for point *D*.
   - **B** Give two fractions to name the point between *A* and *B* on your number line.

2. The figure at the right suggests a shortcut that can be used to see whether two fractions are equivalent. Explain how to use the shortcut for these pairs of numbers.



$3 \cdot 12 = 4 \cdot 9 \Rightarrow \frac{3}{4}$ and $\frac{9}{12}$ are equivalent fractions.

A $\frac{2}{3}, \frac{9}{12}$ **B** $\frac{12}{10}, \frac{6}{5}$ **C** $\frac{8}{3}, \frac{14}{6}$ **D** $\frac{0}{6}, \frac{0}{7}$ **E** $\frac{7}{1}, \frac{14}{2}$

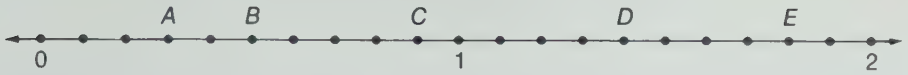
3. When two fractions such as $\frac{2}{3}$ and $\frac{4}{6}$ name the same number, we write $\frac{2}{3} = \frac{4}{6}$. To find x in an equation such as $\frac{3}{4} = \frac{x}{16}$, you can think $4 \cdot x = 3 \cdot 16$, so $4 \cdot x = 48$, and $x = 12$.

   Find x for each of these.

A $\frac{1}{2} = \frac{x}{10}$ **B** $\frac{2}{3} = \frac{x}{15}$ **C** $\frac{x}{8} = \frac{9}{24}$ **D** $\frac{x}{100} = \frac{3}{5}$ **E** $\frac{5}{4} = \frac{x}{24}$

Using the Ideas

1. Give the number for each lettered point on the number line. Use lowest-terms fractions.



2. Give the missing numbers. Then tell whether the fractions are equivalent.

A $\frac{3}{5} \quad \frac{9}{15}$ $\Rightarrow$ $3 \cdot 15 =$ ▒▒, $5 \cdot 9 =$ ▒▒

B $\frac{4}{12} \quad \frac{2}{6}$ $\Rightarrow$ $4 \cdot 6 =$ ▒▒, $12 \cdot 2 =$ ▒▒

C $\frac{6}{10} \quad \frac{15}{20}$ $\Rightarrow$ $6 \cdot 20 =$ ▒▒, $10 \cdot 15 =$ ▒▒

3. Tell whether the pair of fractions are equivalent or not equivalent.

A $\frac{1}{2}, \frac{8}{16}$ **C** $\frac{3}{1}, \frac{12}{4}$ **E** $\frac{20}{30}, \frac{200}{300}$ **G** $\frac{10}{12}, \frac{15}{18}$

B $\frac{3}{4}, \frac{7}{8}$ **D** $\frac{4}{6}, \frac{14}{21}$ **F** $\frac{3}{8}, \frac{8}{3}$ **H** $\frac{0}{11}, \frac{0}{41}$

4. Find the number for **x** in each equation.

A $\frac{1}{8} = \frac{x}{24}$ **D** $\frac{x}{12} = \frac{2}{3}$ **G** $\frac{5}{4} = \frac{x}{20}$ **J** $\frac{x}{78} = \frac{11}{13}$

B $\frac{5}{6} = \frac{x}{18}$ **E** $\frac{3}{4} = \frac{x}{16}$ **H** $\frac{20}{3} = \frac{x}{9}$ **K** $\frac{9}{10} = \frac{x}{150}$

C $\frac{3}{2} = \frac{x}{30}$ **F** $\frac{6}{24} = \frac{x}{4}$ **I** $\frac{8}{28} = \frac{x}{7}$ **L** $\frac{1}{8} = \frac{x}{1000}$

think

The Z-shaped region is called a **tetromino**. The figure shows how seven of these tetrominos may be fitted inside a 6 by 6 square.

Show how eight of the tetrominos can be fitted inside the square.

More practice, page S-13, Set 24

UTILIZATION

Exercises 1, 2, and 3 might be discussed orally, and Exercise 4 assigned as written work. Note that the "cross-products" method of solving these equations reduces the problem to that of solving a simple linear equation. Some students will be able to solve equations such as these mentally by thinking about equivalent fractions. Encourage them to do this whenever possible in solving simple fractional equations.

EXTENSION

Supplementary Exercise Set 24 on page S-13 is recommended for students who would benefit from further practice with the concepts of this lesson.

Enrichment: Ask students to solve the equations below.

1. $\frac{x+1}{16} = \frac{3}{4}$ $(x = 11)$

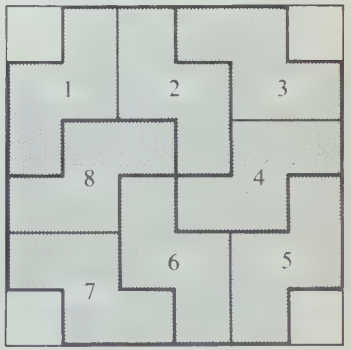
2. $\frac{8}{x-3} = \frac{4}{5}$ $(x = 13)$

3. $\frac{3}{10} = \frac{5x}{100}$ $(x = 6)$

4. $\frac{2}{3} = \frac{x+5}{4x}$ $(x = 3)$

Think Solution

By joining the Z-shaped pieces to form Y-shaped pieces, the puzzle can be solved more readily, as shown below.



Assignments

Minimum 1 and 2, oral; 3A–D, 4A–F.
Average 1–3, oral; 4A–I.
Maximum 1–3, oral; 4.

Objective

The student can name the opposite of any rational number and can apply the opposites principle.

PREPARATION

Up to this point, the student has dealt only with *fractional numbers*, that is, the subset of all rational numbers that are *non-negative*. This lesson introduces the complete set of rationals in much the same way as the set of integers was introduced.

INVESTIGATION

Since the Investigation requires relatively little student activity, you might prepare a large demonstration number line from a long strip of paper and on it show additional points and positive rational numbers. Then combine the Investigation with the Discussion exercises, having students record on the demonstration number line the symbols for the opposites of the positive rational numbers at the matching points on the number line.

DISCUSSION

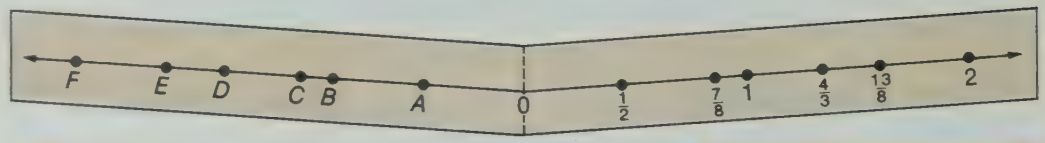
In Exercise 2, students should observe the use of a centred dash to denote the opposite of a positive rational number. Other symbols for fractional numbers will be introduced in succeeding lessons.

The importance of the opposites principle should not be underestimated (see Exercise 3). As was the case in the set of integers, the opposites principle will play a central role in developing computational rules for rational numbers.

The Set of Rational Numbers

Investigating the Ideas

Think about folding a number line at zero so that the two "halves" touch.

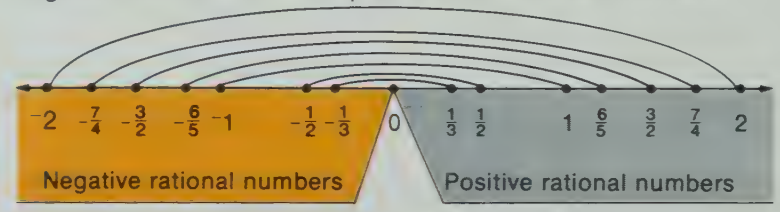


Can you give the letter that each of the labelled points (to the right of zero) would touch?

Discussing the Ideas

1. Suppose each fractional number has an opposite found by "touching" number line points. The number for the point where $\frac{1}{2}$ touches is called **negative** $\frac{1}{2}$. Give the name of each of the other points above.

2. The figure below shows two sets of numbers and the symbols we use for the negatives. These two sets together with zero make up the set of **rational numbers**.


-2 $-\frac{7}{4}$ $-\frac{3}{2}$ $-\frac{6}{5}$ -1 $-\frac{1}{2}$ $-\frac{1}{3}$
0
$\frac{1}{3}$ $\frac{1}{2}$ 1 $\frac{6}{5}$ $\frac{3}{2}$ $\frac{7}{4}$ 2
Negative rational numbers
Positive rational numbers

Write symbols for each of the letters *A* through *F* in the Investigation.

3. We accept the following principle for rational numbers:

The Opposites Principle

Each rational number has an opposite. The sum of any rational number and its opposite is zero.

A What is the opposite of $\frac{2}{3}$?

B What is the sum $\frac{2}{3} + \left(-\frac{2}{3}\right)$?

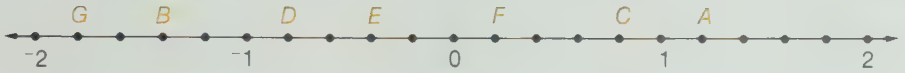
Using the Ideas

1. Sometimes the term **additive inverse** is used instead of *opposite*. Thus, the additive inverse of $\frac{3}{4}$ is $-\frac{3}{4}$, and $\frac{3}{4} + (-\frac{3}{4}) = 0$. Give the additive inverse of each rational number.

   A $\frac{3}{8}$ **B** $-\frac{5}{6}$ **C** $-\frac{1}{4}$ **D** $\frac{11}{5}$ **E** $-\frac{37}{45}$

2. Does zero have an additive inverse? What is it?

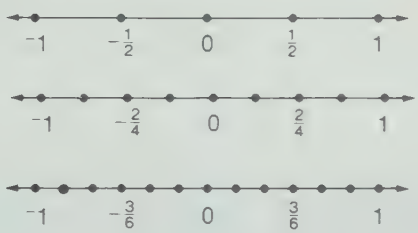
3. Give the rational number for each lettered point.



4. Give the rational number for n.

   A $\frac{3}{4} + n = 0$
   B $n + (-\frac{5}{8}) = 0$
   C $\frac{11}{4} + n = 0$
   D $n + (-\frac{7}{8}) = 0$
   E $n + (-\frac{11}{6}) = 0$
   F $\frac{8}{3} + n = 0$
   G $2 + n = 0$
   H $1\frac{3}{4} + n = 0$
   I $-\frac{2}{3} + n = 0$
   J $\frac{7}{8} + (-\frac{7}{8}) = n$
   K $-\frac{5}{4} + n = 0$
   L $1\frac{5}{6} + (-\frac{11}{6}) = n$
   M $-4 + n = 0$
   N $-\frac{13}{8} + 1\frac{5}{8} = n$
   O $n + \frac{5}{2} = 0$

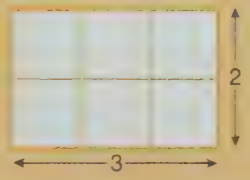
5. The number lines show that $-\frac{1}{2}$, $-\frac{2}{4}$, and $-\frac{3}{6}$ are equivalent fractions. Give three more fractions equivalent to these three.



6. Complete each sentence.
   A The opposite of a positive rational number is a ? rational number.
   B The opposite of a negative rational number is a ? rational number.
   C The opposite of zero is ?.

Think

Trace this figure without lifting your pencil from the paper. You will have to trace some segments twice. What is the shortest possible path for the tracing?



UTILIZATION

Since students have just finished studying the integers in Learning Unit E, they should have little difficulty extending the idea of opposites, or additive inverses, to the set of rational numbers. If your students grasp the ideas of the lesson readily, the exercises might be discussed and answered orally.

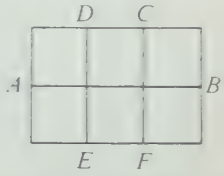
Note that Exercise 5 introduces equivalent sets of negative rational numbers. Contrasting the set $\{-\frac{1}{2}, -\frac{2}{4}, -\frac{3}{6}, -\frac{4}{8}, \ldots\}$ with the set $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\}$ may help students see that a negative rational number can be associated with a set of equivalent fractions in the same way as can a positive rational number.

EXTENSION

The exercises on page 36 of the *Workbook* may be assigned to provide further practice with the ideas presented in this lesson.

Think Solution

If the sides of the small squares are unit length, the minimum path is 19 units. The total length of all the sides of the squares in the figure is 17 units.



The figure would be traceable without repetition if points A and B were the only vertices formed by the intersection of an odd number of segments. We must have one extra path from points C, D, E, and F because these vertices are also vertices formed by the intersection of an odd number of segments. If the parts of the paths retraced are only $\overline{CD}$ and $\overline{EF}$ the length of the total path will be a minimum. One such tracing is shown on the annotated student page.

Assignments

Minimum 1–3, oral; 4. Average 1–3, oral; 4, 5. Maximum 1–3 and 6, oral; 4; 5.

Objective

The student can express the quotient of two integers as a rational number.

PREPARATION

Conceptually, this is a very important lesson for the student. The problem of deciding whether or not two different symbols represent the same rational number becomes increasingly difficult when the numerator or denominator is a negative integer. In this lesson the student learns the theorem $-\frac{a}{b} = \frac{^-a}{b} = \frac{a}{^-b}$ and $\frac{^-a}{^-b} = \frac{a}{b}$, where a and b are integers and $b \neq 0$. (See the Mathematics section of the introduction to this module for other remarks.)

INVESTIGATION

It may be useful for students to make numeral cards like those shown in the Investigation so that they can place one card above the other to represent a fraction. A total of 30 fractions representing 13 distinct rational numbers can be formed from the cards. You should not expect every student to be able to find all of them.

A. $\{\frac{^-4}{2}, \frac{4}{^-2}\}$	B. $\{\frac{^-3}{2}, \frac{3}{^-2}\}$	C. $\{\frac{^-4}{3}, \frac{4}{^-3}\}$
D. $\{\frac{^-2}{2}, \frac{2}{^-2}, \frac{^-3}{3}, \frac{3}{^-3}, \frac{^-4}{4}, \frac{4}{^-4}\}$		E. $\{\frac{^-3}{4}, \frac{3}{^-4}\}$
F. $\{\frac{^-2}{3}, \frac{2}{^-3}\}$	G. $\{\frac{^-2}{4}, \frac{2}{^-4}\}$	H. $\{\frac{2}{4}, \frac{^-2}{4}\}$
I. $\{\frac{2}{3}, \frac{^-2}{3}\}$	J. $\{\frac{3}{4}, \frac{^-3}{4}\}$	K. $\{\ \}$
L. $\{\frac{4}{3}, \frac{^-4}{3}\}$	M. $\{\frac{3}{2}, \frac{^-3}{2}\}$	N. $\{\frac{4}{2}, \frac{^-4}{2}\}$

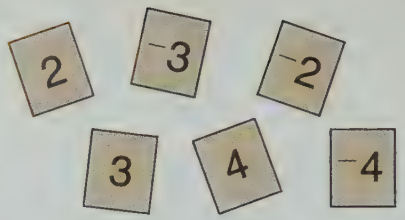
DISCUSSION

In Exercise 2A, $\frac{2}{3} + (-\frac{2}{3}) = 0$ because of the opposites principle. Thus, from Exercises 1 and 2A, we can conclude that $\frac{^-2}{3}$ and $-\frac{2}{3}$ are two different symbols for the additive inverse of $\frac{2}{3}$. Hence $\frac{^-2}{3} = -\frac{2}{3}$ (Exercise 2B).

In the remaining portion of the module the most frequently used symbol for negative rational numbers will be $\frac{^-a}{b}$, where a and b are positive integers.

Integer Pairs and Rational Numbers

Investigating the Ideas



Suppose we assume that the quotient of every pair of nonzero integers is a rational number. Give as many rational numbers as you can (written in fraction form) using the integers on these cards.

EXAMPLES: $\frac{^-2}{3}, \frac{4}{^-2}, \frac{^-3}{^-4}, \frac{2}{3}$

How many of your numbers can you locate on the number line above?

Discussing the Ideas

1. Assuming the basic principles hold for all the rational numbers, can you find this sum? $\frac{2}{3} + \frac{^-2}{3} = n$

2. A Find this sum. $\frac{2}{3} + \left(-\frac{2}{3}\right) = n$
   B Now can you explain why $\frac{^-2}{3} = -\frac{2}{3}$?

3. Can you tell what rational number $2 \div {}^-3$ might be? HINT: Write it as a fraction and multiply the numerator and denominator by $^-1$.

4. Explain why $\frac{^-3}{^-4}$ and $\frac{3}{4}$ are symbols for the same rational number. (Use the hint in Exercise 3.)

5. Since you cannot divide by zero, there is no rational number with denominator zero. What rational number do you think $\frac{0}{^-2}$ or $\frac{0}{^-3}$ might be?

6. Each rational number below is either positive $\frac{5}{8}$ or negative $\frac{5}{8}$. Tell which.
   A $5 \div 8$ B $^-5 \div {}^-8$ C $5 \div 8$ D $\frac{^-5}{8}$ E $\frac{5}{^-8}$ F $\frac{^-5}{^-8}$

Using the Ideas

1. Write each quotient as a fraction with a positive denominator.

EXAMPLE: $\frac{2}{-3} = \frac{2 \cdot (-1)}{-3 \cdot (-1)} = \frac{-2}{3}$

A $4 \div -3$ **B** $-5 \div -6$ **C** $3 \div -5$ **D** $-6 \div -10$ **E** $0 \div -7$

2. Answer **T** (true) or **F** (false) for each part.

A $\frac{-2}{3} = -\frac{2}{3}$ **B** $\frac{2}{-3} = \frac{2}{3}$ **C** $\frac{-2}{-3} = -\frac{2}{3}$ **D** $\frac{-3}{-4} = \frac{3}{4}$ **E** $\frac{-3}{-4} = -\frac{3}{4}$

3. Give the point P or Q for each rational number below.

A $\frac{8}{5}$ **B** $\frac{-8}{-5}$ **C** $8 \div -5$ **D** $8 \div 5$ **E** $-8 \div -5$ **F** $\frac{8}{5}$

4. Give the point (R, S, T, or U) for each number below.

A $\frac{7}{-10}$ **D** $-7 \div 10$ **G** $\frac{-7}{-10}$ **J** -0.7

B $\frac{-10}{7}$ **E** $\frac{-7}{10}$ **H** $7 \div 10$ **K** $\frac{-7}{10}$

C $-10 \div -7$ **F** $-10 \div 7$ **I** $\frac{10}{-7}$ **L** $\frac{10}{7}$

think

Turn over any two adjacent coins at a time in arrangement **A**. What is the fewest number of moves to get arrangement **B**?

UTILIZATION

One of the desired outcomes of this lesson is that the student be able to represent any rational number as a fraction with a positive integral denominator. Addition and subtraction of rational numbers can be performed with less difficulty when all denominators are positive. Comparing two rational numbers or deciding whether a fraction represents a positive or negative rational number is simple when the denominators are positive.

EXTENSION

Workbook page 37 and *Duplicator Masters* page 27 provide additional exercises suitable for use with this lesson.

Enrichment: Have students use the "cross-products" method to tell which pairs of fractions are equivalent. Examples:

$\frac{-4}{3}, \frac{20}{-15}$ *Solution:* $3 \cdot 20 = -4 \cdot -15 = 60$.
The fractions are equivalent.

$\frac{-2}{-5}, \frac{-6}{-15}$ *Solution:* $-2 \cdot -15 = 30$, $-5 \cdot 6 = -30$.
The fractions are not equivalent.

A. $\frac{-3}{4}, \frac{-9}{12}$
B. $\frac{-3}{-9}, \frac{10}{30}$
C. $\frac{1}{-4}, \frac{-20}{80}$
D. $\frac{5}{10}, \frac{-13}{-26}$
E. $\frac{18}{24}, \frac{-8}{-12}$
F. $\frac{-56}{100}, \frac{14}{-25}$

Think Solution

Arrangement B can be formed from arrangement A in three moves.

H	H	H	T	T	T
1	2	3	4	5	6

If the coins of arrangement A are numbered 1 through 6 as shown, then the steps are given below.

Step 1: Turn over 2 and 3.
Step 2: Turn over 3 and 4.
Step 3: Turn over 4 and 5.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can find the sum of two rational numbers represented by fractions which have the same positive denominators.

PREPARATION

Review the basic addition theorem for positive rational numbers having the same denominator. That is,

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}.$$

INVESTIGATION

The purpose of the Investigation is to help students realize that, if two rational numbers have the same positive denominator, then the problem of finding the sum of the two numbers is reduced to that of finding the sum of the two integers that are the numerators.

Some students may benefit by actually making slips of paper for the numerals and placing them over the blanks in order to find and record the possible equations.

Solutions:

$\frac{2}{10}+\frac{3}{10}=\frac{5}{10}$ $\quad\frac{3}{10}+\frac{2}{10}=\frac{5}{10}$ $\quad\frac{-2}{10}+\frac{-3}{10}=\frac{-5}{10}$ $\quad\frac{-3}{10}+\frac{-2}{10}=\frac{-5}{10}$ $\quad\frac{5}{10}+\frac{-3}{10}=\frac{2}{10}$ $\quad\frac{-3}{10}+\frac{5}{10}=\frac{2}{10}$ $\quad\frac{3}{10}+\frac{-5}{10}=\frac{-2}{10}$ $\quad\frac{-5}{10}+\frac{3}{10}=\frac{-2}{10}$ $\quad\frac{3}{10}+\frac{5}{10}=\frac{8}{10}$ $\quad\frac{5}{10}+\frac{3}{10}=\frac{8}{10}$

$\frac{-3}{10}+\frac{-5}{10}=\frac{-8}{10}$ $\quad\frac{-5}{10}+\frac{-3}{10}=\frac{-8}{10}$ $\quad\frac{8}{10}+\frac{-5}{10}=\frac{3}{10}$ $\quad\frac{-5}{10}+\frac{8}{10}=\frac{3}{10}$ $\quad\frac{8}{10}+\frac{-3}{10}=\frac{5}{10}$ $\quad\frac{-3}{10}+\frac{8}{10}=\frac{5}{10}$ $\quad\frac{2}{10}+\frac{-2}{10}=\frac{0}{10}$ $\quad\frac{-2}{10}+\frac{2}{10}=\frac{0}{10}$ $\quad\frac{3}{10}+\frac{-3}{10}=\frac{0}{10}$ $\quad\frac{-3}{10}+\frac{3}{10}=\frac{0}{10}$

$\frac{5}{10}+\frac{-5}{10}=\frac{0}{10}$ $\quad\frac{-5}{10}+\frac{5}{10}=\frac{0}{10}$ $\quad\frac{8}{10}+\frac{-8}{10}=\frac{0}{10}$ $\quad\frac{-8}{10}+\frac{8}{10}=\frac{0}{10}$ $\quad\frac{5}{10}+\frac{-2}{10}=\frac{3}{10}$ $\quad\frac{-2}{10}+\frac{5}{10}=\frac{3}{10}$ $\quad\frac{-5}{10}+\frac{2}{10}=\frac{-3}{10}$ $\quad\frac{2}{10}+\frac{-5}{10}=\frac{-3}{10}$

Since there are a large number of equations that can be formed, you should not expect individual students to find all of them.

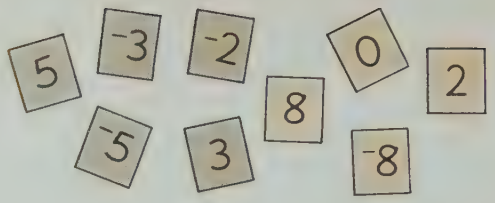
DISCUSSION

Exercise 3 contains examples of the type of problem that students will encounter in the Utilization stage. If a denominator is negative, the denominator and numerator are both multiplied by $^-1$ in order to get a postive denominator.

Addition of Rational Numbers

Investigating the Ideas

Suppose all the usual basic principles and rules hold for the set of rational numbers. That is, you can use the same rules for adding rational numbers as you used for positive rational numbers.



?	How many ways can you find to place the slips of paper above on the shaded spaces below to form equations?	Record each equation that you find.

$$\frac{\quad}{10}+\frac{\quad}{10}=\frac{\quad}{10}$$

Discussing the Ideas

1. A Would the placement of the slips of paper be any different if the denominator were 15 instead of 10?
   B Explain how adding positive and negative rational numbers is much like adding positive and negative integers.

2. In the Investigation, each denominator was positive. Can you write a fraction for any rational number so that it has a positive denominator?

3. How would you change these problems before trying to add?

   A $\frac{-5}{12}+\frac{1}{-12}=x$ $\qquad$ B $\frac{-3}{8}+\frac{-2}{-8}=n$ $\qquad$ C $-\frac{3}{7}+\frac{2}{7}=d$

4. What is the sum in each part of Exercise 3?

5. Explain why the sum in A and B is the same. $\qquad$ A $\frac{-7}{10}+\frac{5}{10}$ $\qquad$ B $\frac{7}{-10}+\frac{-5}{-10}$

6. Give the missing numerator.

   If $\frac{a}{b}$ and $\frac{c}{b}$ are any two rational numbers with $b>0$, then $\frac{a}{b}+\frac{c}{b}=\frac{\square}{b}$.

Using the Ideas

1. Find the sums.

A $\frac{3}{6} + \frac{2}{6} = n$ **B** $\frac{3}{10} + \frac{-2}{10} = r$ **C** $\frac{-3}{8} + \frac{2}{8} = m$ **D** $\frac{-3}{4} + \frac{-2}{4} = x$

E $\frac{7}{12} + \frac{5}{12} = c$ **F** $\frac{7}{10} + \frac{-5}{10} = y$ **G** $\frac{-7}{100} + \frac{5}{100} = s$ **H** $\frac{-7}{10} + \frac{-5}{10} = d$

I $\frac{-7}{8} + \frac{3}{8} = p$ **J** $\frac{7}{5} + \frac{-3}{5} = t$ **K** $\frac{7}{6} + \frac{3}{6} = q$ **L** $\frac{-7}{9} + \frac{-3}{9} = a$

2. Find the sums. HINT: Change to positive denominators before adding.

A $\frac{3}{7} + \frac{-4}{-7}$ **B** $\frac{1}{-4} + \frac{3}{4}$ **C** $\frac{-4}{-10} + \frac{-3}{-10}$

D $-\frac{6}{8} + \frac{1}{8}$ **E** $\frac{5}{6} + \frac{-1}{-6}$ **F** $\frac{-2}{-5} + \frac{-4}{-5}$

G $\frac{-7}{-12} + \frac{5}{12}$ **H** $-\frac{1}{4} + \frac{-5}{-4}$ **I** $\frac{6}{-10} + \frac{-3}{10}$

J $\frac{-3}{-8} + \frac{-3}{-8}$ **K** $\frac{12}{100} + \frac{7}{100}$ **L** $\frac{-2}{-9} + \frac{4}{-9}$

3. Find the sums.

A $\frac{3}{4} + \frac{-1}{4} + \frac{5}{4}$ **B** $\frac{-1}{8} + \frac{-3}{8} + \frac{7}{8}$ **C** $\frac{11}{10} + \frac{-3}{10} + \frac{-7}{10}$

D $1 + \frac{-1}{3} + \frac{-2}{3}$ **E** $-1 + \frac{1}{4} + \frac{1}{4}$ **F** $-\frac{1}{6} + \frac{-5}{6} + \frac{-1}{6}$

G $\frac{-9}{10} + \frac{4}{10} + \frac{3}{10}$ **H** $\frac{11}{3} + \frac{-8}{3} + \frac{-2}{3}$ **I** $\frac{-3}{4} + \frac{-2}{4} + \frac{9}{4}$

4. Solve the equations. The replacement set is the set of rational numbers.

A $\frac{-3}{4} + x = \frac{1}{4}$ **B** $\frac{11}{6} + r = \frac{5}{6}$ **C** $y + \frac{-1}{3} = -\frac{2}{3}$ **D** $\frac{5}{8} + z = \frac{-1}{8}$ **E** $\frac{-1}{2} + s = \frac{1}{2}$ **F** $\frac{9}{10} + t = 0$

G $\frac{-17}{20} + \frac{-2}{20} = x$ **H** $\frac{1}{7} + \frac{-8}{7} = y$ **I** $n + \frac{-4}{15} = 0$ **J** $\frac{-13}{10} + k = \frac{-3}{10}$ **K** $\frac{7}{6} + s = 1$ **L** $m + \frac{-3}{4} = \frac{3}{4}$

M $\frac{-1}{6} + q = \frac{-5}{6}$ **N** $2 + \frac{-1}{2} = t$ **O** $h + \frac{3}{10} = \frac{-7}{10}$ **P** $\frac{37}{100} + r = \frac{50}{100}$ **Q** $\frac{-1}{10} + j = \frac{-4}{10}$ **R** $p + \frac{-63}{100} = \frac{-29}{100}$

think

Find the missing base in each equation.

A $13_{(ten)} = 31_{(a)}$ **B** $23_{(7)} = 32_{(b)}$ **C** $21_{(3)} = 12_{(c)}$ **D** $12_{(9)} = 21_{(d)}$

UTILIZATION

In some of the exercises negative rational numbers are indicated by a fraction with a centred minus sign. In such instances, most students will find it more convenient to express such fractions with a negative numerator before adding. Thus, in Exercise 2D,

$$-\frac{6}{8} + \frac{1}{8} = \frac{-6}{8} + \frac{1}{8} = \frac{-6+1}{8} = \frac{-5}{8}$$

After sufficient experience with this type of problem, students will probably be able to do the intermediate step mentally.

Students are expected to solve the equations in Exercise 4 by finding the missing addends rather than by using subtraction, since the subtraction operation has not yet been discussed.

EXTENSION

Suitable additional practice assignments may be made from *Workbook* page 38 and *Arithmetic Skill Card* F-5.

Think Solution

These equations can be solved by expressing the first term as a base-ten numeral and the second term in expanded notation.

A. $13_{(ten)} = 3a + 1 \Rightarrow 3a = 12 \Rightarrow a = 4$

B. $23_{(7)} = 17_{(ten)}$. $17_{(ten)} = 3b + 2 \Rightarrow 3b = 15 \Rightarrow b = 5$

C. $21_{(3)} = 7_{(ten)}$. $7_{(ten)} = c + 2 \Rightarrow c = 5$

D. $12_{(9)} = 11_{(ten)}$. $11_{(ten)} = 2d + 1 \Rightarrow 2d = 10 \Rightarrow d = 5$

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

Given two or more rational numbers represented by fractions with unlike denominators, the student can find a common denominator and find the sum of the numbers.

PREPARATION

The student should already be familiar with the problem of finding common denominators for positive rational numbers. Therefore, the only new idea in this lesson is that the same methods are used when negative fractions occur. Present a brief review of choosing common denominators for pairs of positive rational numbers.

INVESTIGATION

The degree of success that students have with the Investigation should help you decide how much time should be spent on a review of finding common denominators.

DISCUSSION

In discussing Exercise 1, excourage students to relate the process of finding the LCD of two fractions to that of finding the LCM of two whole numbers.

Exercise 5 reviews the process of adding several mixed numerals. Your emphasis on this type of problem should be directed by the abilities and needs of your students. If your students have difficulty with this type of problem, you may need to make selective assignments from the exercise sets suggested in the Extension section on the following page.

Finding Common Denominators

Investigating the Ideas

Can you match the equations so that the one on the right will help you solve the one on the left?

1. $\frac{3}{4} + \frac{-2}{3} = n$
2. $\frac{-4}{5} + \frac{1}{3} = r$
3. $\frac{-3}{8} + \frac{-1}{6} = t$
4. $\frac{5}{6} + \frac{-3}{4} = a$
5. $\frac{-7}{10} + \frac{23}{100} = y$

A $\frac{-9}{24} + \frac{-4}{24} = \frac{-13}{24}$

B $\frac{-70}{100} + \frac{23}{100} = \frac{-47}{100}$

C $\frac{9}{12} + \frac{-8}{12} = \frac{1}{12}$

D $\frac{-12}{15} + \frac{5}{15} = \frac{-7}{15}$

E $\frac{10}{12} + \frac{-9}{12} = \frac{1}{12}$

Discussing the Ideas

1. Explain your matchings for each part of the Investigation.

2. Give the **least common denominator** (LCD) that you would use for adding each pair of fractions.

   A $\frac{\square}{4} + \frac{\square}{3}$ B $\frac{\square}{2} + \frac{\square}{6}$ C $\frac{\square}{5} + \frac{\square}{4}$ D $\frac{\square}{2} + \frac{\square}{10}$ E $\frac{\square}{10} + \frac{\square}{100}$

3. Explain how to find the missing numerators.

   A $\frac{5}{8} = \frac{\square}{24}$ B $\frac{-1}{3} = \frac{\square}{12}$ C $\frac{2}{3} = \frac{\square}{30}$ D $\frac{-3}{4} = \frac{\square}{12}$ E $\frac{-3}{10} = \frac{\square}{30}$

4. Study the example: $\frac{2}{3} + \frac{-1}{2} = \frac{2 \cdot 2}{3 \cdot 2} + \frac{-1 \cdot 3}{2 \cdot 3} = \frac{4}{6} + \frac{-3}{6} = \frac{4 + -3}{6} = \frac{1}{6}$

   The LCD is 6.

   What is the sum $\frac{-2}{3} + \frac{1}{4}$?

5. When adding several positive rational numbers using mixed numerals, it is helpful to arrange your work as shown in the example.

   A Explain how $29\frac{23}{12}$ was found.

   B Why does $29\frac{23}{12}$ represent the same number as $30\frac{11}{12}$?

EXAMPLE:

$$\begin{array}{rr} 16\frac{2}{3} & 16\frac{8}{12} \\ 3\frac{3}{4} \rightarrow & 3\frac{9}{12} \\ +\,10\frac{1}{2} & +\,10\frac{6}{12} \\ \hline & 29\frac{23}{12} = 30\frac{11}{12} \end{array}$$

Using the Ideas

1. Copy the equations and give the numbers for ***a***, ***b***, ***c***.

A $\frac{1}{4}+\frac{1}{3}=\frac{a}{12}+\frac{b}{12}=\frac{c}{12}$

B $\frac{1}{6}+\frac{3}{8}=\frac{a}{24}+\frac{b}{24}=\frac{c}{24}$

C $\frac{-3}{10}+\frac{7}{100}=\frac{a}{100}+\frac{b}{100}=\frac{c}{100}$

D $\frac{-3}{16}+\frac{-5}{12}=\frac{a}{48}+\frac{b}{48}=\frac{c}{48}$

2. Find the sums.

A $\frac{5}{7}+\frac{-1}{7}$

B $\frac{-3}{4}+\frac{1}{4}$

C $\frac{3}{10}+\frac{-7}{10}$

D $\frac{-1}{2}+\frac{-1}{4}$

E $\frac{-1}{3}+\frac{1}{2}$

F $\frac{1}{4}+\frac{-5}{8}$

G $\frac{-1}{5}+\frac{-3}{10}$

H $\frac{1}{6}+\frac{-3}{2}$

I $\frac{3}{4}+\frac{3}{8}$

J $\frac{1}{4}+\frac{1}{5}$

K $\frac{-1}{2}+\frac{1}{3}$

L $\frac{1}{3}+\frac{-1}{4}$

M $\frac{-5}{8}+\frac{-1}{6}$

N $\frac{2}{3}+\frac{3}{4}$

O $\frac{5}{6}+\frac{-1}{2}$

P $\frac{-5}{8}+\frac{3}{20}$

3. Find the sums.

A $42\frac{5}{6} + 21\frac{1}{3} + 37\frac{2}{3}$

B $9\frac{3}{8} + 7\frac{1}{4} + 5\frac{3}{16}$

C $16\frac{1}{2} + 15\frac{1}{3} + 14\frac{1}{6}$

D $23\frac{1}{6} + 18\frac{1}{4} + 12\frac{1}{2}$

E $31\frac{5}{8} + 17\frac{1}{2} + 19\frac{1}{4}$

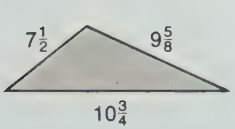
F $123\frac{1}{10} + 243\frac{3}{10} + 366\frac{3}{5}$

G $23\frac{1}{2} + 76\frac{9}{10} + 84\frac{17}{100}$

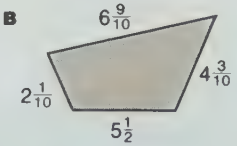
H $243\frac{3}{10} + 184\frac{8}{25} + 769\frac{61}{100}$

4. Find the perimeter (distance around) of each region. The units of length for each side are marked on the figure.

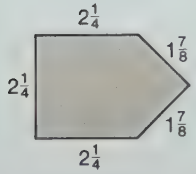
A



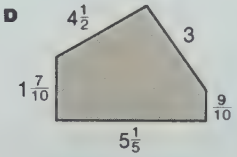
B



C



D



think

Give the missing numbers so that this will be a magic square.

7	$^{-}7$	$^{-}6$	4
$^{-}4$	2	1	
	$^{-}2$		3
		6	

More practice, page S-14, Set 25

UTILIZATION

Assign the exercises according to the needs and abilities of your students. Altogether there are 32 responses called for in this set of exercises.

EXTENSION

For a variety of possible additional practice assignments, see Supplementary Exercise Set 25 on page S-14, *Workbook* page 39, *Duplicator Masters* page 28, and *Arithmetic Skill Cards* F-5 and F-7.

Enrichment: Ask students to complete some addition squares by finding the sums in both rows and columns. (For your convenience, the sums are shown in parentheses.)

$\frac{-3}{5}$	$\frac{1}{2}$	$(\frac{-1}{10})$
$\frac{7}{10}$	$\frac{-3}{4}$	$(\frac{-1}{20})$
$(\frac{1}{10})$	$(\frac{-1}{4})$	$(\frac{-3}{20})$

$\frac{2}{3}$	$\frac{3}{4}$	$(1\frac{7}{12})$
$\frac{-1}{6}$	$\frac{-1}{2}$	$(\frac{-2}{3})$
$(\frac{1}{2})$	$(\frac{1}{4})$	$(\frac{3}{4})$

Think Solution

The sum of the integers in the first row is $^{-}2$. Once this is determined, most students should be able to find the integers for the remaining squares quite readily.

Assignments

Minimum 1, 2A–H, 3A–D.
Average 1, 2A–H, 3A–D, 4A, B.
Maximum 2–4.

Objective

The student can find the difference of two rational numbers expressed by fractions with like or unlike denominators.

PREPARATION

Subtraction with positive rational numbers will be familiar to students at this level. However, you might review the process before students begin the Investigation.

INVESTIGATION

The purpose of the Investigation is to relate addition to subtraction in the set of rational numbers and to provide the opportunity for students to discover that subtraction of rational numbers can be reduced to a problem of subtraction of integers.

Thus, if students know that $^-7 + 2 = {}^-5$, they also know that $^-5 - 2 = {}^-7$ and can relate this to rational numbers with like denominators:

$\frac{^-7}{10} + \frac{2}{10} = \frac{^-5}{10}$ and $\frac{^-5}{10} - \frac{2}{10} = \frac{^-7}{10}$.

DISCUSSION

Throughout the Discussion, stress the fact that finding a difference in a subtraction problem is equivalent to finding a missing addend in an addition problem. Thus,

$\frac{6}{10} - \frac{^-2}{10} = \frac{8}{10}$ because $\frac{8}{10} + \frac{^-2}{10} = \frac{6}{10}$.

Exercise 3 contains the generalization for subtraction using fractions with like denominators:

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b},\ b \neq 0.$$

In discussing Exercise 4, note that after a common positive denominator is selected, the generalization stated in Exercise 3 can be applied.

Subtracting Rational Numbers

Investigating the Ideas

Select one of these sets of numbers. Write two addition and two subtraction equations using your numbers.

$\{^-7, 2, ^-5\}$

$\{8, ^-2, 6\}$

$\{^-3, ^-4, ^-7\}$

? Can you use your numbers as numerators in the two addition and the two subtraction equations below?

$\frac{\square}{10} + \frac{\square}{10} = \frac{\square}{10}$ $\quad$ $\frac{\square}{10} - \frac{\square}{10} = \frac{\square}{10}$

$\frac{\square}{10} + \frac{\square}{10} = \frac{\square}{10}$ $\quad$ $\frac{\square}{10} - \frac{\square}{10} = \frac{\square}{10}$

Discussing the Ideas

1. Find these differences. Explain how to check your answers by adding.

   A $6 - {}^-2 = n$ $\quad$ B $^-4 - 3 = s$ $\quad$ C $^-2 - {}^-6 = x$

2. Now try to find these differences. Remember, you should be able to check your answers by adding.

   A $\frac{6}{10} - \frac{^-2}{10} = m$ $\quad$ B $\frac{^-4}{10} - \frac{3}{10} = t$ $\quad$ C $\frac{^-2}{10} - \frac{^-6}{10} = y$

3. Give the missing numerator.

   If $\frac{a}{b}$ and $\frac{c}{b}$ are any two rational numbers with $b > 0$, then $\frac{a}{b} - \frac{c}{b} = \frac{\square}{b}$.

4. Explain how to find these differences.

   A $\frac{3}{4} - \frac{^-2}{3} = q$ $\quad$ B $\frac{^-5}{10} - \frac{1}{2} = a$ $\quad$ C $\frac{^-3}{8} - \frac{^-1}{4} = r$

Using the Ideas

1. Find the differences.

A $\frac{5}{8} - \frac{-3}{8}$	**D** $\frac{-7}{10} - \frac{3}{10}$	**G** $\frac{5}{6} - \frac{-1}{6}$	**J** $\frac{3}{8} - \frac{-7}{8}$	**M** $\frac{27}{100} - \frac{11}{100}$
B $\frac{-2}{10} - \frac{6}{10}$	**E** $\frac{3}{8} - \frac{-5}{8}$	**H** $\frac{-7}{12} - \frac{5}{12}$	**K** $\frac{-6}{10} - \frac{9}{10}$	**N** $\frac{-139}{100} - \frac{-257}{1000}$
C $\frac{4}{5} - \frac{-8}{5}$	**F** $\frac{-1}{4} - \frac{-3}{4}$	**I** $\frac{-8}{10} - \frac{-4}{10}$	**L** $\frac{1}{16} - \frac{-7}{16}$	**O** $\frac{3}{10} - \frac{7}{10}$

2. Find the differences.

A $\frac{7}{8} - \frac{1}{2}$	**C** $\frac{5}{6} - \frac{1}{4}$	**E** $\frac{8}{10} - \frac{3}{5}$	**G** $\frac{9}{10} - \frac{3}{4}$	**I** $\frac{63}{100} - \frac{1}{2}$
B $\frac{3}{4} - \frac{1}{3}$	**D** $\frac{7}{10} - \frac{1}{2}$	**F** $\frac{5}{12} - \frac{1}{3}$	**H** $\frac{7}{8} - \frac{5}{6}$	**J** $\frac{257}{1000} - \frac{-1}{10}$

3. Find the differences.

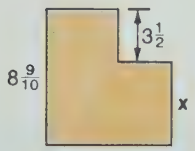
A $\frac{7}{8} - \frac{-1}{4}$	**D** $\frac{2}{3} - \frac{-1}{6}$	**G** $\frac{-9}{10} - \frac{2}{5}$	**J** $\frac{-3}{8} - \frac{3}{6}$	**M** $\frac{2}{3} - \frac{-1}{2}$
B $\frac{-6}{10} - \frac{2}{5}$	**E** $\frac{7}{4} - \frac{3}{8}$	**H** $\frac{1}{10} - \frac{-1}{4}$	**K** $\frac{2}{5} - \frac{-1}{4}$	**N** $\frac{2}{3} - \frac{1}{2}$
C $\frac{5}{12} - \frac{3}{4}$	**F** $\frac{5}{6} - \frac{-6}{10}$	**I** $\frac{-4}{5} - \frac{-3}{10}$	**L** $\frac{-4}{10} - \frac{-1}{2}$	**O** $\frac{-2}{3} - \frac{-1}{2}$

4. Find the differences.

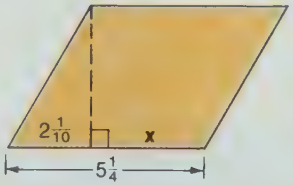
A $16\frac{5}{6} - 11\frac{1}{4}$ **B** $43\frac{1}{2} - 19\frac{5}{16}$ **C** $66\frac{9}{10} - 44\frac{3}{5}$ **D** $107\frac{1}{3} - 94\frac{1}{2}$ **E** $412\frac{1}{8} - 365\frac{1}{4}$

5. Find the distance **x** for each figure.

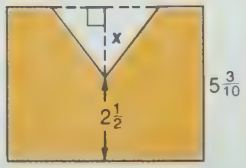
A



B



C ★



think

1. Arrange five ones (1) and four negative ones ($^-1$) in the nine squares so that the product of the numbers in each row, each column, and the two diagonals is 1.
2. Arrange four ones and five negative ones in the nine squares so that the product of the numbers in each row, each column, and the two diagonals is $^-1$.

More practice, page S-14, Set 26

UTILIZATION

Since there are nearly 50 subtraction exercises in the lesson, your assignment should be made selectively, according the students' needs. Suggest that students use addition to check some of the exercises.

EXTENSION

Supplementary Exercise Set 26 on page S-14, *Workbook* page 40, *Duplicator Masters* page 29, and *Arithmetic Skill Cards* F-6 and F-8 provide exercises that would be suitable for students who would benefit from further practice.

Enrichment: Have students complete these subtraction squares. They should subtract the numbers in column B from the numbers in Column A and then subtract the numbers in Row E from the numbers in row D. (Answers are shown in parentheses.)

	A	*B*	*C*
D	$\frac{7}{10}$	$\frac{1}{2}$	$(\frac{1}{5})$
E	$\frac{-1}{3}$	$\frac{5}{6}$	$(\frac{-7}{6})$
F	$(1\frac{1}{30})$	$(\frac{-1}{3})$	$(1\frac{11}{30})$

A	*B*	*C*
$\frac{-1}{4}$	$\frac{-2}{3}$	$(\frac{5}{12})$
$\frac{3}{8}$	$\frac{-3}{5}$	$(\frac{39}{40})$
$(\frac{-5}{8})$	$(\frac{-1}{15})$	$(\frac{-67}{120})$

Think Solution

1	$^-1$	1
$^-1$	$^-1$	$^-1$
1	$^-1$	1

Assignments

Minimum 1A–F, 2A–D, 3A–F.
Average 1G–O, 2E–J, 3G–O, 4A–C, 5A.
Maximum 1–5, alternate parts.

Objective

The student can find the product of two unit fractions and can find the product of a unit fraction and an integer.

PREPARATION

Review the basic principles for multiplication of rational numbers. The principles are assumed to hold for negative rational numbers as well as non-negative rational numbers.

Commutative Principle:
For all rational numbers a and b,
$a \cdot b = b \cdot a$.

One Principle:
For every rational number a,
$a \cdot 1 = a$.

Associative Principle:
For all rational numbers a, b, and c,
$a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Distributive Principle:
For all rational numbers a, b, and c,
$a \cdot (b + c) = a \cdot b + a \cdot c$.

After this review, move on to the Discussion Exercises, which are devoted to developing the generalizations known as the *unit fraction principles*.

DISCUSSION

The Discussion Exercises emphasize the meaning of the term *unit fraction* and the use of the unit fraction principles.

This lesson provides the necessary background for development of the multiplication algorithm in the next lesson.

Basic Principles for Multiplication

Discussing the Ideas

1. The fraction $\frac{1}{4}$ compares the shaded portion of the square region to the whole region. The fraction $\frac{1}{4}$ is an example of a unit fraction. A **unit fraction** is a fraction whose numerator is 1 and whose denominator is a positive integer.

What unit fractions are suggested by these regions?

A

B

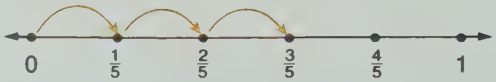
C

2. The regions and the number line below suggest multiplication with unit fractions. Study each figure and complete each equation.

A

$\frac{1}{3}$ of a region $\quad$ $\frac{1}{2}$ of $\frac{1}{3}$ $\quad$ $\frac{1}{6}$ of a region

We write: $\frac{1}{2} \cdot \frac{1}{3} =$ ▨.



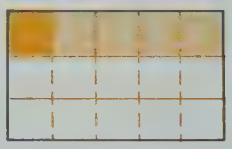
3 jumps of $\frac{1}{5}$

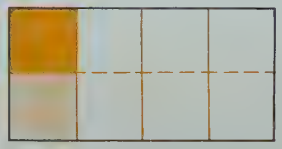
We write: $3 \cdot \frac{1}{5} =$ ▨.

3. Discussion Exercise 2 suggests the following generalizations concerning unit fractions.

Unit Fraction Principles	
For every pair of unit fractions, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$.	If a is a non-negative integer, then $a \cdot \frac{1}{b} = \frac{a}{b}$.

Give the multiplication fact illustrated by each region or number line.

A 

B 

C

D

Using the Ideas

1. Solve the equations.

A $\frac{1}{4} \cdot \frac{1}{3} = n$ **C** $\frac{1}{2} \cdot n = \frac{1}{10}$ **E** $\frac{1}{7} \cdot \frac{1}{6} = n$ **G** $\frac{1}{100} \cdot n = \frac{1}{1000}$

B $\frac{1}{8} \cdot \frac{1}{2} = n$ **D** $n \cdot \frac{1}{4} = \frac{1}{12}$ **F** $\frac{1}{10} \cdot n = \frac{1}{100}$ **H** $n \cdot \frac{1}{100} = \frac{1}{200}$

2. Find the products.

A $4 \cdot \frac{1}{5}$ **B** $6 \cdot \frac{1}{8}$ **C** $0 \cdot \frac{1}{8}$ **D** $6 \cdot \frac{1}{6}$ **E** $12 \cdot \frac{1}{3}$

3. If a is a nonzero whole number, then $a \cdot \frac{1}{a} =$ ▥.

4. Assume that the basic principles hold for the set of rational numbers. Give the principle illustrated by each equation.

A $\frac{2}{7} \cdot \frac{8}{8} = \frac{2}{7}$ **C** $\frac{-3}{4} \cdot \frac{7}{6} = \frac{7}{6} \cdot \frac{-3}{4}$ **E** $\frac{2}{3} \cdot \left(\frac{4}{5} + \frac{3}{4}\right) = \left(\frac{4}{5} + \frac{3}{4}\right) \cdot \frac{2}{3}$

B $\frac{-7}{5} \cdot 1 = \frac{-7}{5}$ **D** $\frac{-1}{2} \cdot \left(\frac{1}{3} \cdot \frac{1}{4}\right) = \left(\frac{-1}{2} \cdot \frac{1}{3}\right) \cdot \frac{1}{4}$ **F** $\left(\frac{3}{4} \cdot \frac{2}{3}\right) + \left(\frac{3}{4} \cdot \frac{1}{5}\right) = \frac{3}{4} \cdot \left(\frac{2}{3} + \frac{1}{5}\right)$

5. The example at the right proves that $^-1 \cdot \frac{2}{3} = \frac{-2}{3}$. Find the products.

A $^-1 \cdot \frac{1}{4}$ **D** $^-1 \cdot \frac{3}{8}$

B $^-1 \cdot \frac{3}{10}$ **E** $^-1 \cdot \frac{5}{6}$

C $^-1 \cdot \frac{2}{5}$ **F** $^-1 \cdot \frac{17}{100}$

EXAMPLE:

$$^-1 \cdot \frac{2}{3} + \frac{2}{3} = {}^-1 \cdot \frac{2}{3} + 1 \cdot \frac{2}{3}$$
$$= (^-1 + 1) \cdot \frac{2}{3}$$
$$= 0 \cdot \frac{2}{3} = 0$$

Since $\frac{-2}{3} + \frac{2}{3} = 0$ and $^-1 \cdot \frac{2}{3} + \frac{2}{3} = 0$, then $^-1 \cdot \frac{2}{3} = \frac{-2}{3}$.

6. The product of a positive rational number and negative 1 is the ? of the rational number.

7. Give the missing number for each ▥.

A $^-1 \cdot \left(\frac{-1}{3}\right) = {}^-1 \cdot \left(^-1 \cdot \frac{1}{3}\right) = (^-1 \cdot {}^-1) \cdot \frac{1}{3} =$ ▥

B $^-1 \cdot \left(\frac{-3}{8}\right) = {}^-1 \cdot \left(^-1 \cdot \frac{3}{8}\right) = (^-1 \cdot {}^-1) \cdot \frac{3}{8} =$ ▥

C $^-1 \cdot \left(\frac{-9}{10}\right) = {}^-1 \cdot \left(^-1 \cdot \frac{9}{10}\right) =$ ▥

D $^-1 \cdot \left(\frac{-3}{4}\right) =$ ▥

8. Give the number for n in each sentence.

A $^-1 \cdot \frac{1}{4} = n$ **B** $\frac{1}{10} \cdot {}^-1 = n$ **C** $^-1 \cdot n = \frac{-7}{8}$ **D** $n \cdot \frac{5}{6} = \frac{-5}{6}$

9. Study the examples then find the products.

A $\frac{-1}{3} \cdot \frac{1}{2}$ **D** $\frac{-1}{5} \cdot \frac{-1}{2}$

B $\frac{1}{8} \cdot \frac{-1}{3}$ **E** $\frac{1}{10} \cdot \frac{-1}{100}$

C $\frac{-1}{6} \cdot \frac{-1}{3}$ **F** $\frac{-1}{10} \cdot \frac{-1}{10}$

EXAMPLES:

$$\frac{-1}{2} \cdot \frac{1}{3} = {}^-1 \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right) = {}^-1 \cdot \frac{1}{6} = \frac{-1}{6}$$

$$\frac{-1}{4} \cdot \frac{-1}{2} = \left(^-1 \cdot \frac{1}{4}\right) \cdot \left(^-1 \cdot \frac{1}{2}\right) = (^-1 \cdot {}^-1) \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

UTILIZATION

A careful discussion of Exercises 5 and 7 will help to clarify multiplication involving negative rational numbers. Note the use of the basic principles in the example accompanying Exercise 5:

$$^-1 \cdot \tfrac{2}{3} + \tfrac{2}{3} = {}^-1 \cdot \tfrac{2}{3} + 1 \cdot \tfrac{2}{3} \quad \text{One Prin.}$$
$$= (^-1 + 1) \cdot \tfrac{2}{3} \quad \text{Distributive Prin.}$$
$$= 0 \cdot \tfrac{2}{3} = 0 \quad \text{Opposites Prin.}$$

This proves that $^-1 \cdot \frac{2}{3}$ is the opposite of $\frac{2}{3}$ (because it adds to $\frac{2}{3}$ to give 0). Hence, $\frac{-2}{3} = {}^-1 \cdot \frac{2}{3}$.

EXTENSION

Remedial: Review the fact that $^-1$ times any integer gives the opposite of that integer:

$^-1 \cdot 7 = {}^-7$, $^-1 \cdot {}^-6 = 6$, $^-1 \cdot {}^-23 = 23$, etc.

This may help students understand that multiplication of a unit fraction by $^-1$ gives the opposite of the unit fraction.

Assignments

Minimum 1–3, oral; 5–8.
Average 1–4, oral; 5–9.
Maximum 1–4, oral; 5–9.

Objective

Given two rational numbers, the student can use the usual multiplication algorithm to find the product.

PREPARATION

To prepare students for this Investigation, you might briefly review the ideas of the preceding lesson or conduct a short review of multiplication of integers.

INVESTIGATION

The factors in the given set are such that students should be able to find the product of any two of them because these cases have been studied previously in the module or in earlier years. Finding the product of all 5 factors in part D is equivalent to finding a product like $\frac{2}{3} \cdot \frac{-4}{5}$ because $2 \cdot \frac{1}{3} \cdot {}^{-}1 \cdot \frac{1}{5} \cdot 4$ can be grouped $(2 \cdot \frac{1}{3}) \cdot ({}^{-}1 \cdot 4 \cdot \frac{1}{5})$.

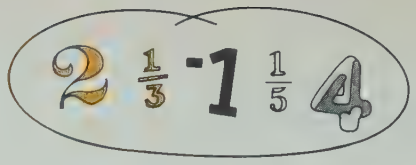
DISCUSSION

Exercises 2 and 3 contain examples of multiplication of rational numbers in which each step is justified by means of the basic principles. Once the student understands this, the "numerator times numerator, denominator times denominator" algorithm should be presented and used.

Multiplying Rational Numbers

Investigating the Ideas

Using the numbers in this set as factors, many different products can be found.



How many different products can you find for each part below?	Record the factors and products.

- **A** The product of any two of the factors.
- **B** The product of any three of the factors.
- **C** The product of any four of the factors.
- **D** The product of all five factors.

Discussing the Ideas

1. Explain how you could arrange and group the factors $^{-}1, \frac{1}{3}$, 4, and $\frac{1}{5}$ to find their product easily.

2. The unit fraction principles and the basic principles can be used to justify the rule for multiplying two positive rational numbers. Study the example below.

$$\frac{2}{3} \cdot \frac{4}{5} = \left(2 \cdot \frac{1}{3}\right) \cdot \left(4 \cdot \frac{1}{5}\right)$$ Unit fraction principle

$$= (2 \cdot 4) \cdot \left(\frac{1}{3} \cdot \frac{1}{5}\right)$$ Associative and commutative principles of multiplication for rational numbers

$$= 8 \cdot \frac{1}{15}$$ Multiplication fact for integers; Unit fraction principle

$$= \frac{8}{15}$$ Unit fraction principle

   - **A** How would you find the product $\frac{3}{5} \cdot \frac{3}{8}$?
   - **B** Notice that $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$ and $\frac{5}{6} \cdot \frac{7}{8} = \frac{35}{48}$. What shortcut for multiplying two rational numbers do these examples suggest?

3. Study the example Then explain how to find the products below.
   - **A** $\frac{2}{3} \cdot \frac{-4}{5}$
   - **B** $\frac{-7}{8} \cdot \frac{-3}{5}$

$$\frac{-3}{5} \cdot \frac{4}{7} = \left({}^{-}1 \cdot \frac{3}{5}\right) \cdot \frac{4}{7}$$
$$= {}^{-}1 \cdot \left(\frac{3}{5} \cdot \frac{4}{7}\right) = {}^{-}1 \cdot \frac{12}{35}$$
$$= \frac{-12}{35}$$

4. Explain why the product $\frac{-5}{3} \cdot \frac{-3}{4}$ is the same as $\frac{5}{3} \cdot \frac{3}{4}$.

Using the Ideas

1. Find the products.

A $\frac{2}{5} \cdot \frac{4}{3}$	**D** $\frac{3}{8} \cdot \frac{3}{4}$	**G** $\frac{9}{4} \cdot \frac{4}{9}$	**J** $\frac{5}{6} \cdot \frac{3}{10}$	**M** $\frac{2}{3} \cdot 5$
B $\frac{5}{7} \cdot \frac{1}{3}$	**E** $\frac{4}{5} \cdot \frac{5}{8}$	**H** $\frac{15}{16} \cdot \frac{8}{5}$	**K** $\frac{3}{8} \cdot \frac{0}{5}$	**N** $\frac{3}{5} \cdot \frac{3}{2}$
C $\frac{9}{10} \cdot \frac{5}{3}$	**F** $\frac{7}{3} \cdot \frac{5}{7}$	**I** $\frac{2}{3} \cdot \frac{4}{4}$	**L** $4 \cdot \frac{5}{9}$	**O** $\frac{9}{10} \cdot \frac{3}{10}$

2. Find the products.

A $\frac{-3}{8} \cdot \frac{3}{2}$	**F** $-1 \cdot \frac{-1}{2}$	**K** $\frac{5}{2} \cdot -1$	**P** $-6 \cdot \frac{7}{10}$
B $\frac{5}{7} \cdot \frac{-1}{3}$	**G** $\frac{3}{8} \cdot \frac{-8}{3}$	**L** $-4 \cdot \frac{-5}{9}$	**Q** $\frac{-5}{3} \cdot -27$
C $\frac{-1}{2} \cdot \frac{-1}{2}$	**H** $5 \cdot \frac{-2}{5}$	**M** $12 \cdot \frac{-1}{2}$	**R** $\frac{-7}{2} \cdot \frac{-2}{7}$
D $\frac{-2}{3} \cdot \frac{2}{5}$	**I** $\frac{-7}{8} \cdot \frac{-2}{7}$	**N** $-30 \cdot \frac{7}{100}$	**S** $\left(\frac{-1}{2} \cdot \frac{1}{4}\right) \cdot -8$
E $\frac{-1}{4} \cdot \frac{-1}{3}$	**J** $\frac{-2}{3} \cdot \frac{-2}{3}$	**O** $\frac{-3}{10} \cdot 25$	**T** $\left(\frac{9}{10} \cdot \frac{-1}{2}\right) \cdot -140$

3. Complete each sentence with one of the words **positive** or **negative.**

A The product of two positive rational numbers is a ? rational number.

B The product of two negative rational numbers is a ? rational number.

C The product of a positive rational number and a negative rational number is a ? rational number.

4. Solve the equations.

A $6 \cdot \left(2 + \frac{1}{3}\right) = 6 \cdot 2 + 6 \cdot \frac{1}{3} = n$	**C** $10 \cdot \left(1 + \frac{1}{5}\right) = n$	**E** $4 \cdot 2\frac{1}{4} = n$
B $-4 \cdot \left(3 + \frac{1}{2}\right) = n$	**D** $-8 \cdot 3\frac{1}{8} = n$	**F** $-6 \cdot -2\frac{2}{3} = n$

5. Write each mixed numeral as an improper fraction. Then find the product.

A $2\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{3} \cdot \frac{4}{5} =$ ▒	**D** $\frac{5}{6} \cdot -4\frac{2}{3} =$ ▒	**G** $33\frac{1}{3} \cdot \frac{-3}{4} =$ ▒
B $1\frac{1}{2} \cdot \frac{7}{8} =$ ▒	**E** $-1\frac{1}{2} \cdot -2\frac{1}{2} =$ ▒	**H** $-10 \cdot -2\frac{1}{2} =$ ▒
C $-2\frac{1}{2} \cdot \frac{4}{5} =$ ▒	**F** $6\frac{2}{3} \cdot 6 =$ ▒	**I** $-3\frac{3}{4} \cdot -5\frac{1}{3} =$ ▒

think

A postman is to deliver letters to each of 10 houses located all in a row and equally spaced. If he can start at any house, what is the **longest** path he can take?

More practice, page S-15, Set 27

UTILIZATION

Negative rational numbers in mixed numeral form are given in Exercises 4 and 5. Since a mixed numeral like $2\frac{1}{2}$ is understood to mean $2 + \frac{1}{2}$, a symbol such as $-2\frac{1}{2}$ may cause some confusion. We agree that this symbol represents $-(2\frac{1}{2}) = -(2 + \frac{1}{2})$. If some students think the symbol means $-2 + \frac{1}{2}$, show that this sum is $-\frac{3}{2}$, but $-2\frac{1}{2} = -(2\frac{1}{2}) = -\frac{5}{2}$.

EXTENSION

Further practice may be provided by assignments from Supplementary Exercise Set 27 on page S–15, *Workbook* page 41, *Duplicator Masters* page 30, and *Arithmetic Skill Cards* F-9 through F-11.

Enrichment: Ask students to complete the array to make a multiplication magic square. The product in each row, column, and diagonal is the same number. (Answers appear in parentheses.)

24	$(\frac{-3}{16})$	6
$(\frac{-3}{4})$	-3	-12
$(\frac{3}{2})$	-48	$\frac{3}{8}$

Think Solution

Some students will assume that the longest route is from 1 to 10, 10 to 2, 2 to 9, and so on, (45 lots), but this is not true. Starting at 5 and moving to 10, 10 to 4, 4 to 9, 9 to 3, 3 to 8, 8 to 2, 2 to 7, 7 to 1, and 1 to 6 will cause the postman to walk past 49 lots (units denoted on the line in the illustration).

Assignments

Minimum 1A–F, 2A–J, 3.
Average 1A–I, 2A–O, 3, 4.
Maximum 1G–O, 2K–T, 3–5.

Objective

The student can name the reciprocal of a rational number and solve problems involving reciprocals of rational numbers.

PREPARATION

Since the definition of the reciprocal of a number depends upon an understanding of multiplication, review multiplication of rational numbers before introducing the Investigation.

INVESTIGATION

Although the Investigation should require only a few minutes for students to complete, it will bring them to grips with the basic concept of reciprocals: two numbers are reciprocals of each other if their product is 1. This also leads to the idea that the reciprocal of a positive rational number is positive, and the reciprocal of a negative rational number is negative.

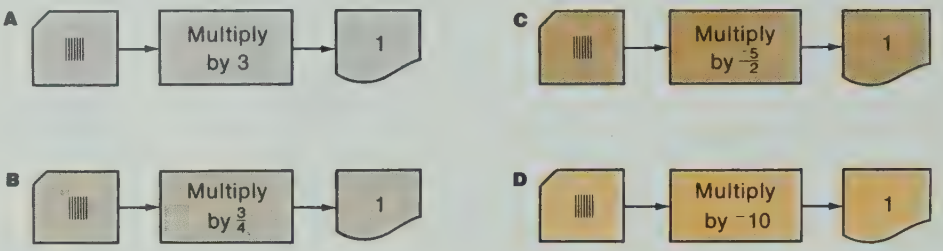
DISCUSSION

Each nonzero rational number has a unique multiplicative inverse (reciprocal). This fact is used in the lesson on pages F-24 and F-25 to define division of rational numbers and to show that, when the divisor is not zero, the quotient of two rational numbers is another rational number. Thus, this lesson on the concept of reciprocal, or multiplicative inverse, is essential to the development of division of rational numbers. Most of your students will have had prior experience with division of rational numbers and reciprocals. However, for most students, reciprocals of negative rational numbers will be a new idea.

Reciprocals

Investigating the Ideas

Each of these flow charts has an output number of 1.



? How many of the input numbers can you find?

Discussing the Ideas

1. If the product of two numbers is 1, each number is the **multiplicative inverse** or **reciprocal** of the other.
   For example, $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$ because $\frac{3}{4} \cdot \frac{4}{3} = \frac{12}{12} = 1$
   and $\frac{-7}{2}$ is the multiplicative inverse of $\frac{-2}{7}$ because $\frac{-7}{2} \cdot \frac{-2}{7} = \frac{14}{14} = 1$.

   Give the reciprocal of each number. Explain your answer.

   A $\frac{11}{10}$ B $\frac{-1}{2}$ C $^-3$ D $2\frac{1}{5}$ E 1

2. One of the basic differences in the structure of the set of rational numbers and the set of integers is that every rational number, except zero, has a multiplicative inverse in the set of rational numbers. However, in the set of integers, only 1 and $^-1$ have multiplicative inverses.

   a is a reciprocal of ***b***, and ***b*** is a reciprocal of ***a*** if $a \cdot b = 1$.

   A Explain why zero does not have a reciprocal.

   B What kind of a number is the reciprocal of a unit fraction?

3. If a rational number is negative, will its reciprocal be a positive or a negative rational number?

4. Since $2\frac{1}{4} = \frac{9}{4}$, the reciprocal of $2\frac{1}{4}$ is $\frac{4}{9}$.
   Explain how to find the reciprocal of $3\frac{7}{10}$.

Using the Ideas

1. Give the reciprocal of each number.

A $\frac{2}{5}$	C $\frac{7}{8}$	E $^{-}4$	G 1	I $3\frac{1}{3}$	K 10
B $\frac{^{-}1}{3}$	D $\frac{16}{15}$	F $\frac{3}{7}$	H $\frac{^{-}11}{6}$	J $7\frac{2}{5}$	L $\frac{^{-}1}{100}$

2. Solve the equations.

A $\frac{3}{8} \cdot n = 1$	D $\frac{6}{5} \cdot n = 1$	G $10 \cdot n = 1$	J $5\frac{1}{8} \cdot n = 1$
B $\frac{5}{4} \cdot n = 1$	E $\frac{12}{7} \cdot n = 1$	H $\frac{1}{6} \cdot n = 1$	K $16\frac{2}{3} \cdot n = 1$
C $n \cdot \frac{8}{7} = 1$	F $\frac{4}{9} \cdot n = 1$	I $2\frac{1}{2} \cdot n = 1$	L $1 \cdot n = 1$

3. Solve each equation.

A $(5 \cdot 3) \cdot x = 1$	C $\left(\frac{1}{2} \cdot \frac{1}{4}\right) \cdot x = 1$	E $\frac{3}{4} \cdot \left(\frac{^{-}1}{2} \cdot x\right) = 1$
B $(^{-}4 \cdot 7) \cdot x = 1$	D $\left(^{-}2 \cdot \frac{1}{5}\right) \cdot x = 1$	F $\frac{^{-}7}{10} \cdot \left(\frac{^{-}1}{3} \cdot x\right) = 1$

4. Study the example. Then solve the equations.

A $\frac{5}{8} \cdot \left(\frac{2}{3} \cdot n\right) = \frac{5}{8}$
B $\frac{3}{4} \cdot \left(\frac{10}{9} \cdot n\right) = \frac{3}{4}$
C $\frac{1}{2} \cdot \left(\frac{5}{6} \cdot n\right) = \frac{5}{6}$
D $\frac{^{-}7}{3} \cdot \left(\frac{8}{3} \cdot n\right) = \frac{7}{3}$
E $\left(\frac{3}{4} \cdot \frac{5}{7}\right) \cdot n = \frac{3}{4}$
F $\left(\frac{6}{5} \cdot \frac{7}{10}\right) \cdot n = \frac{6}{5}$

EXAMPLE:
Solve the equation: $\frac{3}{4} \cdot \left(\frac{5}{7} \cdot n\right) = \frac{3}{4}$.

This number must be 1 because $\frac{3}{4} \cdot 1 = \frac{3}{4}$.

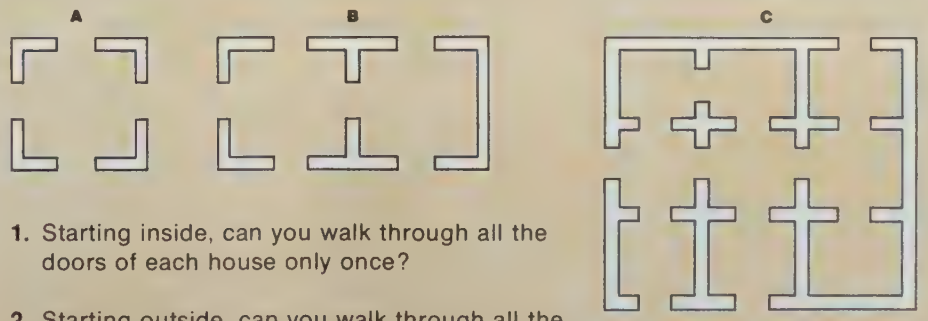
$\frac{3}{4} \cdot \frac{5}{7} \cdot n = \frac{3}{4}$

If $\frac{5}{7} \cdot n = 1$, then $n = \frac{7}{5}$.

SOLUTION: $n = \frac{7}{5}$

Think

Figures A, B, and C represent the floor plans of houses.



1. Starting inside, can you walk through all the doors of each house only once?

2. Starting outside, can you walk through all the doors of each house only once?

UTILIZATION

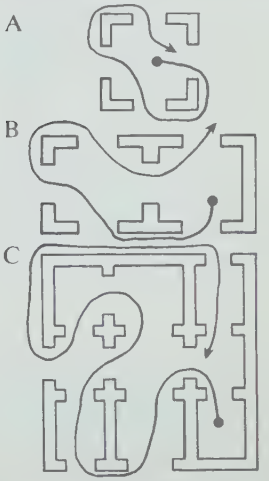
The equations in Exercise 4 involve the type of reasoning that leads to the development of the division algorithm for rational numbers. The missing factor in the multiplication equation $\frac{2}{3} \cdot n = \frac{5}{8}$ is the quotient in the division equation $\frac{5}{8} \div \frac{2}{3} = n$. Using the reciprocal of $\frac{2}{3}$ and the property of 1, we see that $\frac{2}{3} \cdot (\frac{3}{2} \cdot \frac{5}{8}) = \frac{5}{8}$. Hence, $n = \frac{3}{2} \cdot \frac{5}{8}$, so $\frac{5}{8} \div \frac{2}{3} = \frac{5}{8} \cdot \frac{3}{2}$.

EXTENSION

Exercises suitable for further practice assignments are provided on page 42 of the *Workbook*.

Think Solution

1. Yes for Figures A, B, and C.



2. Students should be able to reason as follows to see that the response is "No" to question 2 for floor plan C: Since one is starting outside and there are only 4 outside doors, one must end up *outside* after passing through each outside door only once, but there is one *inside* room with only one door leading into it, which means that in order to pass through that door only once one would have to end up *inside*.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can solve word problems involving applications of rational numbers.

PREPARATION

Before assigning the exercises on this page, you might discuss ways of conserving electrical energy in family households. You might also have available some brochures or news reports from the local electrical plant showing amounts of electrical power and the cost of the use of electricity.

UTILIZATION

Since the meaning of the term *kilowatt-hour* may be unfamiliar to most students, plan to discuss this term in detail, using examples such as:

1. A 60-watt light bulb that burns for 3 hours consumes 60 watts · 3 hours = 180 watt-hours of electrical energy.
2. One 25-watt light bulb in use for 4 hours consumes 25 watts · 4 hours = 100 watt-hours. Therefore, ten 25-watt bulbs burning 4 hours will use ten times as much energy, or 10 · 100 watt-hours = 1000 watt-hours = 1 kW · h

EXTENSION

Enrichment: Some students may be interested in contacting the local electrical utility company and obtaining information on the amount of electrical power used by other home appliances. They could also obtain tables of costs of electrical service and learn how to compute the cost of electrical service.

Other students might be encouraged to write problems similar to the ones in the text based on usage of electrical appliances in their homes during specific hours, such as from the time they return home from school until the time they go to bed.

Assignments

Minimum 1–5. Average 1–8. Maximum 1–9.

ELECTRICAL ENERGY

Electrical power is measured in **watts**. A **kilowatt** is 1000 watts. A **kilowatt-hour** (kW.h) is 1000 watts of power in use for one hour. For example, one kilowatt-hour will run a 100-watt light bulb 10 hours. (That is, 100 watts × 10 hours = 1000 watt-hours = 1 kilowatt-hour.)

The table below shows the approximate number of kilowatt-hours used by some electrical appliances.

APPLIANCE	kW.h	APPLIANCE	kW.h
Water heater	13 per day	Hi-fi set	$\frac{1}{10}$ per hour
Refrigerator	6 per day	Washing machine	$\frac{1}{3}$ per load
Clothes dryer	3 per load	Toaster	$1\frac{1}{3}$ per hour
Kitchen range	1 per meal	Disposal	$\frac{1}{100}$ each time
Color television	$\frac{1}{3}$ per hour	Hair dryer	$\frac{1}{3}$ per hour
Iron	$1\frac{1}{4}$ per hour	Vacuum cleaner	$\frac{2}{3}$ per hour

1. If a color television is used 5 hours per day for 30 days, how many kW.h are used?

2. If you listen to a hi-fi while ironing for one hour, how many kW.h are used?

3. Five loads of clothes are washed and dried. It takes 2 hours to iron the clothes. How many kW.h?

4. Suppose a disposal is used about 15 times per day. How many kW.h are used in 30 days?

5. A toaster is in use for 15 minutes at breakfast. How many kW.h?

6. How many kW.h are used to make three meals per day for one week?

7. The average residential consumer uses about 100 kW.h in home lighting in one month's time. How many 100-watt light bulbs burning for one hour would use the same amount of kW.h?

8. If a vacuum cleaner is used for $\frac{1}{2}$ hour, the range is used for a meal, and the disposal used four times, how many kW.h?

9. Suppose the cost of one kW.h is 3 cents. What does it cost to do the following?

- **A** Heat water for 30 days.
- **B** Watch a 90-minute color television show.
- **C** Run a refrigerator for a year.
- **D** Go on vacation for a week with a 100-watt light bulb left burning.

Improving Computational Skills

1. Find the sums.

A	B	C	D	E
$16\frac{3}{10}$ $29\frac{1}{2}$	$8\frac{3}{4}$ $11\frac{3}{4}$	$123\frac{7}{10}$ $96\frac{9}{10}$	$83\frac{1}{3}$ $16\frac{2}{3}$	$110\frac{5}{6}$ $88\frac{1}{5}$

F	G	H	I	J
$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{5}$	$\frac{3}{10}$ $\frac{37}{100}$ $\frac{377}{1000}$	$\frac{1}{4}$ $\frac{2}{3}$ $\frac{3}{10}$	$\frac{11}{10}$ $\frac{5}{2}$ $\frac{13}{100}$	$\frac{1}{2}$ $\frac{1}{20}$ $\frac{1}{200}$

2. Find the differences.

A	B	C	D	E
$\frac{7}{10}$ $\frac{14}{100}$	$\frac{3}{4}$ $\frac{27}{100}$	$\frac{2}{3}$ $\frac{3}{5}$	$\frac{13}{10}$ $\frac{746}{100}$	$\frac{5}{6}$ $\frac{8}{45}$

F	G	H	I	J
$15\frac{9}{10}$ $7\frac{7}{10}$	28 $19\frac{1}{4}$	$35\frac{7}{10}$ $28\frac{3}{5}$	$72\frac{1}{2}$ $26\frac{9}{10}$	100 $8\frac{7}{8}$

3. Find the products.

A	B	C	D	E
$\frac{7}{10} \cdot \frac{2}{5}$	$\frac{5}{4} \cdot \frac{2}{15}$	$\frac{3}{8} \cdot \frac{5}{6}$	$\frac{1}{3} \cdot 52$	$45 \cdot \frac{4}{9}$

F	G	H	I	J
$2\frac{1}{2} \cdot 6\frac{1}{10}$	$8\frac{2}{3} \cdot \frac{3}{2}$	$42\frac{1}{2} \cdot 5\frac{1}{2}$	$6 \cdot 16\frac{2}{3}$	$9\frac{1}{2} \cdot 9\frac{1}{3}$

think

Can you find these products quickly? Look for some shortcuts.

A $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \frac{8}{9} =$ ▮

B $\frac{-4}{2} \cdot \frac{-3}{2} \cdot \frac{-2}{2} \cdot \frac{-1}{2} \cdot \frac{0}{2} \cdot \frac{1}{2} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{4}{2} =$ ▮

C $\frac{7}{8} \cdot \frac{3}{10} \cdot \frac{6}{13} \cdot 1\frac{4}{7} \cdot \frac{1}{2} \cdot \frac{7}{11} \cdot \frac{13}{6} \cdot \frac{10}{3} \cdot 1\frac{1}{7} =$ ▮

D $\left(1+\frac{1}{2}\right) \cdot \left(1+\frac{1}{3}\right) \cdot \left(1+\frac{1}{4}\right) \cdot \left(1+\frac{1}{5}\right) \cdot \left(1+\frac{1}{6}\right) \cdot \left(1+\frac{1}{7}\right) =$ ▮

Objective

The student can compute sums, differences, and products using positive rational numbers.

UTILIZATION

This page of exercises contains addition, subtraction, and multiplication problems involving only positive rational numbers. The purpose is to provide review and maintenance of arithmetic skills using positive rational numbers.

The assignments for this page should be dictated by the needs of your students.

EXTENSION

Think Solution

A. The numerator and denominator of the product contain the common factor $(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8)$. "Dividing out" this common factor gives the product in lowest terms: $\frac{1}{9}$.

B. The product must be 0 because one factor, $\frac{0}{2}$, is 0.

C. The shortcut for this product involves pairing factors which are reciprocals of each other. The problem may be regrouped as follows to find the product.

$$\left(\tfrac{7}{8} \cdot 1\tfrac{1}{7}\right) \cdot \left(\tfrac{6}{13} \cdot \tfrac{13}{6}\right) \cdot \left(1\tfrac{4}{7} \cdot \tfrac{7}{11}\right) \cdot \left(\tfrac{3}{10} \cdot \tfrac{10}{3}\right) \cdot \tfrac{1}{2} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot \tfrac{1}{2} = \tfrac{1}{2}$$

D. If each sum is expressed as an improper fraction, this problem is similar to A above and the product can be computed mentally.

$$\tfrac{3}{2} \cdot \tfrac{4}{3} \cdot \tfrac{5}{4} \cdot \tfrac{6}{5} \cdot \tfrac{7}{6} \cdot \tfrac{8}{7} = \tfrac{8}{2} = 4$$

Note that $(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7)$ is a common factor to the numerator and denominator of the product.

Assignments

Minimum 1A–D, 2A–D, 3A–D.
Average 1A–G, 2A–G, 3A–G.
Maximum 1–3.

Objective

The student can find the quotient of two rational numbers by using the usual division algorithm.

PREPARATION

Place some multiplication equations having missing factors on the chalkboard along with the related division equations. Use examples like the ones below, in which the missing factor can be found mentally.

A. $\frac{1}{2} \cdot n = \frac{1}{4}$ $\quad \frac{1}{4} \div \frac{1}{2} = n$ $\quad (n = \frac{1}{2})$

B. $\frac{2}{3} \cdot t = \frac{10}{21}$ $\quad \frac{10}{21} \div \frac{2}{3} = t$ $\quad (t = \frac{5}{7})$

C. $\frac{1}{5} \cdot x = \frac{3}{10}$ $\quad \frac{3}{10} \div \frac{1}{5} = x$ $\quad (x = \frac{3}{2})$

DISCUSSION

Even though most students can probably recall and use the division algorithm for rational numbers, you should carefully work through the development in Exercises 1 and 2 to promote understanding. Have some students show the parts of Exercise 3 or other examples on the chalkboard. Be sure that students understand the rule (algorithm) for finding the quotient of two rational numbers, which can be written as

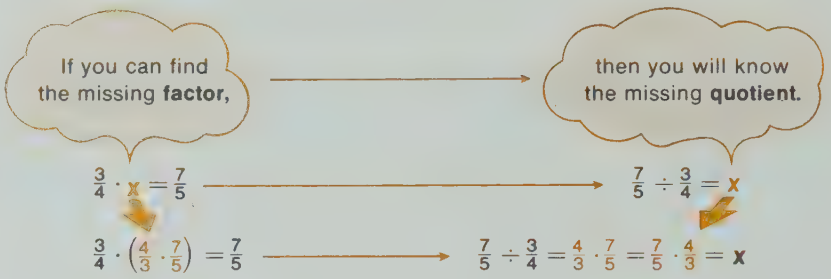
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Before assigning the exercises on page F-25, attempt to find out how much practice the students need in this type of computation. Certainly, most students will have had considerable work with division of positive rational numbers. Therefore, you may need to give special attention only to the problems involving negative rational numbers.

Division of Rational Numbers

Discussing the Ideas

1. Finding missing factors in multiplication problems may help you to understand division of rational numbers. Study the example and then give the number for x.


If you can find the missing factor,
then you will know the missing quotient.
$\frac{3}{4} \cdot x = \frac{7}{5}$
$\frac{7}{5} \div \frac{3}{4} = x$
$\frac{3}{4} \cdot (\frac{4}{3} \cdot \frac{7}{5}) = \frac{7}{5}$
$\frac{7}{5} \div \frac{3}{4} = \frac{4}{3} \cdot \frac{7}{5} = \frac{7}{5} \cdot \frac{4}{3} = x$

2. If the divisor is not zero, the same argument may be made for any pair of rational numbers. Study the example. Then give the missing fraction for ▥.

$\frac{c}{d} \cdot x = \frac{a}{b}$
$\frac{a}{b} \div \frac{c}{d} = x$
$\frac{c}{d} \cdot (\frac{d}{c} \cdot \frac{a}{b}) = \frac{a}{b}$
$\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \cdot \frac{a}{b} = \frac{a}{b} \cdot \frac{d}{c}$

Therefore, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot$ ▥.

3. The example shows how dividing with negative rational numbers is carried out.

$\frac{-2}{5} \div \frac{3}{4} = \frac{-2}{5} \cdot \frac{4}{3} = \frac{-8}{15}$

Try these division problems.

A $\frac{-1}{2} \div \frac{5}{4}$ $\quad$ B $\frac{-3}{8} \div \frac{-2}{5}$ $\quad$ C $\frac{10}{3} \div \frac{-5}{2}$ $\quad$ D $\frac{1}{10} \div \frac{-1}{5}$

4. To subtract one rational number from another, you can add the additive inverse.

$\frac{3}{4} - \frac{2}{3} = \frac{3}{4} + \frac{-2}{3}$

Notice the division example and then complete this statement:

$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \cdot \frac{3}{2}$

To divide one rational number by another, you can multiply ? ? ?.

Using the Ideas

1. Solve the equations.

A $\frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \cdot n$ **D** $\frac{1}{4} \div \frac{2}{3} = \frac{1}{4} \cdot n$ **G** $\frac{15}{8} \div \frac{5}{4} = \frac{4}{5} \cdot n$

B $\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \cdot n$ **E** $\frac{5}{6} \div \frac{-1}{2} = \frac{5}{6} \cdot n$ **H** $\frac{9}{2} \div \frac{3}{4} = \frac{4}{3} \cdot n$

C $\frac{7}{10} \div \frac{5}{2} = \frac{7}{10} \cdot n$ **F** $\frac{-2}{3} \div \frac{-3}{8} = \frac{-8}{3} \cdot n$ **I** $\frac{-12}{5} \div \frac{3}{5} = \frac{5}{3} \cdot n$

2. Find the quotients.

A $\frac{1}{3} \div \frac{1}{2}$ **D** $\frac{7}{12} \div \frac{3}{4}$ **G** $\frac{3}{5} \div \frac{-3}{5}$ **J** $\frac{11}{10} \div \frac{11}{4}$

B $\frac{5}{6} \div \frac{1}{3}$ **E** $\frac{1}{2} \div \frac{-1}{3}$ **H** $\frac{6}{5} \div \frac{3}{10}$ **K** $\frac{2}{3} \div 1$

C $\frac{9}{10} \div \frac{3}{5}$ **F** $\frac{1}{4} \div \frac{-1}{3}$ **I** $\frac{3}{8} \div \frac{1}{4}$ **L** $\frac{-9}{10} \div 3$

3. Write the mixed numerals as improper fractions and then find the quotients.

A $1\frac{2}{3} \div \frac{1}{2}$ **C** $6\frac{1}{4} \div 3\frac{1}{2}$ **E** $\frac{1}{3} \div -1\frac{1}{3}$ **G** $-2\frac{1}{2} \div -1\frac{1}{4}$

B $4\frac{1}{2} \div 2\frac{1}{2}$ **D** $1\frac{1}{8} \div \frac{1}{2}$ **F** $-10 \div 3\frac{1}{3}$ **H** $9\frac{1}{3} \div 3\frac{1}{2}$

4. Use division to find the number for r.

EXAMPLE: $\frac{3}{8} \cdot r = \frac{5}{2} \rightarrow r = \frac{5}{2} \div \frac{3}{8} = \frac{5}{2} \cdot \frac{8}{3} = \frac{40}{6} = \frac{20}{3}$

A $\frac{3}{5} \cdot r = \frac{1}{2}$ **C** $\frac{1}{4} \cdot r = \frac{3}{4}$ **E** $\frac{-2}{3} \cdot r = \frac{-5}{6}$ **G** $\frac{-1}{2} \cdot r = \frac{2}{7}$

B $\frac{1}{10} \cdot r = \frac{5}{2}$ **D** $\frac{-1}{2} \cdot r = \frac{3}{8}$ **F** $\frac{3}{4} \cdot r = \frac{-1}{2}$ **H** $\frac{1}{3} \cdot r = \frac{-5}{6}$

★ **5.** Find the number for x in each equation.

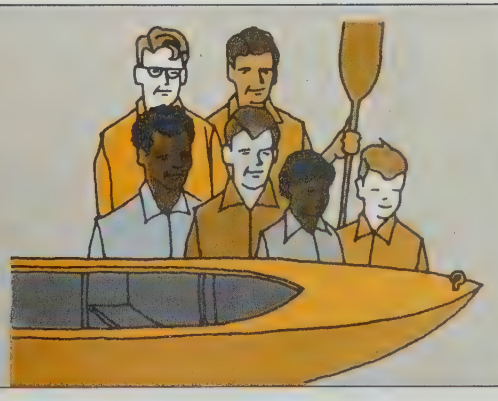
A $\left(\frac{3}{4} + \frac{1}{2}\right) \div \frac{2}{3} = x$ **D** $\left(\frac{3}{4} \div \frac{2}{3}\right) + \left(\frac{1}{2} \div \frac{2}{3}\right) = x$

B $\left(\frac{2}{5} \div \frac{1}{2}\right) + \left(\frac{1}{3} \div \frac{1}{2}\right) = x$ **E** $\frac{2}{3} \div \left(\frac{1}{4} + \frac{1}{3}\right) = x$

C $\left(\frac{2}{5} + \frac{1}{3}\right) \div \frac{1}{2} = x$ **F** $\left(\frac{2}{3} \div \frac{1}{4}\right) + \left(\frac{2}{3} \div \frac{1}{3}\right) = x$

think

Four men and two boys must cross a river in a canoe that will hold only one of the men or the two boys at one time. It will not carry one man and one boy. Tell how the six people can get across the river. What is the smallest number of times the canoe must cross the river?



More practice, page S-15, Set 28

UTILIZATION

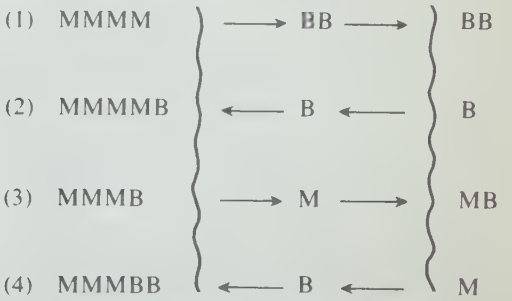
Note the gradual increase in difficulty of the problems on this page. Exercise 1 requires that the student find a missing factor so that a division problem is rewritten as a multiplication problem.

EXTENSION

To provide additional practice, make selective assignments from Supplementary Exercise Set 28 on page S-15, *Workbook* page 43, *Duplicator Masters* page 31, and *Arithmetic Skill Card* F-12.

Think Solution

(1) The two boys must row across first; thus (2) the canoe can be returned by one of them. (3) A man rows across, and (4) the boy left from the previous trip returns with the canoe. Situation (4) reveals three men and two boys still on the wrong side of the stream and one man on the right side.



If this set of procedures is continued three more times, all four men will be on the right side, and the two boys will be on the wrong side. The final crossing will take the two boys across the stream. Since the procedure required four crossings to get one man across, a total of 16 crossings plus the final crossing by the boys, or 17 crossings in all, were needed.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

Given two rational numbers, the student can write a statement showing either equality or inequality of the two numbers.

PREPARATION

Review renaming fractions which have negative denominators with equivalent fractions having positive denominators. You might also review inequalities for integers, inasmuch as comparing rational numbers requires an understanding of comparison of integers.

INVESTIGATION

The Investigation provides the students with a situation in which they can use intuitive methods to order a finite set of rational numbers.

Parts A, B, and C could be discussed orally. Then students could record the correspondence between the numeral tags and the lettered points on the number line.

DISCUSSION

Although a formal definition of the relations "is greater than" and "is less than" can be stated for rational numbers, we rely on more intuitive methods. The number line is first used to effect a comparison between rational numbers. Then the common positive denominator method is introduced. The latter method is particularly useful since it reduces the problem of comparing two rational numbers in fractional form to that of comparing two integers.

Comparing Rational Numbers

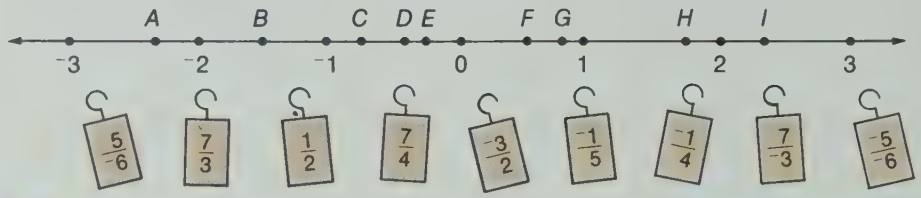
Investigating the Ideas

Give the missing numbers.

A Since $\frac{3}{2}$ is between 1 and 2, then $\frac{-3}{2}$ must be between ? and ?.

B Since $\frac{7}{3}$ is between 2 and 3, then $\frac{-7}{3}$ must be between ? and ?.

C Since $\frac{15}{4}$ is between 3 and 4, then $\frac{-15}{4}$ must be between ? and ?.



Can you match the tags with the proper points?

Discussing the Ideas

1. The smallest rational number named by any of the fraction tags was the number that goes with point *A*, the point farthest to the left. What was that number?

2. The largest number goes with point *I*, the point farthest to the right. What was that number?

3. Using the definition at the right, which symbol ($<$ or $>$) should be placed between each pair of fractions?

   A $\frac{7}{3}$ ◯ $\frac{7}{4}$ C $\frac{-3}{2}$ ◯ $\frac{-1}{5}$

   B $\frac{-5}{-6}$ ◯ $\frac{1}{2}$ D $\frac{-5}{6}$ ◯ $\frac{-5}{6}$

> If the point for $\frac{a}{b}$ lies to the left of the point for $\frac{c}{d}$ on the number line, we say that $\frac{a}{b}$ is less than $\frac{c}{d}$, $\frac{a}{b} < \frac{c}{d}$.
> If $\frac{a}{b} < \frac{c}{d}$, then $\frac{c}{d}$ is greater than $\frac{a}{b}$, $\frac{c}{d} > \frac{a}{b}$.

4. Instead of using the number line, you can write each fraction with a **positive common denominator** and then compare the numerators. Study the examples.

   A $\frac{-5}{-6} = \frac{5}{6} = \frac{10}{12}$ and $\frac{7}{4} = \frac{21}{12}$

   Since $10 < 21$, then $\frac{-5}{-6} < \frac{7}{4}$.

   B $\frac{-1}{5} = \frac{-6}{30}$ and $\frac{-5}{-6} = \frac{5}{6} \cdot \frac{-5}{-5} = \frac{-25}{30}$

   Since $-6 > -25$, then $\frac{-1}{5} > \frac{-5}{-6}$.

   Compare these pairs of rational numbers by this method.

   A $\frac{3}{8}$ and $\frac{5}{12}$ B $\frac{-4}{3}$ and $\frac{-5}{4}$ C $\frac{-1}{2}$ and $\frac{2}{3}$ D $\frac{-2}{-3}$ and $\frac{-3}{5}$

Using the Ideas

1. Give the integer for n.

A $\frac{1}{8} = \frac{n}{16}$ **C** $\frac{-2}{3} = \frac{n}{12}$ **E** $\frac{7}{-3} = \frac{35}{n}$ **G** $\frac{9}{n} = \frac{45}{35}$ **I** $\frac{n}{3} = \frac{20}{30}$

B $\frac{3}{4} = \frac{9}{n}$ **D** $\frac{12}{n} = \frac{24}{10}$ **F** $\frac{5}{-2} = \frac{n}{10}$ **H** $\frac{-4}{-11} = \frac{28}{n}$ **J** $\frac{-8}{28} = \frac{-2}{n}$

2. Give the numbers for a and b. Then give the correct symbol ($<$ or $>$) for each ◯.

A $\left.\begin{array}{l}\frac{1}{5} = \frac{a}{20} \\ \frac{1}{4} = \frac{b}{20}\end{array}\right\}$ $\frac{1}{5}$ ◯ $\frac{1}{4}$ **C** $\left.\begin{array}{l}\frac{-6}{5} = \frac{a}{20} \\ \frac{-5}{4} = \frac{b}{20}\end{array}\right\}$ $\frac{-6}{5}$ ◯ $\frac{-5}{4}$ **E** $\left.\begin{array}{l}\frac{-20}{-3} = \frac{a}{21} \\ \frac{40}{7} = \frac{b}{21}\end{array}\right\}$ $\frac{-20}{-3}$ ◯ $\frac{40}{7}$

B $\left.\begin{array}{l}\frac{5}{12} = \frac{a}{24} \\ \frac{3}{8} = \frac{b}{24}\end{array}\right\}$ $\frac{5}{12}$ ◯ $\frac{3}{8}$ **D** $\left.\begin{array}{l}\frac{7}{11} = \frac{a}{44} \\ \frac{3}{4} = \frac{b}{44}\end{array}\right\}$ $\frac{7}{11}$ ◯ $\frac{3}{4}$ **F** $\left.\begin{array}{l}\frac{5}{16} = \frac{a}{48} \\ \frac{-5}{12} = \frac{b}{48}\end{array}\right\}$ $\frac{5}{16}$ ◯ $\frac{-5}{12}$

3. Copy each problem and give the correct symbol ($<$, $=$, or $>$) for each ◯.

A $\frac{2}{3}$ ◯ $\frac{3}{4}$ **B** $\frac{3}{10}$ ◯ $\frac{1}{3}$ **C** $\frac{3}{8}$ ◯ $\frac{1}{6}$ **D** $\frac{7}{3}$ ◯ $\frac{9}{4}$

E $\frac{-5}{12}$ ◯ $\frac{-7}{16}$ **F** $\frac{-11}{15}$ ◯ $\frac{-7}{10}$ **G** $\frac{3}{8}$ ◯ $\frac{-2}{5}$ **H** $\frac{-11}{-4}$ ◯ $\frac{22}{8}$

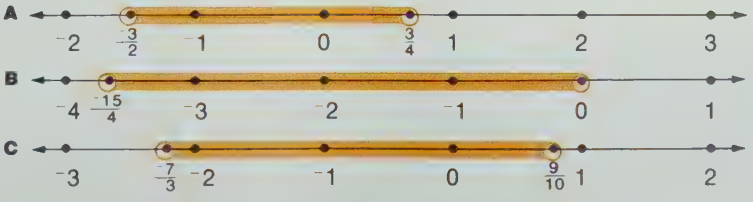
I $\frac{1}{11}$ ◯ $\frac{1}{10}$ **J** $\frac{7}{3}$ ◯ $\frac{-5}{2}$ **K** $\frac{-7}{10}$ ◯ $\frac{-5}{8}$ **L** $\frac{30}{100}$ ◯ $\frac{3}{10}$

M $\frac{-1}{8}$ ◯ $\frac{-1}{9}$ **N** $\frac{2}{9}$ ◯ $\frac{3}{9}$ **O** $\frac{-11}{-14}$ ◯ $\frac{11}{15}$ **P** $\frac{7}{8}$ ◯ $\frac{5}{6}$

4. The graph in the example illustrates the idea that we can think about all rational numbers x such that $x < \frac{9}{16}$ and $x > \frac{1}{8}$. Recall that we write such inequalities as $\frac{1}{8} < x < \frac{9}{16}$.

EXAMPLE:

Write similar inequalities for each number line.

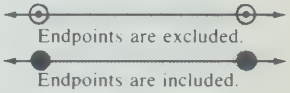


5. Illustrate each inequality by using a number line.

A $\frac{3}{8} < x < \frac{7}{8}$ **B** $\frac{-5}{3} < x < 0$ **C** $\frac{-3}{4} < x < \frac{-1}{2}$ **D** $\frac{-3}{10} < x < \frac{1}{10}$

UTILIZATION

You may need to discuss graphs of rational numbers before assigning Exercises 4 and 5. In the graph of the inequality $\frac{1}{8} < x < \frac{9}{16}$, circles are used at the endpoints, $\frac{1}{8}$ and $\frac{9}{16}$, to show that they are not included in the solution set. If the inequality were $\frac{1}{8} \leq x \leq \frac{9}{16}$, the endpoints would be included and this could be shown on the graph by making the circles at the endpoints solid dots.

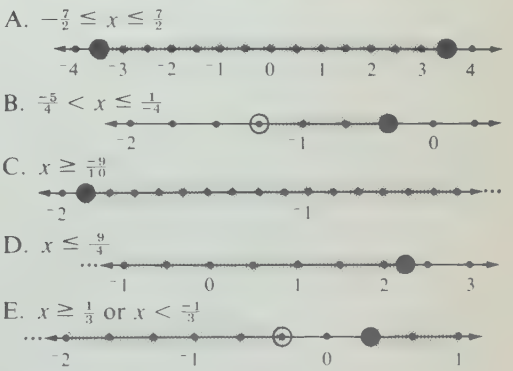


Answers, Exercise 5

EXTENSION

Further practice exercises are provided on page 44 of the *Workbook*.

Enrichment: Ask students to graph each inequality on the number line.



Assignments

Minimum 1A–F, 2A–D, 3A–H.
Average 1A–H, 2, 3A–L, 4.
Maximum Alternate parts of 1–3; all of 4 and 5.

Objective

The student can solve a word problem involving one or more operations with rational numbers.

PREPARATION

In preparation for this lesson, be sure that most of the students have adequate computational skill in working with problems involving rational numbers. You might initiate a ten-minute drill in which a number of students go to the board and work several problems involving addition, subtraction, multiplication, and division of rational numbers. The students who remain at their seats should also attempt these problems.

INVESTIGATION

The ability to solve word problems is one of the most important objectives in any mathematics course. The Investigation presents a practical problem situation which should be familiar to the student. Allow students to use their own methods of solution. It is not necessary for students to formalize their methods of problem solving at this stage.

DISCUSSION

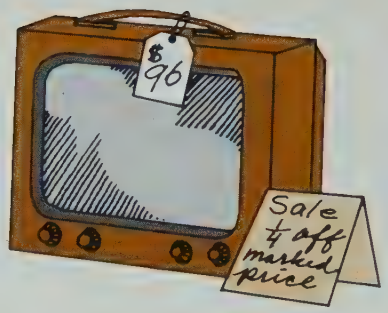
In discussing the Investigation you may need to emphasize what some terms mean; for example:

$96 *marked price* of television set
$\frac{1}{4}$ of $96 = $24 *reduction* in price (also called *discount*)
$96 − $24 = $72 *sales price* (marked price minus discount)
$39.95 *regular price*

Solving Problems

Investigating the Ideas

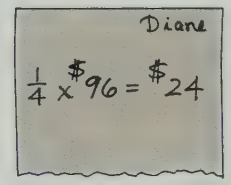
Study the advertisement at the right.



How much would you have to pay for the television set if you bought it during the sale?

Discussing the Ideas

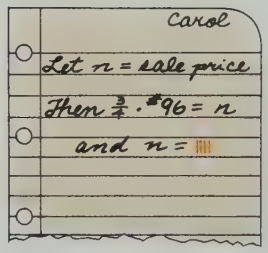
1. Diane found that $\frac{1}{4}$ of $96 was $24. Is this the sale price of the television set or the amount that the marked price is reduced?



2. How can you use Diane's work to find the sale price?

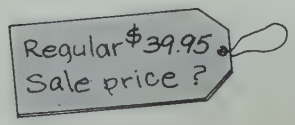
3. Carol thought about the problem differently. She thought about it this way: "Since there is $\frac{1}{4}$ off the marked price, you must pay $\frac{3}{4}$ of the marked price during the sale."

   What is the solution to Carol's equation?



4. **A** How much would you save on the television set if it were a "$\frac{1}{3}$-off" sale?
   B What would be the sale price of the television set?

5. Prices are reduced for a sale by $\frac{1}{5}$. How would you find the sale price for this tag?



Using the Ideas

1. A 10-speed bicycle regularly priced at $109 is on sale during a "$\frac{1}{3}$-off" sale. What is the sale price of the bicycle?

2. A reclining chair regularly priced at $150 is reduced in price by $\frac{1}{10}$ for a sale. What is the sale price of the chair?

3. Jane works in a dress shop which is having a "$\frac{1}{4}$-off" sale. What should she mark on the tag as the sale price of a pair of slacks that is regularly marked $12?

4. A Estimate the cost of $39\frac{7}{10}$ litres of gasoline at $11\frac{9}{10}$¢ per litre.
   B Compute the actual cost.

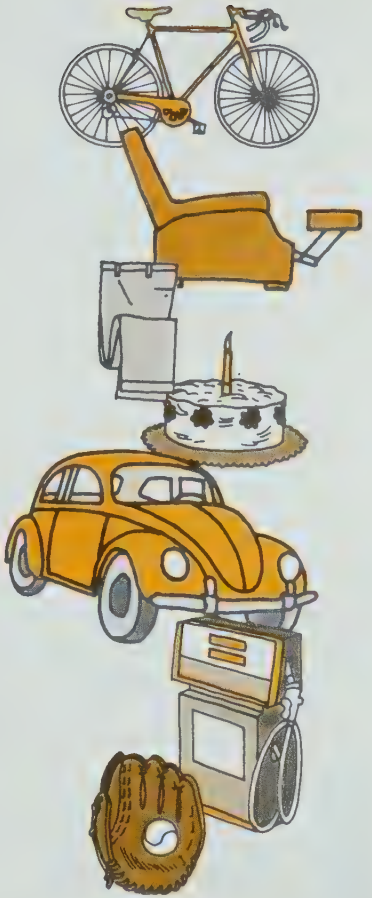
5. When Anne's sister had her twelfth birthday, Anne said, "You are $\frac{3}{4}$ as old as I am." How old is Anne?

6. A car was driven 90 kilometres in $1\frac{1}{2}$ hours. What was the average speed in kilometres per hour?

★7. A tank is $\frac{3}{4}$ full of fuel oil. After 250 litres are used, the tank is $\frac{1}{2}$ full. How many litres does the tank hold?

★8. A school hockey team won $\frac{2}{3}$ of its games. It won 3 more games than it lost. How many games did the team play?

★9. The average of two rational numbers is $\frac{7}{8}$. One number is $\frac{3}{2}$. What is the other number?



think

A bought an article for $10 and sold it to B for $\frac{1}{5}$ more than he paid for it. B sold the article to C for $\frac{1}{6}$ more than he paid for it. Then C sold it to D for $\frac{1}{7}$ more than he paid for it.

A How much did D pay for the article?

B How much profit did A make? B? C?

More practice, page S-16, Set 29

UTILIZATION

Since students differ greatly in their problem solving abilities, you will want to allow considerable freedom in the methods used to solve the problems. However, it would be beneficial to encourage students to write an equation for each problem as well as solve the problem.

EXTENSION

Supplementary Exercise Set 29 on page S-16 and page 32 of the *Duplicator Masters* offer exercises suitable for further practice.

Enrichment: Make a collection of advertisements which offer articles discounted by certain fractional amounts. Students can then write and solve problems based on these advertisements.

Think Solution

Original Price: $10
A's sales price to B: $1\frac{1}{5} \times \$10 = \$12$
B's sales price to C: $1\frac{1}{6} \times \$12 = \$14$
C's sales price to D: $1\frac{1}{7} \times \$14 = \$16$
D paid $16 for the article.
A, B, and C each made $2 profit.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–9.

Objective

Given a function rule and a set of input numbers, the student can compute the output numbers and draw a graph of the function.

PREPARATION

Review the idea of graphing intergers. You might use the function $f(n) = n + 1$ as an example, with integers as inputs. Use other examples of functions as needed.

INVESTIGATION

The words *domain* and *range* are introduced in the Investigation. Point out that mathematicians call the set of input numbers for a function the domain.The set of output numbers is called the range. It is unnecessary to give a more elaborate definition of these two words at this time.

DISCUSSION

The work with graphs of functions using rational numbers is a natural extension of previous work with integers and should not prove difficult for students.

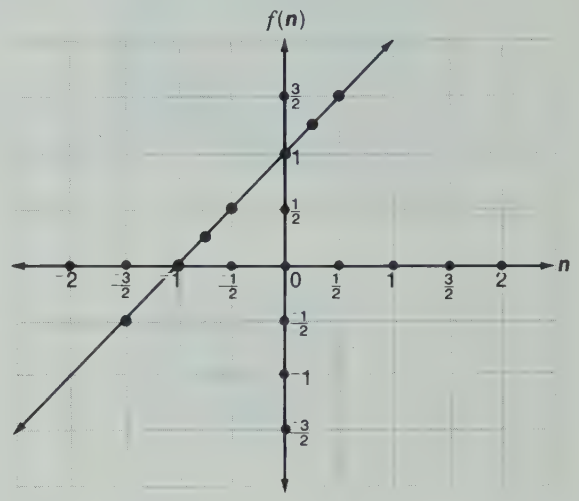
Technically, the graphs for functions with rational number domains should be drawn dotted since points with irrational coordinates are omitted. But because the rational numbers are *dense* (between any two rational numbers there are infinitely many more rational numbers), we indicate the graphs as solid lines.

Rational Numbers and Functions

Investigating the Ideas

The set of all input numbers for a function is called the **domain** of the function. The set of all output numbers is called the **range** of the function.

Function Rule			
$n + 1$			Domain: All rational numbers
n	$f(n)$		Co-ordinates
0	1	→	(0, 1)
$\frac{1}{4}$	$1\frac{1}{4}$	→	$\left(\frac{1}{4}, 1\frac{1}{4}\right)$
$\frac{1}{2}$	$1\frac{1}{2}$	→	$\left(\frac{1}{2}, 1\frac{1}{2}\right)$
$\frac{-1}{2}$	$\frac{1}{2}$	→	$\left(\frac{-1}{2}, \frac{1}{2}\right)$
$\frac{-3}{4}$	$\frac{1}{4}$	→	$\left(\frac{-3}{4}, \frac{1}{4}\right)$
$^{-}1$	0	→	$(^{-}1, 0)$
$\frac{-3}{2}$	$\frac{-1}{2}$	→	$\left(\frac{-3}{2}, \frac{-1}{2}\right)$



If the domain of the function above is the set of all rational numbers, we cannot list all number pairs for the function. Instead, we find a few number pairs and locate the points with those co-ordinates on the co-ordinate plane. Then we connect the points with a line to show where other points for the function must be located.

Can you prepare a function table and draw the graph for the function whose rule is $\frac{1}{2} \cdot n$ when the domain is the set of rational numbers?

Discussing the Ideas

1. Are the pairs that you found for the function rule necessarily the same as those found by another classmate?

2. Will your graph look the same as the graph of another classmate if neither of you has made any mistakes?

3. If the domain of the function graphed in the Investigation had been all rational numbers n such that $^{-}1 \leq n \leq 1$, would the graph have been a *line* or a *segment* or a *ray*?

Using the Ideas

Complete the function tables for Exercises 1 through 6. Then draw a graph of the function. The domain is the set of all rational numbers.

1.

Function Rule: $n - 1$	
n	$f(n)$
0	$^-1$
$\frac{1}{2}$	$\frac{^-1}{2}$
1	0
$1\frac{1}{2}$	$\frac{1}{2}$
A $^-1$	
B $^-1\frac{1}{2}$	

2.

Function Rule: $2 \cdot n$	
n	$f(n)$
$^-2$	$^-4$
$^-1\frac{1}{2}$	$^-3$
$^-1$	$^-2$
A $\frac{^-1}{2}$	
B 0	
C $1\frac{1}{2}$	

3.

Function Rule: $2 - n$	
n	$f(n)$
0	2
1	1
2	0
A $^-1$	
B $^-2$	
C $\frac{^-1}{2}$	

4.

Function Rule: $(\frac{1}{2} \cdot n) + 1$	
n	$f(n)$
0	1
1	$1\frac{1}{2}$
A 2	
B 3	
C $^-1$	

5.

Function Rule: $^-3 \cdot n$	
n	$f(n)$
A 0	
B $\frac{1}{3}$	
C $\frac{1}{2}$	
D $^-1$	
E $\frac{^-1}{3}$	

6.

Function Rule: $\frac{1}{4} \cdot (n^2)$	
n	$f(n)$
A 0	
B 1	
C 2	
D 3	
E $^-1$	

Complete the function tables for Exercises 7, 8, and 9. Then draw the graph of each function. In each case, the domain is given.

7. Domain: $^-3 < n < 3$

Function Rule: $n + 3$	
n	$f(n)$
$^-2\frac{1}{2}$	$\frac{1}{2}$
A $^-2$	
B $^-1$	
C $\frac{^-1}{2}$	
D 0	
E 1	
F $^-1\frac{1}{2}$	$1\frac{1}{2}$

★ **8.** Domain: $^-2 \leq n \leq 2$

Function Rule: $4 - n^2$	
n	$f(n)$
2	0
1	3
0	4
A $^-1$	
B $^-2$	
C $1\frac{1}{2}$	
D $^-1\frac{1}{2}$	

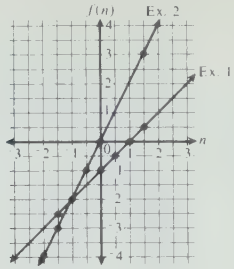
★ **9.** Domain: All rationals except 0

Function Rule: $1 \div n$	
n	$f(n)$
1	1
$\frac{1}{2}$	2
A $\frac{1}{4}$	
B 2	
$\frac{^-1}{2}$	$^-2$
C $\frac{^-1}{4}$	
D $^-2$	

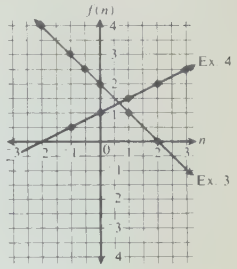
UTILIZATION

Answers, Exercises 1–9

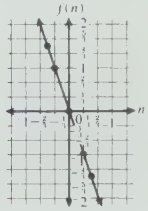
Exercises 1, 2



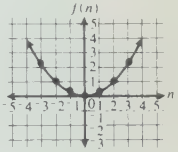
Exercises 3, 4



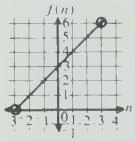
Exercise 5



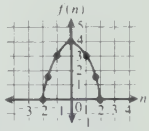
Exercise 6



Exercise 7



Exercise 8



Exercise 9

EXTENSION

Workbook page 45 offers further practice exercises.

Enrichment: Have students make function tables and draw graphs for the function rules below. The domain is the set of all rational numbers.

1. $f(n) = n^2 + 2$
2. $f(n) = n^3$
3. $f(n) = n^2 - 4n + 2$
4. $f(n) = \frac{n}{n+1}$, $n \neq ^-1$

Assignments

Minimum 1–3. Average 4–7.
Maximum Odd-numbered exercises.

Objective

The student can find ordered pairs of numbers for linear equations and draw graphs of the equations.

PREPARATION

Students have had experience in graphing linear equations using ordered pairs of whole numbers (Learning Unit E, Module 2) and using ordered pairs of integers (Learning Unit E, Module 4). You may find that a brief review of graphing will help students to prepare for this lesson on graphing using ordered pairs of rational numbers.

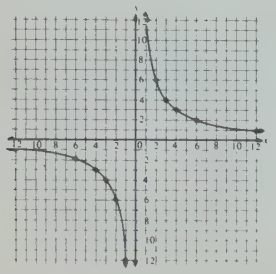
INVESTIGATION

It is quite likely that students will name only ordered pairs of integers. Suggest that they also name some non-integral pairs that satisfy the equation $x + y = 2$ and observe that the graph of these pairs appears to fall in a line with the other points graphed.

DISCUSSION

If an equation is known to be a linear equation, then it is sufficient to graph only two points to determine the location of the line for the equation. However, you should encourage students to graph several points (at least three) satisfying the equations, and then draw the line through the points to indicate all the points on the line. If a particular point does not seem to fall on the line, computation should be checked to see if a mistake has been made.

In Exercise 6, note that it is important that both positive and negative integers be considered for x and y; otherwise, only half of the graph will be drawn.

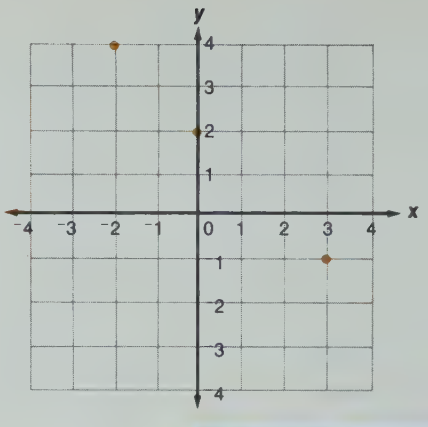


Graphing Linear Equations

Investigating the Ideas

The co-ordinates of three points for the graph of the equation $x + y = 2$ have been found and graphed.

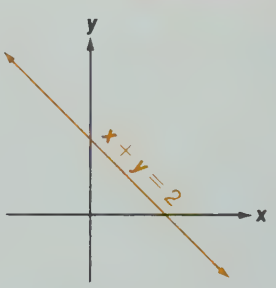
equation $x + y = 2$		co-ordinates
x	y	(x, y)
0	2	→ (0, 2)
3	$^-1$	→ (3, $^-1$)
$^-2$	4	→ ($^-2$, 4)



Can you find the co-ordinates of at least six more points for the equation and show them on a grid?

Discussing the Ideas

1. Do all the points that you found seem to lie in one line? If they do, draw a line through them. This shows that the replacement set of x and y is the set of rational numbers.



2. A Is it true that $\frac{5}{4} + \frac{7}{8} = 2$?
   B Is the pair $\left(\frac{5}{4}, \frac{7}{8}\right)$ the co-ordinates of some point on the graph of $x + y = 2$?

3. Is the pair $\left(\frac{^-3}{2}, \frac{7}{2}\right)$ the co-ordinates of some point on the graph of $x + y = 2$? How can you tell?

4. The point with co-ordinates $\left(^-1\frac{9}{10}, y\right)$ is a point on the graph of $x + y = 2$. What rational number is y?

5. Equations which have graphs that are straight lines are called **linear equations**. Is $x + y = 2$ a linear equation?

6. Find some co-ordinates for the equation $x \cdot y = 12$. Show that $x \cdot y = 12$ is not a linear equation.

Using the Ideas

For Exercises 1 through 6, complete each table and graph the equation. The replacement set for x and y is the set of rational numbers.

1. equation: $x + y = 3$

	x	y
	1	2
	$^-1$	4
A	2	
B	$^-2$	

2. equation: $x - y = 2$

	x	y
	5	3
	3	1
A	0	
B	$^-2$	

3. equation: $2x - y = 0$

	x	y
	2	4
	$^-3$	$^-6$
A	0	
B	4	

4. equation: $x - 2y = {}^-2$

	x	y
A	0	
B	4	
C	$^-4$	

5. equation: $2y + y = {}^-5$

	x	y
A	0	
B	$^-5$	
C	1	

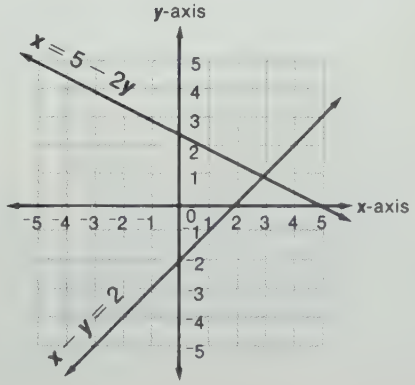
6. equation: $3x - 2y = 8$

	x	y
A	2	
B	0	
C	$^-4$	

7. Make your own table and then graph each equation. Find at least three pairs of co-ordinates for each equation.

A $x + y = 5$ B $x - y = {}^-1$ C $2x - y = 1$

★**8.** Use the figure to give the number pair that is common to the graphs of both equations. **Substitute** this pair of values into each equation to check this choice.



★**9.** Graph both of the following equations on the same grid:

$x + y = 0$ $x - 2y = 15$

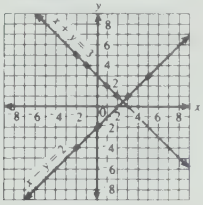
Find the co-ordinates of the point that the two graphs have in common. Check to see that this number pair "works" in both equations.

UTILIZATION

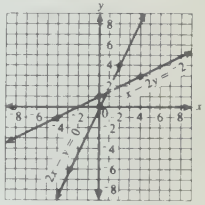
Exercises 8 and 9 involve graphs of *simultaneous* linear equations. The intersection of the lines of the two graphs is the point whose co-ordinates satisfy both equations.

Answers, Exercises 1–7, 9

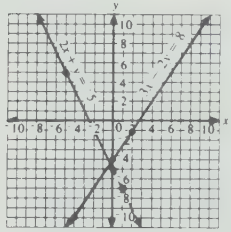
Exercises 1, 2



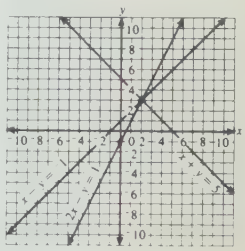
Exercises 3, 4



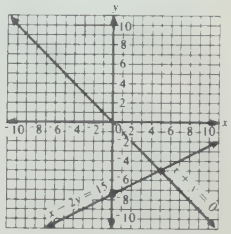
Exercises 5, 6



Exercise 7A–C



Exercise 9



EXTENSION

Additional practice exercises are provided on *Duplicator Masters* page 33.

Enrichment: Ask students to find the point of intersection of the graphs of these pairs of equations.

1. $x - y = 2$, $x + y = 3$ $(2\frac{1}{2}, \frac{1}{2})$
2. $y = 2x$, $x + y = 6$ $(2, 4)$
3. $x - y = {}^-1$, $x - 2y = 3$ $({}^-5, {}^-4)$

Assignments

Minimum 1–3. Average 1–4, 7A.
Maximum 4–9.

Objective

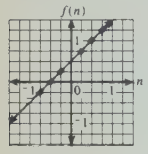
The student can demonstrate the ability to work with the major concepts and topics of this module.

UTILIZATION

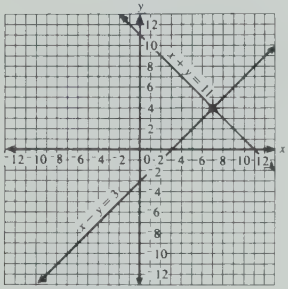
The review exercises can be utilized in any manner that will be beneficial to your students. Some exercises should be discussed orally. Students could show completed solutions for other exercises on the chalkboard. There are over sixty separate responses required in these exercises, so it is not necessary that all parts of every exercise be assigned.

Answers, Exercises 12 and 14

Exercise 12



Exercise 14



EXTENSION

Additional review exercises are provided on page 46 of the *Workbook*.

REVIEWING THE IDEAS

1. A The opposite of $\frac{3}{7}$ is ?.
B The opposite of $\frac{-2}{3}$ is ?.
C $\frac{-5}{7}$ is the additive inverse of ?.
D The additive inverse of 0 is ?.

2. Find the sums.

A $\frac{3}{8}+\frac{-3}{8}$ D $\frac{1}{3}+\frac{-1}{2}$
B $\frac{5}{7}+\frac{-2}{7}$ E $\frac{3}{4}+\frac{-1}{8}$
C $\frac{-2}{3}+\frac{-1}{6}$ F $\frac{1}{4}+\frac{-5}{8}$

3. Find the differences.

A $\frac{3}{7}-\frac{4}{7}$ D $\frac{1}{3}-\frac{-4}{3}$
B $\frac{-7}{8}-\frac{1}{8}$ E $\frac{5}{12}-\frac{7}{12}$
C $\frac{5}{7}-\frac{-1}{7}$ F $\frac{-2}{5}-\frac{-4}{5}$

4. Find the products.

A $\frac{3}{5}\cdot\frac{4}{3}$ D $\frac{3}{8}\cdot\frac{-4}{9}$ G $3\frac{2}{3}\cdot\frac{3}{11}$
B $\frac{2}{3}\cdot\frac{-3}{5}$ E $\frac{-8}{3}\cdot\frac{-6}{5}$ H $1\frac{1}{2}\cdot\frac{-5}{6}$
C $\frac{-5}{6}\cdot\frac{-2}{3}$ F $\frac{9}{10}\cdot\frac{-5}{3}$ I $15\cdot\frac{-3}{5}$

5. What is the reciprocal of each number?

A $\frac{3}{4}$ C $\frac{-1}{6}$ E 4 G $\frac{-4}{3}$
B $\frac{-2}{3}$ D $\frac{-3}{7}$ F $^{-}5$ H $1\frac{2}{5}$

6. Find the quotients.

A $\frac{9}{10}\div\frac{3}{2}$ D $\frac{14}{15}\div\frac{2}{3}$ G $3\frac{1}{3}\div\frac{-5}{6}$
B $\frac{-1}{4}\div\frac{1}{2}$ E $\frac{-15}{16}\div\frac{-3}{8}$ H $-2\frac{2}{3}\div1\frac{1}{2}$
C $\frac{1}{2}\div\frac{-3}{8}$ F $\frac{5}{12}\div\frac{-5}{4}$ I $\frac{-7}{8}\div1\frac{2}{5}$

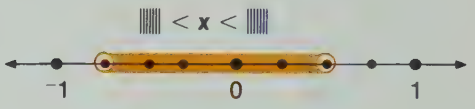
7. Solve the equations.

A $\frac{5}{8}\cdot n=10$ E $\frac{9}{10}\cdot n=\frac{1}{2}$
B $\frac{7}{5}-n=\frac{3}{10}$ F $8\cdot\left(3+\frac{7}{8}\right)=n$
C $n\div\frac{2}{7}=\frac{3}{2}$ G $10\cdot(2+n)=22$
D $\frac{2}{3}+n=\frac{7}{6}$ H $\left(\frac{5}{6}\cdot\frac{1}{2}\right)+\left(\frac{1}{6}\cdot\frac{1}{2}\right)=n$

8. Give the smaller number of each pair.

A $\frac{1}{6}, \frac{3}{8}$ C $\frac{-3}{10}, \frac{-1}{5}$ E $-1\frac{1}{2}, \frac{-1}{2}$
B $\frac{11}{16}, \frac{3}{5}$ D $0, \frac{-1}{2}$ F $-1\frac{1}{3}, \frac{1}{3}$

9. Give the rational numbers for the inequality suggested by this number line.



10. An article originally priced at \$132 is marked for sale at $\frac{1}{4}$ off. What is the sale price?

11. Complete the function rule table.

	Function Rule $n+\frac{1}{2}$	
	n	$f(n)$
	$\frac{3}{4}$	$\frac{5}{4}$
	$\frac{1}{2}$	1
A	$\frac{1}{4}$	
B	$\frac{-1}{4}$	
C	$\frac{-1}{2}$	
D	$\frac{-3}{4}$	

12. Draw a graph of the function for Exercise 11. The domain is the set of rational numbers.

13. Give four number pairs for points on the graph of the equation $2x+y=11$.

14. Give the number pair for the point of intersection of the two lines $x+y=11$, $x-y=3$. Graph both equations.

TEST YOURSELF

1. Give the additive inverse of each rational number.
   A $\frac{3}{4}$ B $\frac{-2}{3}$ C $2\frac{1}{4}$

2. Solve the equations.
   A $\frac{-7}{10} + x = \frac{-7}{10}$ C $\frac{-8}{3} + t = 0$
   B $\frac{5}{8} + \frac{-2}{3} = r + \frac{5}{8}$ D $s \cdot 1 = \frac{-3}{10}$

3. Find the sums.
   A $\frac{3}{5} + \frac{1}{2}$ C $\frac{-7}{3} + \frac{5}{3}$
   B $\frac{-1}{5} + \frac{3}{5}$ D $\frac{-1}{2} + \frac{7}{4}$

4. Find the differences.
   A $\frac{7}{8} - \frac{1}{2}$ B $\frac{-5}{6} - \frac{1}{6}$

5. Find the products.
   A $\frac{1}{5} \cdot \frac{1}{7}$ C $^{-}3 \cdot \frac{1}{4}$
   B $\frac{2}{3} \cdot \frac{-5}{6}$ D $\frac{-3}{2} \cdot \frac{-5}{3}$

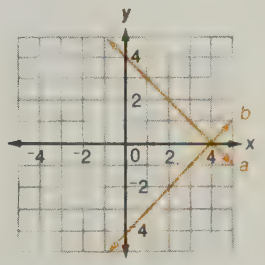
6. What is the reciprocal of $\frac{-5}{8}$?

7. Find the quotients.
   A $\frac{1}{2} \div \frac{3}{4}$ C $1\frac{1}{4} \div \frac{-3}{4}$
   B $\frac{-1}{4} \div \frac{-1}{8}$ D $\frac{-5}{6} \div 1\frac{1}{4}$

8. Give the correct symbol ($<$, $=$, or $>$) for each ▨.
   A $\frac{-5}{6}$ ▨ $\frac{2}{3}$ B $\frac{-3}{2}$ ▨ $\frac{-2}{3}$

9. An article originally priced at \$124 is marked for sale at $\frac{1}{4}$ off. What is the sale price?

10. Which line, *a* or *b*, is the graph of the equation $x - y = 4$?



RESEARCH PROJECTS

A Design a flow chart that will show the steps in finding the sum of any two rational numbers. Try to make flow charts for the other operations using fractional numbers.

B Read about the historical development of the symbols for fractions. Find out what other notation for fractions has been used. Write a report on your findings. *See *History of Mathematics* by David E. Smith; New York: Dover Publications, 1925, vol. II, pp. 208-228; General Publishing)

C Every rational number can be expressed as a **continued fraction**.

For example, $\frac{5}{12} = \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$ and

$$\frac{-11}{5} = -3 + \frac{1}{1 + \frac{1}{4}}$$

Find out what continued fractions are and how they can be computed from ordinary fractions. (See *An Introduction To Continued Fractions* by Charles G. Moore: Washington, D.C.: National Council of Teachers of Mathematics, 1964.)

TEST YOURSELF

Students who are successful with this self-test should be expected to achieve reasonable success with the Module Achievement Tests which accompany *School Mathematics* and are available from the publisher in separate test packets. A correlation of the Test Yourself questions with the objectives stated on the module-opening pages is annotated on the student book page at the left.

RESEARCH PROJECTS

In addition to the projects suggested here, some students might be interested in reading about some other rational number topics. The following references are suggested.

Unit Fractions, by Julia Adkins, Twenty-Seventh Yearbook, National Council of Teachers of Mathematics, Washington, D.C., 1963, pp. 221–226.

Farey Sequences, by George Cunningham, Twenty-Eighth Yearbook, National Council of Teachers of Mathematics, Washington, D.C., 1963, pp. 22–26.

MATHEMATICAL RECREATION

There are several patterns for student discovery in this recreation. The total number of small triangles in each configuration is always a square number. That is, the sequence formed by the number of triangles is 1, 4, 9, 16, 25, . . . , n^2, ...

The pattern of the fractions for the part of each region that is shaded can be shown to follow the sequence,

$$1, \frac{3}{4}, \frac{6}{9}, \ldots, \frac{n(n+1)}{2n^2}, \ldots$$

The part of each region that is not shaded can be found by subtracting the part shaded from 1.

The sequence for the nonshaded part is

$$0, \frac{1}{4}, \frac{3}{9}, \frac{6}{16}, \ldots, \left(1 - \frac{n(n+1)}{2n^2}\right), \ldots$$

3. For a configuation of 100 triangles, we observe that $n^2 = 100$, hence $n = 10$. Substituting in $\frac{n(n+1)}{2n^2}$, we get $\frac{10(11)}{2 \cdot 100} = \frac{11}{20}$ of the region shaded.

4. For 400 small triangles, $n^2 = 400$ and $n = 20$. Hence $\frac{n(n+1)}{2n^2} = \frac{20(21)}{2 \cdot 400} = \frac{21}{40}$.

5. The student will probably guess that as the configuration of triangles becomes larger and larger the part of the region shaded becomes nearer and nearer to $\frac{1}{2}$. This can be proved by writing the general term of the sequence

$$\frac{n(n+1)}{2n^2} \text{ as } \frac{n^2+n}{2n^2} = \frac{n^2}{2n^2} + \frac{n}{2n^2}.$$

Simplifying the last member, we obtain $\frac{1}{2} + \frac{1}{2n}$. When n becomes very large $\frac{1}{2n}$ approaches 0 and the expression $\frac{1}{2} + \frac{1}{2n}$ approaches $\frac{1}{2}$.

MATHEMATICAL RECREATIONS

TRIANGULAR PATTERNS

Figure	Number of small triangles	Part shaded	Part unshaded
△	1	1	0
	4	$\frac{3}{4}$	$\frac{1}{4}$
	9	$\frac{6}{9}$	$\frac{3}{9}$
	16	$\frac{10}{16}$	$\frac{6}{16}$
⋮	⋮	⋮	⋮

1. Draw and shade the next three triangular figures.

2. What fractional part of each region that you drew is shaded?

3. If the large triangular figure contained 100 small triangles, what part of the figure would be shaded? What part of it would be unshaded?

4. If the large triangular figure contained 400 small triangles, what part of the figure would be shaded? What part would be unshaded?

5. If the large triangular figure contained a "very large" number of small triangles, about what part of the figure would be shaded?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $\frac{-3}{4}$; B $\frac{2}{3}$; C $-2\frac{1}{4}$ 2. A $x = 0$; B $r = \frac{-2}{3}$; C $t = \frac{8}{3}$; D $s = \frac{-3}{10}$
3. A $\frac{11}{10}$; B $\frac{2}{5}$; C $\frac{-2}{3}$; D $\frac{5}{4}$ 4. A $\frac{3}{8}$; B -1 5. A $\frac{1}{35}$; B $\frac{-5}{9}$; C $\frac{-3}{4}$; D $\frac{5}{2}$
6. $\frac{-8}{5}$ 7. A $\frac{2}{3}$; B 2; C $\frac{-5}{3}$; D $\frac{-2}{3}$ 8. A $<$; B $<$ 9. $93 10. *b*

UNIT F: RATIONAL NUMBERS

Module 2: Decimals

General Objectives

To provide a necessary review of decimal notation, and computation with decimals.

To emphasize applications of decimals in scientific notation for both large and small rational numbers.

To provide experience in estimation and problem solving using decimals.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
52	**RED** Given powers of ten with negative integral exponents, the student can interpret them.	F-38, 39	**WB** 47 **DM** 34		
53	**RED** Given a decimal numeral, the student can give the place value of each digit.	F-40, 41	**WB** 48 **DM** 35	**SP** h-1	
54	**RED** Given two decimals, the student can find the sum.	F-42, 43	**WB** 49 **DM** 36	**ASC** D-1 **SP** h-2	**PD** h-1 **DS** g-2
55	**RED** Given two decimals, the student can find the difference.	F-42, 43	**WB** 49 **DM** 36	**ASC** D-2 **SP** h-2	**PD** h-1 **DS** g-2
56	**RED** Given two decimals, the student can find the product.	F-46, 47	**WB** 50 **DM** 37	**ASC** D-3 **SP** h-3, 4	**PD** h-1 **DS** g-2, 3
57	**RED** Given two decimals, the student can find the quotient.	F-52, 53	**WB** 53 **DM** 39	**ASC** D-4, 5, 6 **SP** h-3, 4	**PD** h-1 **DS** g-3
58	**RED** Given two decimals, the student can round each decimal to one significant digit in order to estimate the product or quotient.	F-54, 55	**WB** 54	**SP** h-1	
59	**RED** Given a fraction, the student can find a decimal approximation for it.	F-56, 57	**DM** 40	**ASC** D-8	
60	**YELLOW** Given a decimal numeral, the student can express it in scientific notation, and vice versa.	F-48, 49	**WB** 51 **DM** 38	**ASC** AP-3, 4 **SP** a-3	**SWM 3** 196-197
61	**YELLOW** Given a fraction, the student can write a mixed decimal numeral for it.	F-58, 59		**DS** g-2	
62	**YELLOW** Given a rational number, the student can find the repeating decimal for it.	F-60, 61	**WB** 55	**DS** g-2	
63	***GREEN*** *Given a measurement using decimal numerals, the student can find the greatest possible error for it.*	F-50, 51	**WB** 52		
	Reviewing the Ideas	F-64	**WB** 56		

MATHEMATICS

Although students at this level are familiar with decimal notation and computation with decimals, there are sufficient new ideas in the module to warrant careful treatment.

Understanding of the fractional notation for rational numbers and positive and negative integers are prerequisites for a complete understanding of decimal notation. By defining $10^{-n} = \frac{1}{10^n}$ where n is an integer, any decimal can be expressed in *expanded notation*, exponent form. Thus, 312.6145 can be expressed as $3 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0 + 6 \cdot 10^{-1} + 1 \cdot 10^{-2} + 4 \cdot 10^{-3} + 5 \cdot 10^{-4}$. Use of negative exponents permits the extension of the use of scientific notation to all decimals.

Repeating decimals are introduced in the module and by means of examples students are shown that any rational number has a repeating decimal representation. Conversely every repeating decimal represents a rational number. This is an important mathematical idea, especially for students who will later encounter real numbers (Learning Unit H, Module 3). Real numbers are those numbers which can be expressed as infinite decimals, repeating or not. The subset of the real numbers which cannot be represented by repeating decimals are *irrational numbers*.

TEACHING THE MODULE

Vocabulary

accuracy
approximation
Celsius temperature
decimal equivalent
decimals, one-place, two-place, . . .
equivalent decimals
estimation
expanded notation (exponent form)
exponential notation
greatest possible error
mixed decimal numeral
negative exponents
period
precision
relative error
repeating decimal
rounding
scientific notation
significant digits
specific gravity
terminating decimal

This module provides vital background for future work in extending the rational number system to the real numbers. Students should develop an understanding of the decimal system of numeration, develop adequate skills in computation using decimals, and experience a wide variety of applications of decimal notation.

In teaching this module, be particularly sensitive to the needs and abilities of your class. With abler classes, you may want to pass quickly over the material which is strictly review. With slower classes, you may want to spend more time on the development of basic skills in decimal computation. Finally, you should make an effort to help most students gain an understanding of repeating decimals as representations of rational numbers.

Lesson Schedule

Your actual lesson schedule will depend on the amount of time spent in review of basic material and in development of computational skills with decimals, but 12 to 15 days should be enough time for adequate coverage of the module.

Supplementary Exercises are provided in the Appendix for additional practice, and the *Workbook* and *Duplicator Masters* can be used to provide reinforcement, remedial, and extension work.

Evaluation

Evaluation of student progress should focus on three major areas:

1. Understanding of decimal notation
2. Computational skill with decimals
3. Applications of decimals to problem situations.

The review exercises and the Test Yourself quiz should help students prepare for an achievement test covering the module.

RESOURCES FOR ACTIVE LEARNING

Activities

Cloudburst, Vol. 5, Nos. 5700 series (science applications); Nos. 7400 series (distance/-speed), Midwest Publications

Fractions, "Fractionals and decimals," Set 8, Herder and Herder

Let's Play Games in Metrics (games and activities involving temperature), pp. 184–185, National Textbook Co.

Nuffield Project: *Computation and Structure 4* (decimals), pp. 1–12; *Computation and Structure 5* (decimal numbers), pp. 1–6, 45–57; *Decimals 1*, Wiley

The Metric System, Addison-Wesley

Manipulative Devices

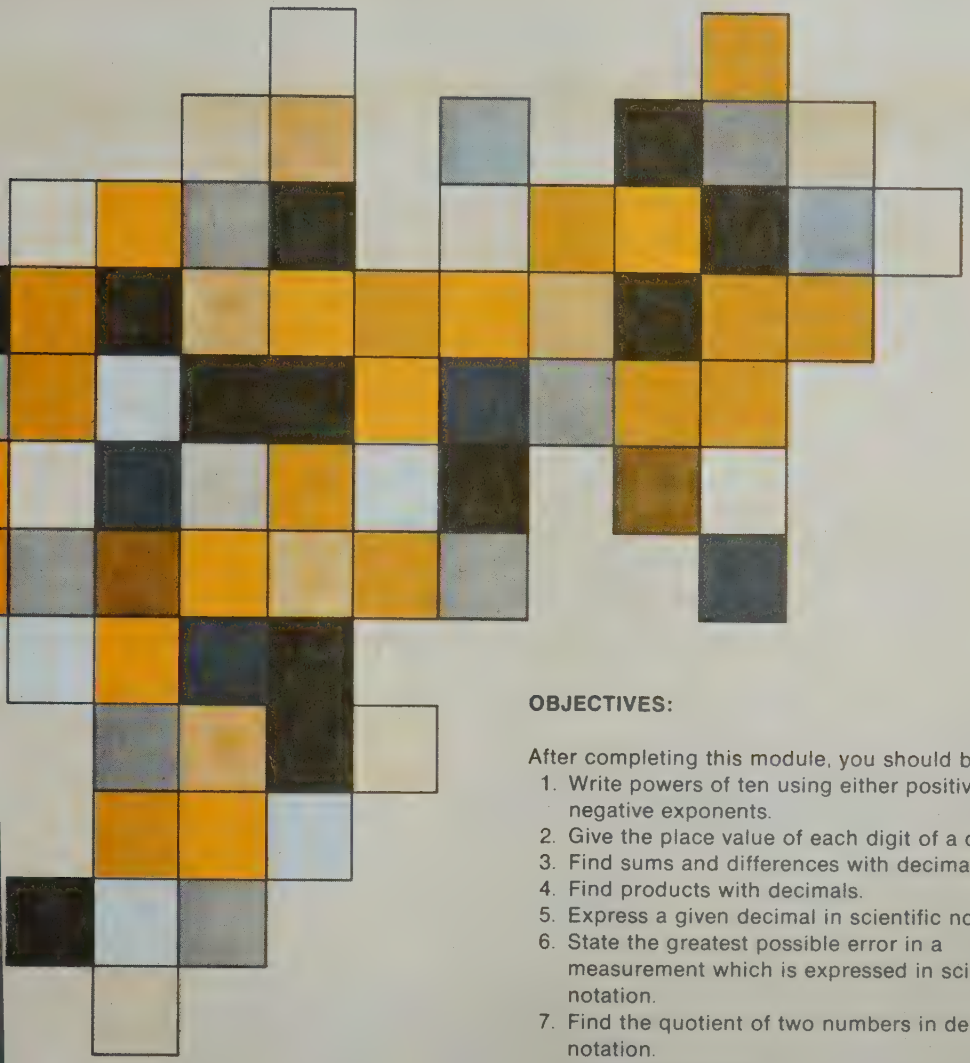
Base-Ten Blocks (Addison-Wesley)

Metric Aids (Addison-Wesley; Balla; Creative Publications; Dick Blick; Educational Teaching Aids)

Slide rules (Creative Publications; Geyer; Pickett)

Games and Puzzles

In Order Game Kit (Midwest Publications)



OBJECTIVES:

After completing this module, you should be able to:

1. Write powers of ten using either positive or negative exponents.
2. Give the place value of each digit of a decimal.
3. Find sums and differences with decimals.
4. Find products with decimals.
5. Express a given decimal in scientific notation.
6. State the greatest possible error in a measurement which is expressed in scientific notation.
7. Find the quotient of two numbers in decimal notation.
8. Estimate products and quotients using decimals.
9. Find a decimal for a given rational number in fraction notation.
10. Find a repeating decimal for a given rational number.

Objective

The student can interpret numbers having exponents which are positive, zero, or negative integers.

PREPARATION

At this time the student is familiar with non-negative exponents, and it is assumed that he knows that the following laws of exponents hold:

If a and b are non-negative integers,

(1) $10^a \cdot 10^b = 10^{a+b}$

(2) $(10^a)^b = 10^{ab}$

(3) $10^a \div 10^b = 10^{a-b}$ if $a > b$

$10^a \div 10^b = \frac{1}{10^{b-a}}$ if $a < b$

By defining $10^{-n} = \frac{1}{10^n}$ for any integer n, laws (1) and (2) can be shown to hold and the necessity of considering the two cases in (3) can be removed: thus $10^a \div 10^b = 10^{a-b}$ for any integers a and b.

INVESTIGATION

The purpose of the Investigation is to prepare students for the definition of negative exponents. To reinforce this idea, you might write a division problem such as $10^2 \div 10^3$ as $\frac{10^2}{10^3} = \frac{1}{10^1}$. But if the division law for exponents holds for all integers, we must have $\frac{10^2}{10^3} = 10^{2-3} = 10^{-1}$. Hence $10^{-1} = \frac{1}{10^1}$.

DISCUSSION

Exercise 2 contains the general definition for powers with integer exponents. You will want to provide additional examples in which the bases are positive integers other than ten. Exercise 3 relates rational numbers to expanded notation using negative powers of ten.

Negative Exponents

Investigating the Ideas

You are already familiar with non-negative integer exponents. The idea of exponents may be extended to include negative integers. Study the table of powers of ten. Notice that each power of ten is $\frac{1}{10}$ as large as the one above it.

Can you discover a pattern for the powers of ten in the table and give the number for each ▒?

POWERS OF TEN

10^4	$= 10\,000$	$\leftarrow \frac{1}{10}$ as large as 10^5
10^3	$= 1000$	$\leftarrow \frac{1}{10}$ as large as 10^4
10^2	$= 100$	$\leftarrow \frac{1}{10}$ as large as 10^3
10^1	$= 10$	$\leftarrow \frac{1}{10}$ as large as 10^2
10^0	$= 1$	$\leftarrow \frac{1}{10}$ as large as 10^1
10^{-1}	$= \frac{1}{10}$	$\leftarrow \frac{1}{10}$ as large as 10^0
10^{-2}	$=$ ▒	$\leftarrow \frac{1}{10}$ as large as 10^{-1}
10^{-3}	$=$ ▒	$\leftarrow \frac{1}{10}$ as large as 10^{-2}
10^{-4}	$=$ ▒	$\leftarrow \frac{1}{10}$ as large as 10^{-3}
10^{-5}	$=$ ▒	$\leftarrow \frac{1}{10}$ as large as 10^{-4}

Discussing the Ideas

1. The pattern of the table suggests that $10^{-1} = \frac{1}{10^1} = \frac{1}{10}$, $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$, ...

   A How could 10^{-6} be rewritten with a positive exponent?

   B How could $\frac{1}{100\,000\,000}$ be expressed as a power of ten?

2. To extend the idea of exponents to include all integers, we make the following definition:

   If b is a positive integer and n is any integer, then $b^{-n} = \frac{1}{b^n}$.

   Thus, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ and $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$.

   A How can you express 7^{-2} with a positive exponent?

   B How can you express $\frac{1}{2^3}$ with a negative exponent?

3. The example shows how $7 \cdot 10^{-1} + 3 \cdot 10^{-2}$ can be expressed as a rational number. Express each of these sums as a rational number.

   EXAMPLE:
   $7 \cdot 10^{-1} + 3 \cdot 10^{-2} = 7 \cdot \frac{1}{10} + 3 \cdot \frac{1}{100}$
   $= \frac{7}{10} + \frac{3}{100}$
   $= \frac{73}{100}$

   A $2 \cdot 10^{-1} + 3 \cdot 10^{-2}$

   B $6 \cdot 10^{-1} + 2 \cdot 10^{-2} + 9 \cdot 10^{-3}$

   C $3 \cdot 10^1 + 2 \cdot 10^0 + 7 \cdot 10^{-1}$

   D $5 \cdot 10^2 + 5 \cdot 10^{-2}$

Using the Ideas

1. Rewrite each power using positive exponents.

A 10^{-2}	**C** 10^{-5}	**E** 6^{-2}	**G** 2^{-8}	**I** 7^{-3}
B 10^{-4}	**D** 10^{-14}	**F** 5^{-3}	**H** 3^{-4}	**J** 8^{-5}

2. Rewrite each numeral using negative exponents.

A $\frac{1}{10^3}$	**C** $\frac{1}{7^5}$	**E** $\frac{1}{10^1}$	**G** $\frac{1}{1000}$	**I** $\frac{1}{2}$
B $\frac{1}{10^4}$	**D** $\frac{1}{3^3}$	**F** $\frac{1}{10^1}$	**H** $\frac{1}{10\,000}$	**J** $\frac{1}{3^5}$

3. Rewrite each numeral as a power of ten.

A 1000	**C** $\frac{1}{100}$	**E** 10 000	**G** 100 000	**I** 1 000 000
B $\frac{1}{1000}$	**D** 100	**F** $\frac{1}{10}$	**H** $\frac{1}{10\,000}$	**J** 1

4. Express each sum as a rational number.

A $6 \cdot 10^{-1} + 4 \cdot 10^{-2}$

B $7 \cdot 10^{-1} + 3 \cdot 10^{-3}$

C $3 \cdot 10^{0} + 5 \cdot 10^{-1}$

D $1 \cdot 10^{0} + 3 \cdot 10^{-1} + 9 \cdot 10^{-2}$

E $8 \cdot 10^{2} + 7 \cdot 10^{0} + 4 \cdot 10^{-1}$

F $5 \cdot 10^{-1} + 6 \cdot 10^{-2} + 3 \cdot 10^{-3} + 1 \cdot 10^{-4}$

G $1 \cdot 10^{1} + 2 \cdot 10^{0} + 3 \cdot 10^{-1} + 7 \cdot 10^{-2}$

H $3 \cdot 10^{3} + 1 \cdot 10^{1} + 6 \cdot 10^{-2}$

★ **5.** What rational number is this sum?

$1 \cdot 2^{2} + 1 \cdot 2^{1} + 1 \cdot 2^{0} + 1 \cdot 2^{-1} + 1 \cdot 2^{-2}$

★ **6.** Which statement is correct? Try to prove your answer.

A $10^{-1} \cdot 10^{-1} = 10^{1}$

B $10^{-1} \cdot 10^{-1} = 10^{-1}$

C $10^{-1} \cdot 10^{-1} = 10^{-2}$

D $10^{-1} \cdot 10^{-1} = 10^{2}$

think

Sixteen pencils are arranged to form five squares. Show how four squares can be formed by moving two of the pencils to new positions. All sixteen pencils must be used, and each must be a side of a square.

UTILIZATION

Although negative exponents can be used with bases other than 10, as suggested in the text, the emphasis in the exercise set is upon powers of ten. This practice is followed because at this time the application of negative exponents will be to decimal fractions.

EXTENSION

Further practice exercises are provided on *Workbook* page 47 and *Duplicator Masters* Page 34.

Enrichment: Ask some students to try problems like the ones that follow. (Answers are in parentheses.)

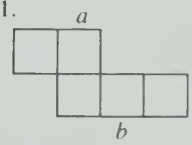
1. Give the integer for a and the rational number for b.
   $10^{4} \cdot 10^{-3} = 10^{a} = b \quad (a = 1, b = 10)$
   $10^{-2} \cdot 10^{6} = 10^{a} = b \quad (a = 4, b = 10\,000)$
   $10^{-1} \cdot 10^{-1} = 10^{a} = b \quad (a = -2, b = \frac{1}{100})$
   $10^{-5} \cdot 10^{4} = 10^{a} = b \quad (a = -1, b = \frac{1}{10})$
   $10^{3} \cdot 10^{-3} = 10^{a} = b \quad (a = 0, b = 1)$
   $10^{2} \cdot 10^{1} = 10^{a} = b \quad (a = 3, b = 1000)$

2. Give the number for x as a power of 10.
   $10^{2} \cdot x = 10^{4} \quad (x = 10^{2})$
   $10^{5} \cdot x = 10^{3} \quad (x = 10^{-2})$
   $10^{-3} \cdot x = 10^{-2} \quad (x = 10^{1})$
   $10^{2} \cdot x = 10^{2} \quad (x = 10^{0})$
   $10^{-4} \cdot x = 10^{0} \quad (x = 10^{4})$
   $10^{-3} \cdot x = 10^{-7} \quad (x = 10^{-4})$

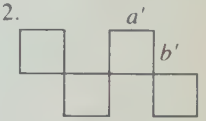
Think Solution

Move pencils a and b, as shown in Figure 1, to positions a' and b', as shown in Figure 2.

1.



2.



Assignments

Minimum 1–3. Average 1–4. Maximum 1–6.

Objectives

The student can write decimal fractions for fractions having denominators which are powers of ten.

The student can state the place value of each digit in a decimal.

PREPARATION

Decimal notation should be quite familiar to all students at this level. Conduct a discussion of the advantages of decimal notation over fractional notation. Discuss what advantages fractional notation might sometimes have over decimal notation. (The fraction $\frac{1}{3}$ might be cited to illustrate that certain fractions are simpler to use than their decimal equivalents.)

INVESTIGATION

The Investigation capitalizes on the students' familiarity with decimals for money notation. In discussing the Investigation, you might bring out the fact that when we write a personal check for a certain amount of money we customarily use decimal notation on one part of the check and a combination of words and fractions on another part of the check.

DISCUSSION

Exercise 1 reviews the use of decimal notation. Note that in the decimal representation of numbers less than 1, a zero is written preceding the decimal point. Encourage students to adopt this usage; they will encounter it more and more frequently as use of the metric system of measure becomes more fully implemented throughout the world.

You can use the decimals given in Exercise 3 to review reading of decimals, if necessary.

Answers, Discussion Exercise 3

A. $75.4 = 7 \cdot 10^1 + 5 \cdot 10^0 + 4 \cdot 10^{-1}$

B. $2.708 = 2 \cdot 10^0 + 7 \cdot 10^{-1} + 0 \cdot 10^{-2} + 8 \cdot 10^{-3}$

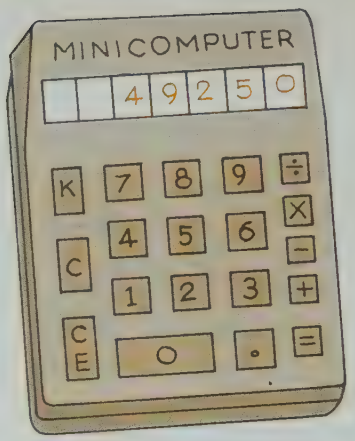
C. $129.35 = 1 \cdot 10^2 + 2 \cdot 10^1 + 9 \cdot 10^0 + 3 \cdot 10^{-1} + 5 \cdot 10^{-2}$

D. $3.1416 = 3 \cdot 10^0 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 1 \cdot 10^{-3} + 6 \cdot 10^{-4}$

Decimal Notation

Investigating the Ideas

Suppose the minicomputer shows a certain amount of money but does not show the decimal point.



Can you place a decimal point in the numeral so that it shows

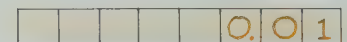
A the cost of a new automobile?

B the cost of a table model AM-FM radio?

C the cost of a color television set?

Discussing the Ideas

1. Instead of $\frac{1}{10}$, a minicomputer would show

Instead of $\frac{1}{100}$, a minicomputer would show 

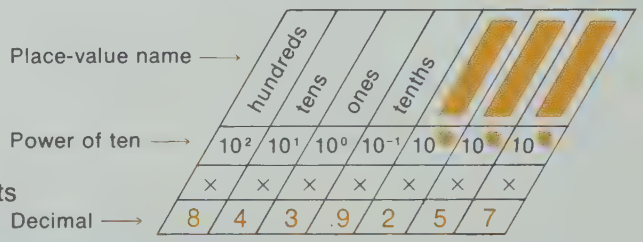
What decimal would a minicomputer show for each of these?

A $\frac{1}{1000}$ B $\frac{1}{10\,000}$ C $\frac{2}{10}$ D $\frac{3}{100}$ E $\frac{27}{100}$ F $\frac{179}{1000}$

2. The diagram will help you review how place value is used in decimal notation.

A Give the missing place-value names.

B Give the missing exponents for the power of ten.



3. The decimal in the diagram in Exercise 2 can be written using **expanded notation** (exponent form) as

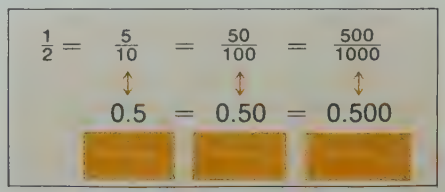
$8 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0 + 9 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3} + 7 \cdot 10^{-4}$.

How would you write each of these decimals in expanded notation?

A 75.4 B 2.708 C 129.35 D 3.1416

4. The diagram at the right shows how to write a one-place decimal, a two-place decimal, and a three-place decimal for $\frac{1}{2}$. Give one-, two-, and three-place decimals for each of these fractions.

A $\frac{9}{10}$ B $\frac{3}{5}$ C $\frac{1}{10}$ D $\frac{4}{5}$



Using the Ideas

1. Give the missing word and number.

EXAMPLE: 32.57: The 7 in the hundredths' place represents 7×10^{-2}

A 325.467: The 3 in the ? place represents ▥.

B 325.467: The 6 in the ? place represents ▥.

C 325.467: The 7 in the ? place represents ▥.

2. Write each fraction in decimal notation.

A $\frac{3}{10} = 0.3$ C $\frac{19}{100}$ E $\frac{9}{10}$ G $\frac{743}{1000}$ I $\frac{39}{100}$ K $\frac{291}{10\,000}$

B $\frac{7}{10}$ D $\frac{239}{1000}$ F $\frac{3}{100}$ H $\frac{6197}{10\,000}$ J $\frac{7}{1000}$ L $\frac{9}{1000}$

3. Write a decimal for each mixed numeral.

A $23\frac{75}{100} = 23.75$ C $56\frac{375}{1000}$ E $25\frac{7}{100}$ G $62\frac{539}{10\,000}$

B $7\frac{142}{1000}$ D $475\frac{3}{10}$ F $27\frac{46}{1000}$ H $3\frac{7}{10\,000}$

4. Write each decimal in expanded notation.

A 3.147 B 298.42 C 0.6129 D 0.0067

5. Write a decimal for each of the following:

A $2 \cdot 10^1 + 3 \cdot 10^0 + 1 \cdot 10^{-1}$

B $8 \cdot 10^0 + 0 \cdot 10^{-1} + 3 \cdot 10^{-2}$

C $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0 + 8 \cdot 10^{-1} + 1 \cdot 10^{-2}$

D $7 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 10^0 + 0 \cdot 10^{-1} + 3 \cdot 10^{-2}$

6. For the fraction $\frac{7}{10}$, write

A a one-place decimal. B a two-place decimal. C a three-place decimal.

7. Write a three-place decimal and a four-place decimal for 0.75.

8. Write each minicomputer numeral with a decimal according to the directions given.

A

A number between 10 and 100

B

A number between 1 and 10

C

A number between $\frac{1}{10}$ and $\frac{1}{100}$

think

Suppose fractional exponents are used according to these examples: $4^2 = 16 \leftrightarrow 16^{\frac{1}{2}} = 4$ and $2^3 = 8 \leftrightarrow 8^{\frac{1}{3}} = 2$
What numbers do these powers represent?

A $25^{\frac{1}{2}}$ B $27^{\frac{1}{3}}$ C $100^{\frac{1}{2}}$ D $49^{\frac{1}{2}}$ E $125^{\frac{1}{3}}$ F $81^{\frac{1}{4}}$

UTILIZATION

Since almost all of the exercises constitute review material, you may want to vary the assignment according to the needs of your students.

EXTENSION

Further practice exercises are provided on *Workbook* page 48 and *Duplicator Masters* page 35.

Enrichment: Your students may be interested in knowing that the greater percent of people in the world do not use the standard dot notation for the decimal point but use a comma instead. Thus 0.5 would be written 0,5. This is common practice in continental Europe, Asia, many African nations, Mexico, Central and South America. When the comma notation is used, the periods of numerals are marked by space. Thus 287 963,51 denotes the decimal that we would write as287 963.53. The United Kingdom uses a raised dot for a decimal point. Thus 3·5 means 3.5 (not 3 times 5).

This would be an opportune time to encourage interested students to explore Research Project A on page F-65, which deals with an earlier kind of decimal notation.

Think Solution

Since negative exponents have just been introduced, students may find fractional exponents a topic of interest. Students can find the solutions to the parts of this problem without knowledge of roots of numbers. The two completed examples should serve as a guide.

A. If $x = 25^{1/2}$, then $x^2 = 25$ and $x = 5$
B. If $y = 27^{1/3}$, then $y^3 = 27$ and $y = 3$
C. If $t = 100^{1/2}$, then $t^2 = 100$ and $t = 10$
D. If $s = 49^{1/2}$, then $s^2 = 49$ and $s = 7$
E. If $r = 125^{1/3}$, then $r^3 = 125$ and $r = 5$
F. If $n = 81^{1/4}$, then $n^4 = 81$ and $n = 3$

Assignments

Minimum 1, 2, 3, 6, 7. Average 1–7.
Maximum 1–8.

Objective

Given two numbers in decimal notation, the student can find their sum or difference.

PREPARATION

Review writing decimals for fractions whose denominators are powers of ten and equivalent decimals of 1, 2, and 3 places, such as $0.8 = 0.80 = 0.800$.

INVESTIGATION

Since addition and subtraction of decimals should be familiar to students, the Investigation can be used as a diagnostic tool to determine which students are in need of reteaching or remedial instruction.

DISCUSSION

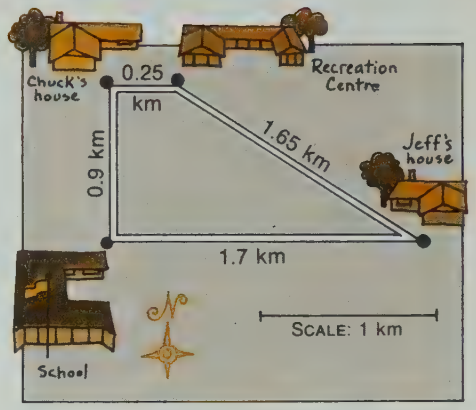
Assuming your students have developed adequate skills in computing with whole numbers, development of skill in adding and subtracting with decimal notation should follow almost automatically. In other words, once they understand the ideas associated with work in decimal notation, the actual computational aspect of the work is reduced to use of algorithms that are the same as those they use for whole number computation.

If your students are well prepared in addition and subtraction, move quickly through the Discussion to the Utilization phase.

Adding and Subtracting with Decimals

Investigating the Ideas

Study the map and the distances shown on it.



How many of the questions below can you answer?

A How far is it from school to Chuck's house to the Recreation Centre?

B How far is it from Jeff's house to school to Chuck's house?

C How much farther is it from Chuck's house to school than it is from his house to the Recreation Centre?

Discussing the Ideas

1. Suppose a student added 0.9 and 0.25 like this: →

$$\begin{array}{r} 0.9 \\ +\,0.25 \\ \hline 0.34 \end{array}\checkmark$$

Since $0.9 = \frac{9}{10} = \frac{90}{100}$ and $0.25 = \frac{25}{100}$, explain why the sum shown is incorrect.

2. To subtract 0.25 from 0.9, it is convenient to replace 0.9 by the equivalent two-place decimal 0.90. This is the same as choosing a common denominator in the problem

$$\begin{array}{rcl} 0.9 & = & 0.90 \\ -0.25 & = & -0.25 \end{array}$$

$\frac{9}{10} - \frac{25}{100} = \frac{90}{100} - \frac{25}{100}$.

Explain how you would subtract 0.347 from 0.4.

3. Explain how we can think of the sum $0.31 + 0.024 + 0.135$ as $\left(\frac{3}{10} + \frac{1}{100}\right) + \left(\frac{2}{100} + \frac{4}{1000}\right) + \left(\frac{1}{10} + \frac{3}{100} + \frac{5}{1000}\right)$.

4. Explain how the associative and commutative principles help us to think of lining up the decimal points to get $\left(\frac{3}{10} + \frac{1}{10}\right) + \left(\frac{1}{100} + \frac{2}{100} + \frac{3}{100}\right) + \left(\frac{4}{1000} + \frac{5}{1000}\right)$.

→

$$\begin{array}{r} 0.31 \\ 0.024 \\ +\,0.135 \end{array}$$

Using the Ideas

1. Find the sums.

A
$$\begin{array}{r} 87.14 \\ 23.87 \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.129 \\ 0.744 \\ 0.668 \\ \hline \end{array}$$

C
$$\begin{array}{r} 135.29 \\ 642.77 \\ 883.56 \\ \hline \end{array}$$

D
$$\begin{array}{r} 9.006 \\ 18.213 \\ 31.747 \\ \hline \end{array}$$

E
$$\begin{array}{r} 0.0714 \\ 0.0068 \\ 0.1429 \\ 0.0009 \\ \hline \end{array}$$

2. Find the differences.

A
$$\begin{array}{r} 74.3 \\ 59.8 \\ \hline \end{array}$$

B
$$\begin{array}{r} 7.05 \\ 1.96 \\ \hline \end{array}$$

C
$$\begin{array}{r} 0.621 \\ 0.583 \\ \hline \end{array}$$

D
$$\begin{array}{r} 109.3 \\ 86.5 \\ \hline \end{array}$$

E
$$\begin{array}{r} 7.002 \\ 0.785 \\ \hline \end{array}$$

3. Write equivalent three-place decimals and then add.

A $5 + 0.6 + 2.13 + 0.255$

B $1.24 + 3.062 + 4 + 2.5$

C $12.46 + 1.5 + 0.262 + 7$

D $14 + 4.32 + 1.256 + 0.7$

EXAMPLE:

$2 + 0.5 + 1.14 + 0.316 \rightarrow$

$$\begin{array}{r} 2.000 \\ 0.500 \\ 1.140 \\ +\,0.316 \\ \hline 3.956 \end{array}$$

4. Line up the decimal points and then add without changing to equivalent decimals.

A $12 + 1.4 + 0.56$

B $37 + 5.8 + 3.14 + 0.226$

C $5.4 + 11.28 + 0.423 + 7$

D $12 + 3.16 + 0.004 + 7.9$

EXAMPLE:

$17 + 0.8 + 4.26$

$$\begin{array}{r} 17. \\ 0.8 \\ +\ 4.26 \\ \hline \end{array}$$

5. Write equivalent decimals and then subtract.

A $4.3 - 1.742$

B $16 - 3.517$

C $8.2 - 4.35$

D $100 - 0.8776$

EXAMPLE:

$5.2 - 1.468 \rightarrow$

$$\begin{array}{r} 5.200 \\ -1.468 \\ \hline 3.732 \end{array}$$

6. Negative rational numbers can be represented by decimals as shown. Find the sums.

A
$$\begin{array}{r} 18.2 \\ {}^{-}12.6 \\ \hline \end{array}$$

B
$$\begin{array}{r} {}^{-}0.765 \\ {}^{-}0.426 \\ \hline \end{array}$$

C
$$\begin{array}{r} {}^{-}9.5 \\ {}^{-}6.4 \\ \hline \end{array}$$

D
$$\begin{array}{r} {}^{-}0.0068 \\ 0.0724 \\ \hline \end{array}$$

E ${}^{-}0.17 + {}^{-}0.09$

F ${}^{-}0.58 + {}^{-}0.36$

G ${}^{-}5.28 + 7.45$

H ${}^{-}23.4 + 19.15$

EXAMPLE:

${}^{-}0.72 + 0.57 \rightarrow$

$$\begin{array}{r} {}^{-}0.72 \\ +\ 0.57 \\ \hline {}^{-}0.15 \end{array}$$

More practice, page S-16, Set 30

UTILIZATION

Encourage students to develop their computational skills in addition and subtraction. If some students have difficulty computing with decimals, it is probably because they lack skill in working with whole numbers, have an inadequate understanding of decimal notation, or are careless in aligning decimal points.

Negative rational numbers in decimal form are introduced in Exercise 6. Such exercises will help the student to maintain skill in computation with negative numbers.

EXTENSION

Supplementary Exercise Set 30 on page S-16, *Workbook* page 49, *Duplicator Masters* page 36, and *Arithmetic Skill Cards* D-1 and D-2 provide additional exercises appropriate for use with this lesson.

Enrichment: To provide practice in reading and comparing decimals, play the "Hi-Lo" game described below.

Direct all players to print the numerals 0 through 9 on appropriate playing cards. Each player should also draw a picture of some kind on one unnumbered card; this will be the "joker." In this game of "Hi-Lo," the joker will represent a decimal point. Each player keeps one joker throughout the game.

Four players should combine their numeral cards to make a 40-card deck. The dealer shuffles and deals four cards face down to each player, and calls "Hi" or "Lo." Each player adds the joker to these cards and, while they are still face down, aligns the five cards in any desired order. Then the dealer asks the players to show their cards, reminding them whether he called "Hi" or "Lo." Each player turns over his cards as they are sitting before him and reads off his number. The player with the highest (or lowest) number gets a point. The winner is the player with the most points after a predetermined number of rounds.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can solve word problems involving decimals.

PREPARATION

If a Celsius thermometer is not available in your classroom, you could make a cardboard model of one for demonstration purposes. Some students might enjoy making such a model as a special project.

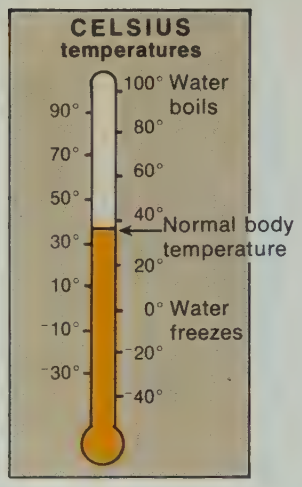
UTILIZATION

The word problems give practice in finding differences between various temperatures including negative temperatures. Allow some time for discussing the exercises and checking the answers.

Assignments

Minimum Even-numbered exercises.
Average 1–7. Maximum 1–10.

CELSIUS TEMPERATURES



1. Normal body temperature for humans is 37° C (Celsius).
   - A How much below normal temperature is 35.7° C?
   - B How much above normal temperature is 39.2° C?

2. The spiny anteater of Australia has a normal body temperature of only 23.2° C. How many degrees lower than normal human body temperature is this?

3. A marathon runner, on a hot day, was found to have a temperature of 40.6° C. How much above normal was his temperature?

4. The lowest recorded temperature for British Columbia is $^{-}58.9°$ C. The highest recorded temperature for British Columbia is 46.7° C. How many degrees difference is there in these two temperatures?

5. The lowest recorded temperature in the world was at the Soviet Antarctic station Vostok on July 24, 1960. The temperature was $^{-}88.3°$ C. The highest temperature in the world, 57.8° C, was recorded at Azizia, Libya, on September 13, 1922. What is the difference in these temperatures?

6. The record high temperature for San Francisco, California, is 41.1° C. The record low temperature is $^{-}6.7°$ C. What is the difference in these temperatures?

7. Water boils at 100° C. **Absolute zero,** the coldest possible temperature, is 373.16° C below the boiling point of water. What is the temperature for absolute zero?

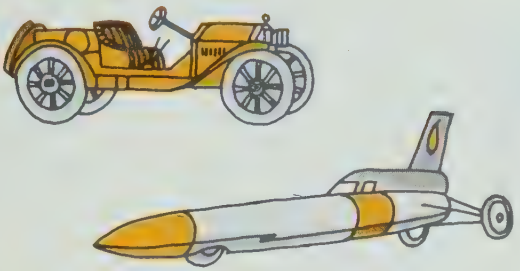
8. "Dry ice" (solid carbon dioxide) will evaporate at temperatures above $^{-}78.5°$ C How much above the temperature for absolute zero is this?

9. Mercury becomes a solid instead of a liquid at $^{-}39°$ C. Can a mercury thermometer be used to find the temperature of "dry ice"?

10. Over a period of nine years, the temperature at Saipan, Mariana Islands, varied from 19.6° C to 31.4° C. How many degrees variation is this?

Speeeeeeeeeeeeeed Records

1. In 1910, Barney Oldfield set a land speed record of 211.983 kilometres per hour (km/h). In 1970, Gary Gabelich set a new record of 1001.64 km/h. How much faster was Gabelich's speed than Oldfield's speed?



2. The greatest speed at which man has travelled was flown by the astronauts during the Apollo X mission. Their maximum speed was 39 892 kilometres per hour. About how many times faster than Gabelich's land speed record is this?

3. In 1972, Mark Donahue set an automobile record speed of 263.064 km/h at Indianapolis. This was 37.291 km/h faster than Rodger Ward's record in 1962. What was Ward's speed in 1962?

4. In the 1968 Olympics, Lee Evans had an average speed of 9.13 metres per second in the 400-metre race and Jim Hines had a speed of 10.1 metres per second in the 100-metre race. How much faster was Hines' speed?

5. A record walking speed was set by Agapor of the USSR in 1969. In a 20 000-metre race, he walked at a rate of 3.84 metres per second. How much slower is this than Lee Evans' running speed? (See Exercise 4.)

6. Mark Spitz had a swimming speed of 1.95 metres per second in setting a record time for the 100-metre swim in the 1972 Olympics. His speed for the 200-metre race was 0.18 metres per second slower. What was his speed in the 200-metre race?

7. Shane Gould set a record for the 100-metre free-style swim by swimming at a rate of 1.71 metres per second. How much slower than Mark Spitz' rate was her rate?

Objective

The student can solve story problems involving decimal notation.

UTILIZATION

These exercises might be assigned in conjunction with those on page F-44 if you so desire.

EXTENSION

Enrichment: New speed records are continually being established. Some students might do some research on current speed records and compare them with the ones given in this lesson.

Assignments

Minimum 1–5. Average 1–7. Maximum 1–7.

Objective

The student can find the product of two rational numbers in decimal notation.

PREPARATION

Materials: Graph paper (*Duplicator Masters*, page 83).

Write several multiplication problems using fractions whose denominators are powers of ten on the chalkboard. Have students compute the products, then give the corresponding decimals for each factor and product; for example:

A. $\frac{3}{10} \times \frac{4}{10} = \frac{12}{100}$ B. $\frac{3}{100} \times \frac{5}{100} = \frac{15}{10000}$

$0.3 \times 0.4 = 0.12$ $0.03 \times 0.05 = 0.0015$

INVESTIGATION

The purpose of the Investigation is to provide a model for multiplication using decimals. The length of a side of each small square is 0.1 and the area of each small square is 0.01 of the whole square region. Students should be provided with graph paper to illustrate the different regions. It is not necessary for students to find all possible regions. The area of the possible regions and their dimensions (Discussion Exercise 1) are given below.

0.07	(0.1 by 0.7)	0.16	(0.2 by 0.8
0.08	(0.1 by 0.8		*or* 0.4 by 0.4)
	or 0.2 by 0.4)	0.18	(0.2 by 0.9
0.09	(0.1 by 0.9		*or* 0.3 by 0.6)
	or 0.3 by 0.3)	0.20	(0.2 by 1.0
0.10	(0.1 by 1.0		*or* 0.4 by 0.5)
	or 0.2 by 0.5)	0.21	(0.3 by 0.7)
0.12	(0.2 by 0.6	0.24	(0.3 by 0.8
	or 0.3 by 0.4)		*or* 0.4 by 0.6)
0.14	(0.2 by 0.7)	0.25	(0.5 by 0.5)
0.15	(0.3 by 0.5)	0.27	(0.3 by 0.9)
		0.28	(0.4 by 0.7)

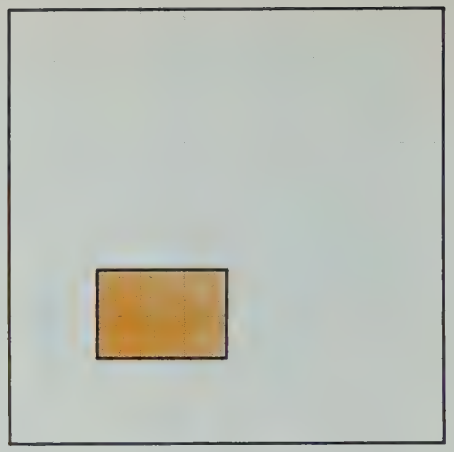
DISCUSSION

The emphasis in the Discussion should be on understanding the placement of the decimal point in multiplication (Exercises 2 and 3). Since this is review, move through this section quickly if your students are well prepared. Exercise 4 reviews the algorithm for multiplication of decimals by powers of ten.

Multiplying with Decimals

Investigating the Ideas

The shaded rectangular region is 0.3 units long and 0.2 units wide. What part of the entire square region does it cover?



How many rectangular regions can you show on a square grid like this that will cover between 0.06 and 0.3 of the square region?

Discussing the Ideas

1. How long and how wide were the rectangles you found?

2. The small rectangle shown in the Investigation suggests the multiplication equation $0.3 \times 0.2 = \mathbf{x}$, where $\mathbf{x}$ denotes the shaded portion of the entire square region. To find the product, you can do all the computing as if working with whole numbers. Then, the last step is to locate the decimal point in the product. What is the product?

Tenths times tenths is hundredths. The product is ▮.

$0.3 \times 0.2 = ▮$

$3 \times 2 = 6$

3. What are the missing numbers in each part?

A A two-place decimal times a two-place decimal is a four-place decimal. The product is ▮.

$0.04 \times 0.12 = ▮$

$4 \times 12 = 48$

B A whole number times a three-place decimal is a three-place decimal. The product is ▮.

$100 \times 0.035 = ▮$

$100 \times 35 = 3500$

C A two-place decimal times a three-place decimal is a five-place decimal. The product is ▮.

$2.13 \times 0.045 = ▮$

$213 \times 45 = 9585$

4. Multiplying a decimal by a power of ten can often be done mentally. Study the examples. Then complete the two rules.

$10 \times 0.23 = 10 \times \frac{23}{100} = \frac{230}{100} = 2.3$

$0.1 \times 0.23 = \frac{1}{10} \times \frac{23}{100} = \frac{23}{1000} = 0.023$

To multiply by 10, we move the decimal point one place to the ?.
To multiply by .1, we move the decimal point one place to the ?.

Using the Ideas

1. Give the missing word.

- **A** 3 tenths times 6 tenths is 18 ? .
- **B** 14 hundredths times 3 tenths is 42 ? .
- **C** 23 thousandths times 31 hundredths is 713 ? .

2. Find the products.

- **A** 21.3×6
- **B** 4.2×0.9
- **C** 318×0.07
- **D** 12.5×0.13
- **E** 6000×0.875
- **F** 7.76×1.9
- **G** 4.28×6.5
- **H** 1.5×3.1416
- **I** 1.6093×48.4

Solve each short story for Exercises 3 through 8.

3. Skirt price: \$18.00.
Sales tax: 0.05 times the price.
How much sales tax?

4. Sheet of paper: 0.016 cm thick.
How thick are 75 sheets of paper?

5. Speed of sound in air:
One metre per 0.00301 seconds.
How long for sound to travel one kilometre?

6. Earnings: \$120.
Income tax: 0.175 of earnings.
How much income tax?

7. Length of walking pace: 0.84 metre.
How far in 1200 paces?

8. Speed of sound: about 0.3 km/s
Speed of light: about one million times as fast.
How fast is the speed of light?

★ **9.** When one or more factors are negative, find the product as if both factors were positive and then decide whether the product is positive or negative.

EXAMPLES: $^{-}0.3 \times 0.7 = {}^{-}0.21$ $\quad {}^{-}2 \times {}^{-}3.4 = 6.8$

Find the products.

- **A** $0.8 \times {}^{-}2.1$
- **B** $^{-}0.04 \times 6$
- **C** $^{-}0.7 \times {}^{-}3.2$
- **D** $^{-}0.13 \times 6.1$
- **E** $12.8 \times {}^{-}0.2$
- **F** $^{-}0.63 \times {}^{-}0.8$
- **G** $0.101 \times {}^{-}31.4$
- **H** $^{-}0.001 \times 68.4$
- **I** $^{-}0.111 \times {}^{-}3.29$

think

The symbol $\sum_{1}^{n}$ means the **sum** of the first ***n*** positive integers.

Thus, $\sum_{1}^{5} = 1 + 2 + 3 + 4 + 5 = 15$.

The symbol ***n*!** means the **product** of the first ***n*** positive integers.
Thus, 5! ... $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

What integer is ***n*** if $\sum_{1}^{n} = n!$?

More practice, page S-17, Set 31

UTILIZATION

The exercises provide practice in finding products as well as word problems involving multiplication using decimals. Plan to discuss the word problems with the class.

Since multiplication of rational numbers involving negative rational numbers has been developed previously, the only new concept in Exercise 9 is the decimal notation for negative rational numbers.

EXTENSION

To provide additional practice, make selective assignments from Supplementary Exercise Set 31 on page S-17, *Workbook* page 50, *Duplicator Masters* page 37, and *Arithmetic Skill Card* D-3.

Remedial: Some students may need extra practice on problems like the ones below. Have students give the products orally.

10×1.23	(12.3)	100×1.23	(123)
10×0.031	(0.31)	100×0.031	(3.1)
10×12.3	(123)	100×12.3	(1230)
0.1×1.23	(0.123)	0.01×1.23	(0.0123)
0.1×0.031	(0.0031)	0.01×0.031	(0.00031)
0.1×12.3	(1.23)	0.01×12.3	(0.123)

Enrichment: Provide some challenge problems like these for your more able students.

1. What mathematical symbol can be placed between the numerals 7 and 8 so that the resulting numeral represents a number between 7 and 8? (Answer: a decimal point)
2. Evaluate the expression $2n^2 + 0.4n + 1.7$ for $n = 0.6$. (Answer: 2.66)
3. What positive number is x if $x^2 = 44.89$? (Answer: 6.7)
4. What is the product? $1.6 \times 10^{-5} \times 6.25 \times 10^4$ (Answer 1.0)

Think Solution

A trivial solution is $n = 1$.
When $n = 3$, we have $\Sigma_1^3 = 1 + 2 + 3 = 6 = 1 \cdot 2 \cdot 3 = 3!$

Assignments

Minimum 1–5. Average 1–8. Maximum 1–9.

Objective

The student can express a given decimal numeral in scientific notation, and conversely.

PREPARATION

Review the meaning of positive and negative exponents. A short oral drill of multiplication of decimals by powers of ten would also be helpful preparation.

INVESTIGATION

The Investigation presents a matching quiz for the student. In each case the product in the right column is a number expressed in scientific notation.

DISCUSSION

Some students may recall scientific notation for numbers larger than one, but scientific notation for positive numbers less than one will be new material for most.

Following a discussion of the Investigation, work through the Discussion Exercises as a class activity. Emphasize that writing a number in scientific notation permits a quick identification of the size of the number and also simplifies writing large and small numbers.

Examples A and B of Exercise 3 suggest a shortcut for writing decimals in scientific notation. If you prefer, you may delay discussion of the shortcut until the students attempt to formulate shortcuts of their own.

Scientific Notation

Investigating the Ideas

Each number in the left column matches one of the products in the right column.

NUMBER	PRODUCT
A 32 000	H 3.2×10^{6}
B 0.0032	I 3.2×10^{5}
C 0.32	J 3.2×10^{4}
D 320 000	K 3.2×10^{0}
E 3.2	L 3.2×10^{-1}
F 3 200 000	M 3.2×10^{-2}
G 0.032	N 3.2×10^{-3}

Can you match each number with the correct product?

Discussing the Ideas

1. Each number in the left column is expressed in **scientific notation** by one of the products in the right column in the Investigation.
   How is 32 000 expressed in scientific notation?

2. To express a number in scientific notation, you write it as **a product of a number from 1 to 10, not including 10, and a power of ten.**
   Thus, 0.00217 written in scientific notation is 2.17×10^{-3}.

   Which numbers below are written in scientific notation?

   A 6.46×10^{-2} C 31.4×10^{3} E 11.4×10^{-1}
   B 3.8×10^{0} D 9.99×10^{-9} F 6.3×8^{3}

3. Examples A and B below illustrate a shortcut for writing decimals in scientific notation.

   A $27\,600 = 2.76 \times 10^{4}$

   To get the decimal point here

   We need this power of ten.

   B $0.0000395 = 3.95 \times 10^{-5}$

   To get the decimal point here

   We need this power of ten.

   Show how each decimal can be written in scientific notation.

   A 5800 B 0.017 C 10 000 D 3.76 E 0.00008

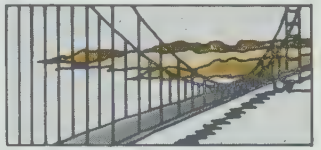
Using the Ideas

1. Copy and give the integer for n.

A $52 = 5.2 \times 10^n$
B $430 = 4.3 \times 10^n$
C $0.52 = 5.2 \times 10^n$
D $0.043 = 4.3 \times 10^n$
E $4350 = 4.35 \times 10^n$
F $7\,000\,000 = 7 \times 10^n$
G $320\,000 = 3.2 \times 10^n$
H $0.000000043 = 4.3 \times 10^n$
I $1.53000 = 1.53 \times 10^n$

2. Express the number in each statement in scientific notation.

A A side of the Great Pyramid is 230 metres long.
B The length of the channel span of the Golden Gate Bridge is 1280 metres.
C The height of Mount McKinley is 6190 metres.
D The air distance from Toronto to St. John's is 3029 kilometres.


Golden Gate Bridge, San Francisco

E The circumference of the Earth is about 40 000 kilometres.
F The circumference of the sun is about 4 440 000 kilometres.
G The circumference of a giant star is about 8 700 000 000 kilometres.
H According to Einstein, the possible radius of the universe is 30 000 000 000 000 000 000 000 kilometres.

3. Numbers represented by scientific notation are given below. Give the ordinary decimal symbol for each number.

A 7×10^3
B 6.2×10^{-2}
C 5.45×10^{-6}
D 9.03×10^7
E 6.43×10^0
F 5×10^{-12}
G 1.1×10^{10}
H 3.37×10^{-8}

4. Write an ordinary decimal for each length. The lengths are measured in centimetres.

A Size of very small insects: 2×10^{-2}
B Size of large bacteria: 3.5×10^{-1}
C Red light wave length: 6.4×10^{-5}
D Green light wave length: 5.3×10^{-5}
E Violet light wave length: 4.5×10^{-5}
F Size of a virus: 2.5×10^{-7}
G Diameter of an atom: 1.06×10^{-8}
H Cosmic ray wave length: 3×10^{-19}

5. Give the products in the form $k \times 10^n$ where $1 \leq k < 10$ and k is an integer.

EXAMPLE: $4 \times 10^2 \times 6 \times 10^3 = 24 \times 10^5 = 2.4 \times 10^6$

A $5 \times 10^2 \times 6 \times 10^4$
B $4 \times 10^{10} \times 5 \times 10^{12}$
C $5 \times 10^{-3} \times 4 \times 10^{-2}$
D $8 \times 10^{-8} \times 7 \times 10^5$
E $2 \times 10^3 \times 5 \times 10^4$
F $2 \times 10^{-3} \times 5 \times 10^4$
G $6 \times 10^5 \times 7 \times 10^7$
H $3 \times 10^{-7} \times 2 \times 10^0$
I $2 \times 10^{-3} \times 5 \times 10^{-4}$

More practice, page S-17, Set 32

UTILIZATION

Exercise 2 relates scientific notation to large numbers while Exercise 4 involves scientific notation for very small measures.

Note in Exercise 5 that a number of the form $k \times 10^n$, where $1 \leq k < 10$ and n is an integer, is a number written in scientific notation.

EXTENSION

Additional appropriate exercises are provided by Supplementary Exercise Set 32 on page S-17, *Workbook* page 51, *Duplicator Masters* page 38, and *Arithmetic Skill Cards* AP-3 and AP-4.

Enrichment: Suggest that students find some references to measures of physical objects which are either very large or very small numbers and express the numbers in scientific notation. They can use science books, almanacs, encyclopedias, or other reference sources for examples. (See Research Project C on page F-65.)

Assignments

Minimum 1A–F, 2, 3A–D, 4.
Average 1–4. Maximum 1–5.

Objective

The student can state the greatest possible error in a reported measurement.

PREPARATION

Review scientific notation for both large and small positive numbers.

INVESTIGATION

The segment in the scale drawing is about 7.7 cm long. Using the scale given, the approximate distance from Earth to the moon is 385 000 km or 3.85×10^5 km.

DISCUSSION

Relate the parts of Exercise 1 to students' findings in the Investigation:

A. The measurement found (385 000) is certainly between the successive multiples of 100 000 km.
B. We would expect our answer to be correct within these limits (nearest 50 000).
C. The error in measure could not be large enough for the moon distance to be between these two measures.
D. We can be reasonably sure that the computed distance will lie between these two measures.

In Exercise 2 students should observe that, unless otherwise specified, the zeros in the numerals in A and B are not significant.

In Exercise 3, be sure to bring out the idea that zeros separating 2 or more nonzero digits are significant. If a number is less than 1, none of the zeros to the left of the first nonzero digit are significant.

Exercise 5 introduces the term *greatest possible error*. Emphasize that if scientific notation is used for a measure, the greatest possible error is determined by the number of digits in the factor of ten and the power of ten; thus:

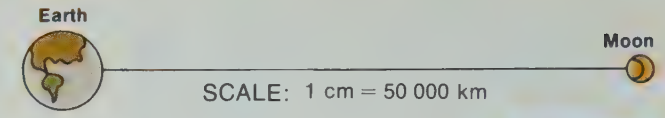
5A. $3.835 \times 10^5 < 3.84 \times 10^5 < 3.845 \times 10^5$.
GPE: 0.005×10^5

5B. $3.8425 \times 10^5 < 3.843 \times 10^5 < 3.8435 \times 10^5$.
GPE: 0.0005×10^5

Precision in Measurement

Investigating the Ideas

Suppose this line segment represents the distance from Earth to the moon.



Can you find the distance from Earth to the moon using the scale drawing?	Express this distance in scientific notation.

Discussing the Ideas

1. Using the scale drawing above, is the moon
   - A 300 000 to 400 000 km from Earth?
   - B 350 000 to 400 000 km from Earth?
   - C 350 000 to 375 000 km from Earth?
   - D 375 000 to 400 000 km from Earth?

2. The **significant digits** of a measurement are those digits which are correct when the measurement is carefully made and properly rounded. All nonzero digits are significant digits.

   300 000 has 1 significant digit. **35**0 000 has 2 significant digits.

   - A How many significant digits in 384 000?
   - B How many significant digits in 283 300?

3. Some zeros are significant. The measure 2.07 metres has 3 significant digits while 0.06 metre has 1 significant digit. Give the number of significant digits for each measure.

   A 6.394 mm B 20.7 cm C 0.7724 km D 0.003 mm

4. When scientific notation is used, all digits in the number times the power of ten are significant.

   EXAMPLES: A 3.8×10^5 (2 significant digits) B 6.30×10^6 (3 significant digits) C 1.6093×10^{-4} (5 significant digits)

   How many significant digits are in 3.05×10^5?

5. When scientific notation is used, you can determine the **greatest possible error** (GPE). For example, if a measure of 3.8×10^5 km is given for a distance, the actual distance is between 3.75×10^5 km and 3.85×10^5 km. The greatest possible error is 0.05×10^5 km or 5000 km.

   - A What is the GPE for 3.84×10^5 km?
   - B What is the GPE for 3.843×10^5 km?

Using the Ideas

1. Give the number of significant digits for each measurement.

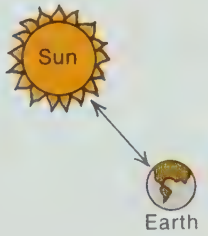
A 5.4×10^3 cm
B 5.42×10^3 cm
C 5.4×10^0 cm
D 5.40×10^{-2} cm
E 5.004×10^{-2} cm
F 5.0004×10^{-2} cm

2. Find the greatest possible error (GPE) for each measurement.

EXAMPLE: 2.4×10^3 m → GPE = $.05 \times 10^3$ m = 50 m

A 3.9×10^4 cm
B 6.28×10^{-2} mm
C 5.3×10^9 year
D 7.04×10^7 kg
E 1.001×10^{-4} cm
F 9.6×10^3 ml
G 3.0×10^6 m
H 5×10^{11} km
I 8.1×10^{-1} cm

★ **3.** The average distance from Earth to the sun is 150 000 000 km, to the nearest million km. We can write this as 150 000 000 ± 500 000 km, or in scientific notation as $(1.50 \pm .005) \times 10^8$ km. The greatest possible error in this measurement is $.005 \times 10^8$ km. The **relative error** in this measurement is the ratio of the greatest possible error to the measure.



$$\text{Relative error} = \frac{\text{GPE}}{\text{measure}} = \frac{.005 \times 10^8}{1.50 \times 10^8} = \frac{.005}{1.50} = \frac{5}{1500} = \frac{1}{300}$$

The relative error is about 1 part in 300.

Find the relative error of each measurement.

A $(6.5 \pm .05) \times 10^3$ m
B $(7.2 \pm .05) \times 10^{-2}$ cm
C $(3.0 \pm .05) \times 10^8$ m/s
D $(1.25 \pm .005) \times 10^{-2}$ cm

★ **4.** The more **precise** of two measures is the one with the smaller relative error. Choose the more precise measure of each pair.

A $(3.2 \pm .05) \times 10^{-2}$ cm or $(3.24 \pm .005) \times 10^2$ cm
B 1.3×10^{-2} ml or 2.14×10^3 ml
C 4.0×10^3 km or 4×10^{-3} km

UTILIZATION

The exercises are designed to introduce the student to terms such as *precision*, *relative error*, and *accuracy*.

Slower students may have difficulty with some of the applications, and you should not expect even your best students to memorize all the definitions. This section is designed more to give students an exposure to a variety of ideas involving measurement than it is to provide a wide range of topics for mastery.

EXTENSION

Workbook page 52 offers exercises appropriate for further practice.

Enrichment: Some students might enjoy the challenge of the problem that follows.

The velocity of light recommended for use by the International Astronomical Union (Hamburg, 1961) is 2.997925×10^5 km/s.

A. What is the GPE of this measurement?
(Answer: 0.0000005×10^5 km/s = 5×10^{-2} km/s = 0.05 km/s = 50 m/s)

B. What is the relative error in the measure?
(Answer: $\frac{5 \times 10^{-2}}{2.997925 \times 10^5} = \frac{5}{2.997925 \times 10^7}$ $= \frac{5}{29979250}$. This is an error of about 1 part in 6 000 000—a very accurate measure.)

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–5.

Objective

The student can find the quotient of two numbers in decimal notation.

PREPARATION

Give students practice in mental multiplication of decimals by 10, 100, and 1000.

INVESTIGATION

The Investigation is designed to lead students to think about multiplication in order to place the decimal point correctly in the quotient. It is quite likely, however, that students will use other techniques since they have had previous experience with division of decimals.

DISCUSSION

The Discussion Exercises are designed to promote student understanding of the placement of decimal points in quotients. If your students have indicated by their performance in the Investigation that they already have a good understanding of the process, you should move quickly ahead to the Utilization stage.

Dividing with Decimals

Investigating the Ideas

Only one of the quotients in each example below is correct.

A	B	C
$37\overline{)20.091}$	$23.5\overline{)1.1045}$	$0.68\overline{)0.017}$
$37\overline{)20.091}$	$23.5\overline{)1.1045}$	$0.68\overline{)0.017}$
$37\overline{)20.091}$	$23.5\overline{)1.1045}$	$0.68\overline{)0.017}$

Can you use multiplication to decide which is the correct quotient for each part?

Discussing the Ideas

1. How can you use the diagram at the right to help you place the decimal point correctly in 543 for example **A** above?

2. Explain how to place the decimal point in examples **B** and **C**.

3. When dividing by a whole number, the decimal point is placed directly above the decimal point in the dividend. Locate the decimal point in each division problem below.

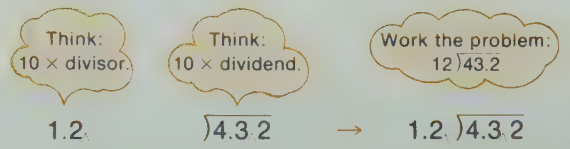
A $12\overline{)43.44}$ **B** $35\overline{)0.08505}$ **C** $142\overline{)3.266}$

EXAMPLE: $23\overline{)3.128}$ → .136

23
82
69
138
138
0

4. Study the example.

EXAMPLE: $1.2\overline{)4.32}$ →



Replace each problem by a problem with a whole number divisor which has the same answer. Then divide.

A $0.7\overline{)29.47}$ **B** $1.1\overline{)0.0352}$ **C** $0.015\overline{)0.00225}$ **D** $21.5\overline{)156.95}$

Using the Ideas

1. Find the quotients.

A $72\overline{)273.6}$ **C** $53\overline{)3.339}$ **E** $23\overline{)0.0529}$ **G** $76\overline{)402.344}$

B $94\overline{)53.58}$ **D** $19\overline{)67.64}$ **F** $18\overline{)0.00270}$ **H** $519\overline{)384.06}$

2. Find the quotients.

A $0.06\overline{)2.142}$ **C** $0.004\overline{).0256}$ **E** $3.01\overline{)1.2341}$ **G** $0.417\overline{)0.7089}$

B $0.21\overline{)7.98}$ **D** $3.18\overline{).19716}$ **F** $9.9\overline{)76.23}$ **H** $0.049\overline{)0.001764}$

Solve the short story problems for Exercises 3 through 8.

3. Paper: 0.008 cm thick. How many sheets are there in a stack 1 cm high?

4. Quarter: about 0.15 cm thick. About how many quarters in a stack 6 cm high?

5. Distance around a tree is about 3.1 times the diameter of the tree. If the distance around the tree is 4.06 m, what is the diameter?

6. Mathematics book: 2.25 cm thick. How many books in a stack that is 432 cm high?

7. Speed: 6.25 metres per second. Distance: 400 metres. How long to run 400 metres?

8. A kangaroo leap is about 12.5 metres. How many leaps in one kilometre?

★ **9.** Select the correct quotient from the three choices.

A $^{-}0.4 \div 0.2 = ?$ $(^{-}2, ^{-}0.2, ^{-}0.02)$ **D** $^{-}0.12 \div ^{-}0.6 = ?$ $(2, 0.2, 0.02)$

B $1.5 \div ^{-}3 = ?$ $(^{-}5, ^{-}0.5, ^{-}0.05)$ **E** $0.035 \div .7 = ?$ $(0.05, 0.5, 5)$

C $^{-}42 \div 0.06 = ?$ $(^{-}7, ^{-}70, ^{-}700)$ **F** $2.4 \div 0.08 = ?$ $(0.3, 3, 30)$

Think

Since $3^3 = 27$, the last digit of the number 3^3 is 7.
What is the last digit of each of these numbers?
1. $(3^3)^3 = 27^3$ **2.** $((3^3)^3)^3 = (27^3)^3$ **3.** $(((3^3)^3)^3)^3$
4. $(((3^3)^3)^3 \ldots)^3$, where there are 100 threes as exponents.

More practice, page S-18, Set 33

UTILIZATION

Note that Exercise 9 involves quotients with negative numbers. You may want to review products and quotients of negative rational numbers before assigning this exercise.

Since much of the exercise material will be review for your students, you should adjust the assignment accordingly.

EXTENSION

Additional practice exercises may be assigned from Supplementary Exercise Set 33 on page S-18, *Workbook* page 53, *Duplicator Masters* page 39, and *Arithmetic Skill Cards* D-4, D-5, and D-6.

Think Solution

1. $(3^3)^3 = 27^3 = 27 \cdot 27 \cdot 27$
   Using only last digits, $7 \cdot 7 = 49$. Then $7 \cdot 9 = 63$, so 27^3 ends in 3.
2. From Exercise 1 we know that $(3^3)^3$ ends in 3. Therefore $((3^3)^3)^3$ must end in the same digit as 3^3, that is 7.
3. $((3^3)^3)^3$ ends in 7 (Exercise 2); therefore the cube of the number ends in 3.
4. The pattern established in the first three parts shows that since there is an even number of exponents of 3 in the expression, the last digit will be 3. If there were an odd number of exponents the last digit would be 7.

Assignments

Minimum 1A–D, 2A–D, 3, 4, 5.
Average 1E–H, 2E–H, 3–8.
Maximum 1A–F, 2A–F, 3–9.

Objective

The student can use rounding or scientific notation as aids in estimating products and quotients.

PREPARATION

Conduct a discussion of estimation of products and quotients involving only whole numbers.

INVESTIGATION

If you have a demonstration slide rule in your classroom, you might show the students how products and quotients are found by using the slide rule. Emphasize the fact that estimation of the product or quotient is vital for efficient use of the slide rule, since the position of the decimal point in an answer must be determined by the person using the slide rule.

The Investigation should require only a few minutes time, so you might wish to give additional estimation problems, including some that involve quotients.

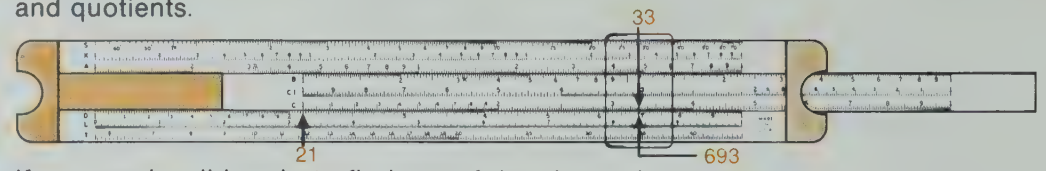
DISCUSSION

Your main objective in this section should be to help the students develop skill in estimating products and quotients. They will also be afforded an opportunity to improve their work with decimals, significant digits, and scientific notation. Students should be encouraged to discuss any special techniques of their own that they use in finding products and quotients.

Estimating Products and Quotients

Investigating the Ideas

A **slide rule** can be used to find approximate products and quotients.



If you used a slide rule to find any of the six products below, you could read 693 for the product. You would then have to decide where to place the decimal point according to the factors involved.

? Can you place the decimal point in 693 to show the product for each problem below?

A $2.1 \cdot 3.3$

B $0.21 \cdot 33$

C $0.021 \cdot 330$

D $21 \cdot 33$

E $0.21 \cdot 0.33$

F $2.1 \cdot 0.0033$

Discussing the Ideas

1. Explain how you decided to place the decimal point in each product.

2. It is often helpful to round each numeral to one significant digit in order to estimate answers.
   - A What is 2.1 rounded to one significant digit?
   - B What is 3.3 rounded to one significant digit?
   - C How can the product of the rounded factors help you place the decimal point in 693 correctly?

3. Explain how to locate the decimal point by estimation.

   A $7.42\overline{)818.93}$ (1104 . . .)

   B $0.0062\overline{)3.165}$ (5105 . . .)

   C $2.3\overline{)0.04352}$ (1892 . . .)

4. Scientific notation is helpful in estimation. For example, to estimate the product 587.3×79.54, round each number to one significant digit and write it in scientific notation as

   $$6 \times 10^2 \times 8 \times 10^1 = 48 \times 10^3.$$

   The product is about 48 000. Explain how to estimate the product 475.21×0.0397 in this way.

Using the Ideas

1. Round each number to one significant digit. Use scientific notation and estimate the products.

A 5846×6214	D 318.7×0.003187	G 0.06752×0.00225
B 691.3×87.65	E 0.0235×0.00829	H 41.689×0.000785
C 91.25×0.4863	F 0.00078×0.0192	I $0.00000692 \times 0.000406$

2. The correct digits are given in parentheses for each quotient. Using estimation, correctly locate the decimal point.

A $9.43 \div 4.1$ (23)	C $8.0698 \div 2.57$ (314)	E $0.153126 \div 0.047$ (3258)
B $16.544 \div 3.52$ (47)	D $0.2856 \div 0.714$ (4)	F $0.01517 \div 4.1$ (37)

3. Find the quotients.

A $10^3 \div 10^0$	D $(6 \times 10^2) \div 10^{-1}$	G $(8 \times 10^0) \div (2 \times 10^3)$
B $10^2 \div 10^{-2}$	E $(3 \times 10^4) \div 10^{-4}$	H $(6 \times 10^{-2}) \div (3 \times 10^2)$
C $10^{-3} \div 10^{-1}$	F $(4 \times 10^{-7}) \div 10^{-7}$	I $(8 \times 10^{-3}) \div (4 \times 10^{-5})$

4. Round each number to one significant digit. Use scientific notation and estimate the quotient.

A $597.2 \div 2.964$	D $0.6354 \div 215.83$	G $16.42 \div 0.000243$
B $79.62 \div 0.03982$	E $0.00396 \div 0.00000207$	H $0.08099 \div 0.03702$
C $0.8743 \div 0.0295$	F $0.000053 \div 987.65$	I $0.08909 \div 0.000251$

5. Study the example and then estimate the reciprocal of each number below.

EXAMPLE: Estimate the reciprocal of 809.4.

SOLUTION: The reciprocal is $\frac{1}{809.4} \approx \frac{1}{800} = \frac{1}{8 \times 10^2} = \frac{1}{8} \times 10^{-2}$

$= 0.125 \times 10^{-2} = 0.00125$

The reciprocal of 809.4 is about 0.0012

A 489.32	B 0.00197	C 0.07216	D 3.765

6. Use scientific notation to estimate each of the powers.

A $(0.00207)^3$	C $(402.6)^3$	E $(0.019832)^5$
B $(3012.876)^4$	D $(0.000178)^4$	F 2^{100} (HINT: $2^{10} = 1024$)

7. Use scientific notation, round each number to one significant digit, and estimate the answer. $\dfrac{(0.021)^2 \times 59.61 \times 0.000713}{694.35 \times (0.00001956)^3}$

More practice, page S-18, Set 34

UTILIZATION

Since much of the computation done in everyday life involves estimation, it may prove worthwhile to have students discuss how they arrived at their estimates for the various exercises. This allows the slower students a chance to observe the process some of the abler students use to estimate answers.

Answers, Exercises 6A and 7

6A. $(0.00207)^3 \approx (2 \times 10^{-3})^3$

$\approx 2^3 \times 10^{-9}$

$\approx 8 \times 10^{-9}$

7. $\dfrac{(0.021)^2 \times 59.61 \times 0.000713}{694.35 \times (0.00001956)^3}$

$\approx \dfrac{(2 \times 10^{-2})^2 \times (6 \times 10^1) \times (7 \times 10^{-4})}{(7 \times 10^2) \times (2 \times 10^{-5})^3}$

$\approx \dfrac{2^2 \times 10^{-4} \times 6 \times 10^1 \times 7 \times 10^{-4}}{7 \times 10^2 \times 2^3 \times 10^{-15}}$

$\approx \dfrac{168 \times 10^{-7}}{56 \times 10^{-13}}$

$\approx 3 \times 10^6$

EXTENSION

Supplementary Exercise Set 34 on page S-18 and *Workbook* page 54 may be used to provide further practice.

Enrichment: Some students may be interested in learning to use a slide rule to find products and quotients. (See Research Project B on page F-65.) Although the slide rule has, to a large extent, been replaced by desk calculators or mini-calculators, it is still a valuable tool when only approximate results are necessary.

Assignments

Minimum 1A–F, 4A–F.

Average 1–4. Maximum 1–7.

Objective

The student can express a fraction as a decimal rounded to a given decimal place.

PREPARATION

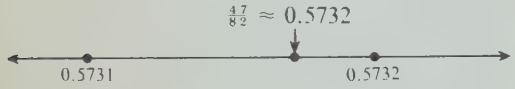
The division method of finding a decimal approximation for a rational number in fraction form will be familiar to almost all students. However, a brief review of the process will be valuable, especially for slower students.

INVESTIGATION

Have the students read the introductory material and study examples A and B. The number lines below the examples may help students understand the rounding of decimals to the nearest tenth and nearest hundredth. After students have carried out the division to the nearest thousandth and ten thousandth, you may want to show points for these decimals on the number line.

To the nearest *thousandth*:

To the nearest *ten thousandth*:



DISCUSSION

In Exercise 1, if the division is carried out to thousandths, $\frac{47}{82} = 0.573\frac{14}{82}$. The remainder, 14, is less than half of 82. Therefore the point for $\frac{47}{82}$ is nearer to 0.573 than to 0.574.

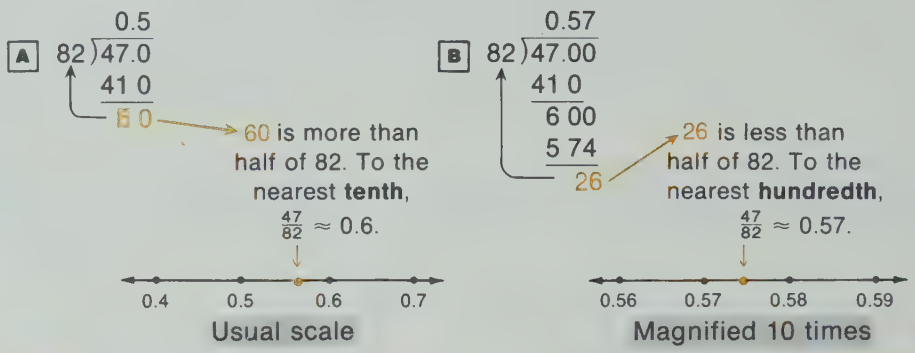
In Exercise 2, if the division is carried out to ten thousandths, $\frac{47}{82} = 0.5731\frac{58}{82}$. Since 58 is more than half of 82, the point for $\frac{47}{82}$ is nearer to 0.5732 than to 0.5731.

You may want to discuss some of the applications of this material. A familiar application involves team standings in sports of various kinds. In sports, many "averages" are given as decimals which must be computed to thousandths in order to decide rankings.

Approximations for Rational Numbers

Investigating the Ideas

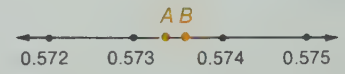
When we use decimals to represent rational numbers, we often round to the nearest tenth, hundredth, thousandth, and so forth. If we divide numerator by denominator to get decimal approximations for the fraction $\frac{47}{82}$, we have



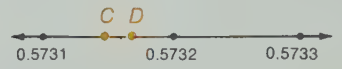
Can you use division to find the decimal approximation of $\frac{47}{82}$ to the nearest **thousandth** and **ten thousandth**?

Discussing the Ideas

1. Which point on the number line, **A** or **B**, is nearer to the point for $\frac{47}{82}$?



2. **A** Which point, **C** or **D**, is nearer to the point for $\frac{47}{82}$?



   B What is $\frac{47}{82}$ to the nearest ten thousandth?

3. What is the decimal for $\frac{1}{3}$ to the nearest
   A tenth? **C** thousandth?
   B hundredth? **D** ten thousandth?

4. The number 4.35 is midway between 4.3 and 4.4. In such cases, we agree always to "round up." We would round 4.35 to 4.4. Round each of the following numbers to the nearest hundredth.
   A 4.735 **B** 3.655 **C** 7.955 **D** 15.455 **E** 0.809

Using the Ideas

1. Use the following division problems to round each of the numbers $\frac{10}{23}$, $\frac{23}{37}$, and $\frac{47}{31}$ to the nearest

A tenth:

$$23\overline{)10.000}$$

0.434

92

80

69

110

92

18

B hundredth:

$$37\overline{)23.000}$$

0.621

222

80

74

60

37

23

C thousandth:

$$31\overline{)47.000}$$

1.516

31

160

155

50

31

190

186

4

2. Find a decimal approximation of each number to the nearest hundredth.

A $\frac{5}{6}$ **B** $\frac{13}{16}$ **C** $\frac{3}{7}$ **D** $\frac{11}{15}$ **E** $\frac{35}{64}$

3. Find a decimal approximation of each number to the nearest thousandth.

A $\frac{5}{10}$ **B** $\frac{7}{9}$ **C** $\frac{7}{12}$ **D** $\frac{65}{99}$ **E** $\frac{1455}{10\,000}$

4. Four league-leading teams have records as shown. Use decimals rounded to the nearest thousandth to find what part of the games played each team has won.

Team	Games won	Games played
A	34	56
B	35	60
C	32	55
D	36	57

5. **A** Which team in Exercise 4 has the best record?

B Which team has the worst record?

6. In football, a tie game is sometimes counted as $\frac{1}{2}$ game *won*. A team which has won one game, lost one game, and tied one game, has won $1\frac{1}{2}$ games out of three games. As a decimal, $1.5 \div 3 = 0.500$. Find the decimal (to the nearest thousandth) for each football team record shown in the table.

Team	Won	Lost	Tied	Decimal
A	6	2	0	
B	5	2	1	
C	3	3	2	
D	3	4	1	
E	2	4	2	

More practice, page S-19, Set 35

UTILIZATION

It may be valuable to present a number of the exercises in this section on the board for further discussion after the students have completed their work.

EXTENSION

Supplementary Exercise Set 35 on page S-19, *Duplicator Masters* page 40, and *Arithmetic Skill Card* D-8 may be used for additional practice assignments.

Enrichment: Obtain records of league standings in baseball, basketball, or other sports. Have students compute the ratio of number of games won to number of games played as a decimal to the nearest thousandth. This ratio is called the "percentage" (often denoted by "Pct.").

Assignments

Minimum 1–4. Average 1–6. Maximum 1–6.

Objective

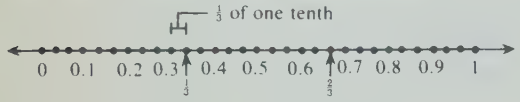
The student can express specified rational numbers as mixed decimal numerals, and conversely.

PREPARATION

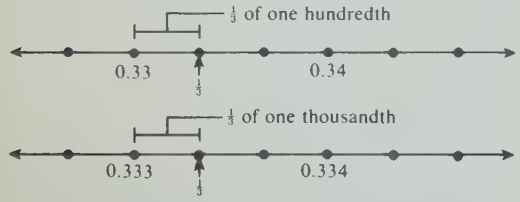
Since there is no Investigation for this lesson, you might use a few minutes to review the process of division for finding a decimal for a fraction.

DISCUSSION

In discussing Exercises 1, 2, and 3, it will be helpful to illustrate that $\frac{1}{3}$ is $0.3\frac{1}{3}$ (3 tenths plus $\frac{1}{3}$ of a tenth) by drawing a portion of the number line from 0 to 1 and dividing it into 30 parts. Then mark off the thirds and tenths and point out that it is easy to see that $\frac{1}{3} = 0.3\frac{1}{3}$.



Also, you might draw portions of magnified number lines to illustrate $0.33\frac{1}{3}$ and $0.333\frac{1}{3}$.



In Discussion Exercise 3, do not overemphasize the importance of being able to read aloud the mixed decimal numerals.

As preparation for the exercise set on page F-59, it may be helpful to supply the students with additional examples similar to those in Discussion Exercises 4 and 5.

Answers, Discussion Exercise 3

A. "two and one half tenths"
B. "sixteen and two thirds hundredths"
C. "sixty-two and one fourth thousandths"
D. "seven hundred nine and one third thousandths"
E. "twenty and two sevenths ten thousandths"

Mixed Decimal Numerals

Discussing the Ideas

1. Since there is no ordinary decimal for the rational number $\frac{1}{3}$, we can represent the number by combining fraction and decimal notation.

$$\frac{1}{3} = \frac{10}{30} = \frac{9}{30} + \frac{1}{30} = 0.3 + \left(\frac{1}{3} \times \frac{1}{10}\right) = 0.3\frac{1}{3}$$
$$\frac{1}{3} = \frac{100}{300} = \frac{99}{300} + \frac{1}{300} = 0.33 + \left(\frac{1}{3} \times \frac{1}{100}\right) = 0.33\frac{1}{3}$$
$$\frac{1}{3} = \frac{1000}{3000} = \frac{999}{3000} + \frac{1}{3000} = 0.333 + \left(\frac{1}{3} \times \frac{1}{1000}\right) = 0.333\frac{1}{3}$$

Symbols like $0.3\frac{1}{3}$ are called **mixed decimal numerals**.

In the example above, explain why $\frac{9}{30} = 0.3$, and $\frac{99}{300} = 0.33$.

2. Give the correct numbers for ***a***, ***b***, ***c***, and ***d***.
Then give the mixed decimal numeral for ***n***.

A $\frac{2}{3} = \frac{a}{30} = \frac{b}{30} + \frac{2}{30} = \frac{c}{10} + \left(\frac{d}{3} \times \frac{1}{10}\right) = n$

B $\frac{2}{3} = \frac{a}{300} = \frac{b}{300} + \frac{2}{300} = \frac{c}{100} + \left(\frac{d}{3} \times \frac{1}{100}\right) = n$

C $\frac{2}{3} = \frac{a}{3000} = \frac{b}{3000} + \frac{2}{3000} = \frac{c}{1000} + \left(\frac{d}{3} \times \frac{1}{1000}\right) = n$

3. Read $0.3\frac{1}{3}$ as, "three and one third *tenths*," not "three and one third hundredths"
Read each mixed decimal numeral.

A $0.2\frac{1}{2}$ B $0.16\frac{2}{3}$ C $0.062\frac{1}{4}$ D $0.709\frac{1}{3}$ E $0.0020\frac{2}{7}$

4. Explain these two ways for finding a fraction for a mixed decimal numeral.

A $0.3\frac{1}{3} = \frac{3\frac{1}{3}}{10} = \frac{10}{3} \div 10 = \frac{1}{3}$

B $0.3\frac{1}{3} = \frac{3\frac{1}{3}}{10} = \frac{3\frac{1}{3} \times 3}{10 \times 3} = \frac{10}{30} = \frac{1}{3}$

5. Explain these two ways for finding a mixed decimal numeral.

A $\frac{1}{7} = \frac{100}{700} = \frac{14\frac{2}{7}}{100} = 0.14\frac{2}{7}$

B $\frac{1}{7} \rightarrow 7\overline{)1.00} \rightarrow 0.14\frac{2}{7}$

6. Study the division and then give the missing fraction in each mixed decimal numeral.

$$11\overline{)4.000} = 0.363$$
$$33,\ 70,\ 66,\ 40,\ 33,\ 7,\ 7,\ 30,\ 28,\ 2$$

tenths: $\frac{4}{11} = 0.3\frac{7}{11}$ ← 70

hundredths: $= 0.36\frac{\text{IIII}}{\text{IIII}}$ ← 40

thousandths: $= 0.363\frac{\text{IIII}}{\text{IIII}}$ ← 7

7. What mixed decimal numeral is named in each part?

A One and two thirds **tenths** B Twelve and one half **hundredths**
C One hundred forty-two and six sevenths **thousandths**

Using the Ideas

1. Write a mixed decimal numeral for the number in each sentence.
   - **A** The sales tax rate in San Francisco is six and one fourth hundredths.
   - **B** Some credit card companies charge interest rates of one and one half hundredths interest on unpaid balances per month.
   - **C** About thirty-four and three tenths hundredths of the people in North America are under 18 years of age.

2. By dividing numerator by denominator, find the mixed decimal numeral in tenths, hundredths, and thousandths.

   A $\frac{7}{12}$ **B** $\frac{5}{6}$ **C** $\frac{4}{3}$ **D** $\frac{15}{16}$ **E** $\frac{8}{37}$ **F** $\frac{5}{27}$

3. Find the fraction for f.

   A $0.1\frac{1}{4} = \frac{1\frac{1}{4}}{10} = \frac{5}{4} \div 10 = f$

   B $0.12\frac{1}{2} = \frac{12\frac{1}{2}}{100} = \frac{25}{2} \div 100 = f$

   C $0.08\frac{1}{3} = \frac{8\frac{1}{3}}{100} = 8\frac{1}{3} \div 100 = f$

   D $0.3\frac{3}{4} = \frac{3\frac{3}{4}}{10} = \frac{3\frac{3}{4} \times 4}{10 \times 4} = f$

   E $0.62\frac{1}{2} = \frac{62\frac{1}{2}}{100} = \frac{62\frac{1}{2} \times 2}{100 \times 2} = f$

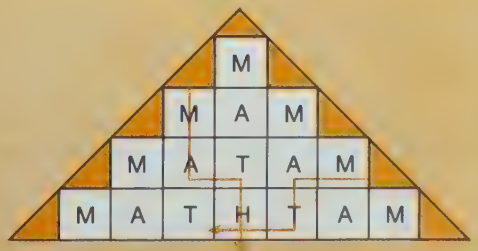
   F $0.4\frac{1}{6} = \frac{4\frac{1}{6}}{10} = \frac{4\frac{1}{6} \times 6}{10 \times 6} = f$

★ 4. The number $\frac{5}{32}$ can be written $0.1\frac{9}{16}$, $0.15\frac{5}{8}$, $0.156\frac{1}{4}$, $0.1562\frac{1}{2}$, and 0.15625. Give corresponding decimals for these.

   A $\frac{15}{64}$ **B** $\frac{224}{625}$ **C** $\frac{3}{16}$ **D** $\frac{77}{125}$ **E** $\frac{39}{40}$ **F** $\frac{39}{80}$

think

The colored arrows show two paths you can follow in order to spell the word **MATH**. Find the **total** number of such paths.



UTILIZATION

The ability to interpret what is meant by a mixed decimal numeral is important, since this type of notation is often used in applications of mathematics. This notation is especially useful when the student has not yet been introduced to repeating decimals. The ability to find a fraction for a given mixed decimal numeral, and vice versa, is also important. All of this work can be viewed as preparation for the introduction of repeating decimals in the next lesson.

EXTENSION

Enrichment: Ask students to give a fraction for each decimal numeral in Exercise 1.

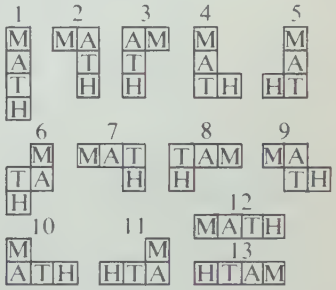
A. $0.06\frac{1}{4} = \frac{6\frac{1}{4}}{100} = \frac{25}{400} = \frac{1}{16}$

B. $0.01\frac{1}{2} = \frac{1\frac{1}{2}}{100} = \frac{3}{200}$

C. $0.34\frac{3}{10} = \frac{34\frac{3}{10}}{100} = \frac{343}{1000}$

Think Solution

There are 15 paths: 13 + 2 (two given in the student text illustration).



Some students might be interested in knowing how many paths there would be for the complete word MATHEMATICS. It can be shown that there would be $2^{11} - 1$, or 2047 paths. In general, if the pattern is extended to a word of n letters, there will be $2^n - 1$ paths.

Assignments

Minimum 1, 2, 3A–C.
Average 1–3. Maximum 1–4.

Objective

The student can find a repeating decimal for a given rational number by dividing the numerator of the fraction by its denominator.

PREPARATION

Since the division process for finding decimals for fractions has been used in previous lessons, no special preparation is needed for the Investigation.

INVESTIGATION

Allow time for each student to carry out the division process for each of the given fractions.

DISCUSSION

In the development of real numbers in Learning Unit H, Module 3, the student will need to know that each rational number can be represented by a repeating or a terminating decimal and that each terminating or repeating decimal represents some rational number. These basic ideas are used in the introduction of irrational numbers and the completion of the set of real numbers.

In discussing Exercise 3, note that a fraction will have a terminating decimal if and only if the only factors of the denominator are powers of 2 or 5. All other fractions will yield repeating decimals which do not terminate in zeros. To illustrate this, use the fraction $\frac{1}{7}$.

The remainders are limited to the numbers 1 through 6. Since these remainders, coupled with the zero brought down from the dividend (10, 20, 30, 40, 50, 60), are not divisible by 7, we know that one of the remainders must repeat before the seventh decimal place in the quotient. As soon as a remainder repeats, then a group of digits will repeat in the quotient. This group of digits will correspond with the group of digits determined by the initial appearance of the repeated remainder.

$$\begin{array}{r} 0.142 \\ 7\overline{)1.0000} \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 6 \end{array}$$

Repeating Decimals

Investigating the Ideas

Every rational number has a repeating decimal representation. (The repeating part is sometimes all zeros.)

$\frac{2}{3} \rightarrow 3\overline{)2.000}$ = 0.666 . . .

18
20
18
20
18
2
⋮

$\frac{5}{8} \rightarrow 8\overline{)5.0000}$ = 0.625000 . . .

48
20
16
40
40
00
0
0
⋮

$\frac{7}{11} \rightarrow 11\overline{)7.0000}$ = 0.6363 . . .

66
40
33
70
66
40
33
7
⋮

The three dots " . . . " means that the pattern indicated continues indefinitely.

Can you find a repeating decimal for each of these? **A** $\frac{5}{6}$ **B** $\frac{9}{25}$ **C** $\frac{5}{9}$ **D** $\frac{3}{7}$

Discussing the Ideas

Given number and its decimal	Repeating decimal notation
$\frac{2}{3} = 0.6666\ldots$	$= 0.\overline{6}$
$\frac{7}{11} = 0.636363\ldots$	$= 0.\overline{63}$
$\frac{5}{27} = 0.185185\ldots$	$=$
$\frac{123}{90} = 1.366666\ldots$	$=$

1. Instead of the dot notation, a bar is often used to indicate the repeating part of a decimal. What would be the bar notation for $\frac{5}{27}$ and $\frac{123}{90}$ in the table?

2. First select the correct decimal for n. Then explain why each of the other decimals is incorrect.
   A $0.\overline{17} = n$ (0.17777 . . . , 0.171111 . . . , 0.171717 . . .)
   B $0.\overline{254} = n$ (0.2545454 . . . , 0.254254 . . . , 0.254444 . . .)

3. Repeating decimals which repeats zeros, from some point on, are called **terminating decimals**.
   EXAMPLES: $\frac{7}{16} = 0.4375000\ldots = 0.4375\overline{0}$ is a terminating decimal.
   $\frac{11}{40} = 0.275000\ldots = 0.275\overline{0}$ is a terminating decimal.

   Can you decide, without dividing, whether or not a fraction will have a terminating decimal representation?

Using the Ideas

1. Write the repeating decimal notation for each decimal.

A 0.3333. . .	D 0.324324 . . .	G 0.024024 . . .	J 0.507507 . . .
B 0.7777 . . .	E 0.565656 . . .	H 0.37153715 . . .	K 0.626262 . . .
C 0.151515 . . .	F 0.9999 . . .	I 0.04260426 . . .	L 0.111111 . . .

2. For some rational numbers, the repeating part does not immediately follow the decimal.

EXAMPLE: $3.2465252\ldots = 3.246\overline{52}$

Write the repeating decimal notation for each decimal.

A 5.2636363 . . .	D 526.1444 . . .	G 56.661515 . . .
B 0.74444 . . .	E 73.4666 . . .	H 0.043212121 . . .
C 3.282828 . . .	F 77.77131313 . . .	I 0.762436243 . . .

3. Give the repeating decimal for each fraction.

A $\frac{1}{3}$	C $\frac{1}{9}$	E $\frac{8}{9}$	G $\frac{3}{11}$	I $\frac{1}{6}$	K $\frac{2}{99}$	M $\frac{1}{7}$
B $\frac{2}{3}$	D $\frac{3}{9}$	F $\frac{1}{11}$	H $\frac{9}{11}$	J $\frac{1}{99}$	L $\frac{71}{99}$	N $\frac{4}{7}$

4. The number of digits in the repeating block is called the **period** of a repeating decimal. The period of $1.42317231\overline{72}\ldots$ is 4. Give the period of each repeating decimal in Exercise 3.

5. Give the fraction or decimal for ***n*** and ***a***.

A If $\frac{1}{9} = 0.\overline{1}$, then $\frac{2}{9} = n$.	F If $\frac{1}{999} = 0.\overline{001}$, then $\frac{285}{999} = n$.
B If $\frac{1}{9} = 0.\overline{1}$, then $n = 0.\overline{7}$.	G If $a = 0.\overline{12}$, then $2 \times a = n$.
C If $\frac{5}{11} = 0.\overline{45}$, then $\frac{1}{11} = n$.	H If $a = 0.\overline{04}$, then $n \times a = 0.\overline{36}$.
D If $\frac{5}{11} = 0.\overline{45}$, then $n = 0.\overline{90}$.	I If $a = 0.\overline{312}$, then $5 \times a = n$.
E If $\frac{1}{99} = 0.\overline{01}$, then $\frac{43}{99} = n$.	J If $a = 0.\overline{473}$, then $n \times a = 0.\overline{946}$.

★**6.** Using the facts that $\frac{1}{9} = 0.1111\ldots$, $\frac{1}{99} = 0.010101\ldots$, and $\frac{1}{999} = 0.001001\ldots$, you can show how every repeating decimal can be written as a fraction. Study the examples and then work the problems.

A $0.\overline{4} = 4 \times (0.111\ldots) = 4 \times \frac{1}{9} = \frac{4}{9}$

B $0.\overline{27} = 27 \times (0.0101\ldots) = 27 \times \frac{1}{99} = \frac{27}{99} = \frac{3}{11}$

C $0.3\overline{27} = .3 + 2.7 \times (0.0\overline{101}\ldots) = \frac{3}{10} + \frac{2.7}{99} = \frac{3}{10} + \frac{27}{990} = \frac{324}{990} = \frac{18}{55}$

A $0.\overline{3} =$	C $0.\overline{28} =$	E $0.8\overline{13} =$	G $0.7034\overline{34} =$
B $0.\overline{71} =$	D $0.9\overline{4} =$	F $0.64\overline{1} =$	H $0.30\overline{6} =$

UTILIZATION

Exercise 5 emphasizes multiples of decimals represented by $\frac{1}{9}$, $\frac{1}{99}$, $\frac{1}{999}$, etc.

Since $\frac{1}{9} = 0.\overline{1}$, it follows that $\frac{2}{9} = 0.\overline{2}$, $\frac{3}{9} = 0.\overline{3}$, etc. Similarly:
$\frac{1}{99} = 0.\overline{01}$, $\frac{2}{99} = 0.\overline{02}$, $\frac{3}{99} = 0.\overline{03}$, etc.
$\frac{1}{999} = 0.\overline{001}$, $\frac{2}{999} = 0.\overline{002}$, $\frac{3}{999} = 0.\overline{003}$, etc.

Exercise 6 illustrates how the process in Exercise 5 can be reversed to find a fraction for a repeating decimal. Thus $0.\overline{27} = \frac{27}{99} = \frac{3}{11}$, $0.\overline{501} = \frac{501}{999} = \frac{167}{333}$, etc. In general,

$$0.\overline{abc\ldots d} = \frac{abc\ldots d}{999\ldots 9},$$

where a, b, c, and d represent digits, not necessarily distinct.

EXTENSION

Workbook page 55 offers appropriate additional practice exercises.

Enrichment: When expressed as a decimal, the fraction $\frac{1}{7}$ has a period of maximum length—six places. (The maximum number of places in a period is 1 less than the divisor.)

$$\frac{1}{7} = 0.\overline{142857}$$

Challenge the students to find some other unit fractions whose decimal representation has a period of maximum length. The fraction $\frac{1}{13}$ might seem like a good example, for its maximum possible period is 12, but $\frac{1}{13} = 0.\overline{076923}$, a period of only six places. A true example would be $\frac{1}{17}$. $\frac{1}{17} = 0.\overline{0588235294117647}$; thus it has a maximum period. Another fraction whose repeating decimal has a period of maximum length is $\frac{1}{23}$. $\frac{1}{23} = 0.\overline{0434782608695652173913}$.

Assignments

Minimum 1A–F, 2A–F, 3A–I, 4.
Average 1–5. Maximum 1–6.

Objective

The student can find sums, differences, and products using repeating decimal numerals.

PREPARATION

This lesson should be regarded as an optional lesson, since the material is not prerequisite for any succeeding material. It does, however, present an unusual treatment of repeating decimals and thus may prove quite motivating.

Place some sample addition and subtraction exercises on the chalkboard involving repeating decimals of the type in the exercise set. Discuss the fact that since the decimals do not terminate, the usual "right to left" method of adding decimals cannot be used. However, we can use a "left to right" process provided that we are careful when regrouping.

UTILIZATION

In Exercise 2B, note that when we work from left to right, we must "plan ahead" and note if regrouping or "borrowing" is necessary in the next place to the right.

$$\begin{array}{r} {\scriptstyle 5\,12\,5} \\ 0.\not{6}\not{2}\not{6}2\ldots \\ -0.1313\ldots \\ \hline 0.4949\ldots \end{array}$$

Exercise 5 suggests two ways to show that $1 = 0.9999\ldots$. This idea is difficult for many students to accept. Another method of showing equality is by subtracting from left to right.

$$\begin{array}{r} 1.00000\ldots \\ -0.99999\ldots \\ \hline 0.00000\ldots \end{array}$$

Since the difference is zero, the two numerals (1 and 0.99999 . . .) must each be a name for 1.

Think Solution

If the multiplication is carried out additional steps, the repeating pattern 0.074074074 . . . will develop. The repeating decimal for $\frac{2}{27}$ is $0.\overline{074}$.

Assignments

Minimum 1A, B; 2A, B; 3A, B.
Average 1–3. Maximum 1–5.

COMPUTING WITH REPEATING DECIMALS

By working from left to right, instead of the usual way, you can do some computing using repeating decimals.

1. Find the sums. Carry out each sum until you get a repeating decimal.

A
$$\begin{array}{r} 0.1111\ldots \\ +0.5555\ldots \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.2323\ldots \\ +0.6464\ldots \\ \hline \end{array}$$

C
$$\begin{array}{r} 1.010101\ldots \\ +0.892892\ldots \\ \hline \end{array}$$

2. Find the differences.

A
$$\begin{array}{r} 0.7777\ldots \\ -0.3333\ldots \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.6262\ldots \\ -0.1313\ldots \\ \hline \end{array}$$

C
$$\begin{array}{r} 0.101010\ldots \\ -0.090909\ldots \\ \hline \end{array}$$

3. Find the products.

A
$$\begin{array}{r} 0.333\ldots \\ \times 2 \\ \hline \end{array}$$

B
$$\begin{array}{r} 0.161616\ldots \\ \times 2 \\ \hline \end{array}$$

C
$$\begin{array}{r} 0.666\ldots \\ \times 2 \\ \hline \end{array}$$

★**4.** In the example, the remainders are given in color. The division has not yet yielded the repeating decimal for $\frac{1}{17}$. Explain why the period of the repeating decimal cannot be as great as 20.

EXAMPLE:

$$\begin{array}{r} 0.05882 \\ 17\overline{)1.00000\ldots} \\ 85 \\ \hline 150 \\ 136 \\ \hline 140 \\ 136 \\ \hline 40 \\ 34 \\ \hline 60 \\ \vdots \end{array}$$

★**5.** Each example illustrates an interesting fact about repeating decimals. Copy each example and complete the computation by adding the decimals in Example A and by multiplying the decimals in Example B.

A
$$\begin{array}{r} \frac{2}{3} = 0.6666\ldots \\ +\frac{1}{3} = 0.3333\ldots \\ \hline 1 = 0.\blacksquare \end{array}$$

B
$$\begin{array}{r} \frac{1}{9} = 0.11111\ldots \\ \times 9 = 9 \\ \hline 1 = 0.\blacksquare \end{array}$$

Strange as it may seem, these examples show that the repeating decimal 0.999 . . . must be another name for 1.

think

The example shows how to multiply $0.\overline{2}$ by $0.\overline{3}$. Copy and continue the multiplication until you are sure that the product is a repeating decimal.

$0.\overline{2} = \frac{2}{9}$ and $0.\overline{3} = \frac{1}{3}$ $\quad \frac{2}{9} \cdot \frac{1}{3} = \frac{2}{27}$.

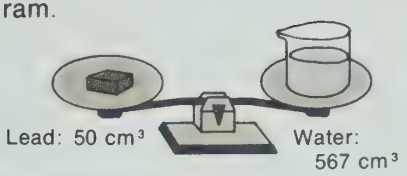
Find the repeating decimal for $\frac{2}{27}$ and check your multiplication.

$$\begin{array}{rr} 0.\overline{2} \rightarrow & 0.22222\ldots \\ 0.\overline{3} \rightarrow & \times 0.33333\ldots \\ \hline 0.3 \times 0.\overline{2} \rightarrow & 0.06666\ldots \\ 0.03 \times 0.\overline{2} \rightarrow & +0.00666\ldots \\ \hline & 0.07333\ldots \\ 0.003 \times 0.\overline{2} \rightarrow & +0.00066\ldots \\ \hline & 0.07399\ldots \\ & \vdots \quad\quad \vdots \end{array}$$

Specific Gravity

The specific gravity of a substance is found by dividing its mass by the mass of an equal volume of water. The mass of one cubic centimetre (cm^3) of water is one gram. Thus, one litre of water has a mass of one kilogram. Lead has a specific gravity of 11.34.

Thus, the mass of 1 cm^3 of lead is 11.34 grams. The mass of 50 cm^3 of lead is $50 \cdot 11.34$ grams $= 567.0$ grams.



Element	Symbol	Specific gravity
Aluminum	Al	2.70
Carbon	C	3.52
Copper	Cu	8.92
Gold	Au	19.3
Helium	He	0.00018
Hydrogen	H	0.00009
Iron	Fe	7.86
Lead	Pb	11.34

Element	Symbol	Specific gravity
Mercury	Hg	13.55
Nitrogen	N	0.00125
Oxygen	O	0.001429
Platinum	Pt	21.37
Potassium	K	0.87
Silver	Ag	10.5
Tin	Sn	7.3
Uranium	U	18.7

1. Find the mass of the given volume of each element.

- **A** 1000 cm^3 of Au
- **B** 10 cm^3 of C
- **C** 76 cm^3 of Ag
- **D** 10 cm^3 of Fe
- **E** 23 cm^3 of Hg
- **F** 100 000 litres of H
- **G** 500 000 litres of He
- **H** 800 litres of N

2. Air is composed mostly of nitrogen and oxygen. It has a specific gravity of about 0.0013.

- **A** What is the mass of one litre of air?
- **B** What is the mass of one cubic metre of air?

3. A classroom has a volume of 384 cubic metres (m^3). What is the mass of the air in the classroom?

4. How many litres of water does it take to have a mass equal to the mass of one litre of mercury?

5. If you found a nugget of pure gold with a mass of one kilogram, what would be the volume of the nugget to the nearest tenth of a cubic centimetre?

Objective

The student can solve problems involving applications of decimals to specific gravity of the elements.

PREPARATION

It would be particularly beneficial in terms of student motivation and understanding if you could introduce this lesson through a specific gravity demonstration like that suggested by the illustration on the student text page. Perhaps the science department of your school might co-operate in arranging such a demonstration, or lend you the needed demonstration materials.

UTILIZATION

Before students begin the exercises, make sure that they understand the introductory text and the meaning of the term *specific gravity*. Explain that, by definition, water at 4°C has a specific gravity of 1 and is the standard for specific gravity of solids and liquids. In order to balance 1 cm^3 of aluminum, one would need about 2.7 cm^3 of water. Thus, aluminum is said to have a specific gravity of 2.7.

Note that chemical symbols for the various elements are included in the table. The symbols are used here merely to add interest and lend additional scientific color to the presentation; students should not be expected to memorize the symbols.

EXTENSION

Enrichment: Have students use a physics or chemistry handbook to find the specific gravity of other elements. Ask them to create problems similar to the ones in this lesson.

Assignments

Minimum 1, 2. Average 1–5. Maximum 1–5.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The exercises on this page can be utilized in various ways. Some students will benefit from demonstrations of completed problems on the chalkboard or overhead projector. Allow sufficient time for students to raise questions on material which has been troublesome to them.

Additional module review problems are provided on page 56 of the *Workbook*.

REVIEWING THE IDEAS

1. Express each numeral as a power of ten.

A $10\,000$ B $\frac{1}{10}$ C $\frac{1}{1000}$

2. Give the usual numeral for $8 \cdot 10^3 + 4 \cdot 10^2 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2}$.

3. Express 563.081 in expanded notation.

4. In the decimal 495.876,

A the 4 tells how many ? .

B the 7 tells how many ? .

5. Find the sums.

A $12.1 + 5.73 + 8.15$

B $160 + 0.376 + 13.5$

C $0.15 + 0.015 + 0.0015$

6. Find the differences.

A $8.59 - 4.87$ C $75 - 69.53$

B $6.1 - 1.39$ D $100 - 3.079$

7. Find the sum.

$5.73 + {}^-6.87 + 10.9 + {}^-0.355$

8. Find the products.

A 0.01×875 D 4.3×0.005

B 1000×4.08 E 0.12×1.3

C 0.5×19.43 F 95.1×2.1

9. Give the usual numeral for each.

A 9×10^5 C 5.3×10^{-2}

B 6.3×10^3 D 3.88×10^{-6}

10. Express each number in scientific notation.

A 2500 C 0.059

B 45 000 D 0.00031

11. Round 385 700 to two significant digits.

12. What is the greatest possible error in the measurement 3.79×10^5 km?

13. Use the fact $224 \div 16 = 14$ to give your answer.

A $2.24 \div 16$ D $0.0224 \div 160$

B $2.24 \div 0.16$ E $0.00224 \div 0.016$

C $22.4 \div 1.6$ F $2240 \div 0.016$

14. Find the quotients.

A $18\overline{)4.5}$ C $0.013\overline{)7.28}$

B $0.6\overline{)0.75}$ D $5.3\overline{)^-48.23}$

15. Round each number to one significant digit. Then estimate each product and quotient.

A 5795×78 C $63\,955 \times 795$

B 32×0.069 D 2005×0.019

16. Find a decimal approximation for each number to the nearest thousandth.

A $\frac{5}{16}$ B $\frac{4}{15}$ C $\frac{8}{9}$

17. Find a mixed decimal numeral for $\frac{7}{11}$ to the nearest hundredth.

18. Give the repeating decimal for each.

A $0.757575\ldots$ B $0.304304\ldots$

19. Give the repeating decimal for each fraction.

A $\frac{5}{6}$ B $\frac{5}{9}$ C $\frac{2}{7}$

TEST YOURSELF

1. Find the exponent n in each equation.
   - A $100 = 10^n$
   - B $10\,000 = 10^n$
   - C $\frac{1}{10} = 10^n$
   - D $0.01 = 10^n$

2. In the decimal 2.7134,
   - A the 7 tells the number of ?.
   - B the 4 tells the number of ?.
   - C the ? means 3×10^{-3}.

3. Find the sum or difference.
   - A $6.24 + 18.1 + 11.729$
   - B $126.4 - 92.48$

4. Find the products.
   - A 1000×0.0628
   - B 0.01×346.4
   - C 19.43×0.08

5. Round 0.007439 to two significant digits and express in scientific notation.

6. The average distance from Earth to the moon is 3.83×10^5 km. What is the greatest possible error in this measurement?

7. Find the quotients.
   - A $0.18\overline{)0.702}$
   - B $168\overline{)0.7896}$

8. Find a decimal approximation for each number to the nearest thousandth.
   - A $\frac{7}{16}$
   - B $\frac{29}{48}$

9. Find a repeating decimal for each rational number.
   - A $\frac{5}{9}$
   - B $\frac{7}{12}$
   - C $\frac{1}{2}$

10. Which number is the best estimate for the quotient $12.49 \div 0.00618$?
    - A 2
    - B 20
    - C 200
    - D 2000

RESEARCH PROJECTS

A Simon Stevin (1548–1620) is credited with inventing decimal notation for rational numbers. Stevin's notation was somewhat different than our modern notation. Find out about Stevin's decimal symbols and report on your research to the class. (See *A History of Mathematics* by Carl B. Boyer; New York: John Wiley and Sons, 1968, pp. 347–350)

B Find out how to use a slide rule to multiply and divide. Demonstrate and explain to your classmates how it is used.

C Many physical constants that occur in scientific work are either very large or very small. Therefore, they are often expressed in scientific notation. Find some reference books in chemistry or physics and make a list of some of these constants. For example, find the speed of light, the length of a light year, Avogadro's number, the mass of the Earth, the size of a molecule or an atom, and so on. Make a display showing these magnitudes expressed in scientific notation.

TEST YOURSELF

This self-evaluation test will enable students to assess for themselves their grasp of the basic ideas and skills developed in the module. After they have checked their answers against those given at the bottom of the next page, they should return to the appropriate lessons of the module and review any topics which gave them difficulty.

Note that annotations at the left of the test items on the student text page denote the related objectives as listed on the module-opening page.

RESEARCH PROJECTS

Students who elect to undertake Project A will find that most mathematical history books contain some information concerning Simon Stevin. Some encyclopedias may also be useful.

Project B will appeal to students who own or have access to a slide rule. Most educational supply houses stock student slide rules that are usable and quite inexpensive. Instruction booklets usually accompany each slide rule.

Research Project C offers students who are artistically inclined a chance to show their creativity by making posters to illustrate large numbers and how they are represented in scientific notation.

MATHEMATICAL RECREATION

Students who have enjoyed working in other bases will find this recreational page enjoyable and challenging. Since this material is highly theoretical, not all students will benefit from it.

Answers

1. $2.2_{(1/3)} = 2 \cdot (\frac{1}{3})^0 + 2 \cdot (\frac{1}{3})^{-1} = 2 + 2(\frac{3}{1})$
$= 8_{(ten)}$

$0.01_{(1/3)} = 1 \cdot (\frac{1}{3})^{-2} = 1 \cdot 3^2 = 9_{(ten)}$
$1.01_{(1/3)} = 1 + (\frac{1}{3})^{-2} = 1 + 9 = 10_{(ten)}$
$2.01_{(1/3)} = 2 + (\frac{1}{3})^{-2} = 2 + 9 = 11_{(ten)}$
$0.11_{(1/3)} = 1 \cdot (\frac{1}{3})^{-1} + 1 \cdot (\frac{1}{3})^{-2} = 3 + 9$
$= 12_{(ten)}$

2. Age $13_{(ten)} = 1.11_{(1/3)}$
Age $14_{(ten)} = 2.11_{(1/3)}$
Age $15_{(ten)} = 0.21_{(1/3)}$

3. Addition with base-$\frac{1}{3}$ numerals is quite different from addition with integer bases. The example given may lead some students to think that base-$\frac{1}{3}$ numerals can be added almost like ordinary decimals. However, an example like the following should convince them of the difference.

$2.1_{(1/3)} + 1.2_{(1/3)} = 0.11_{(1/3)}$

Some very good students may be able to develop some rules for adding base-$\frac{1}{3}$ numerals.

4. Base $\frac{1}{3} \rightarrow 2 \times 0.1 = 0.2$
$\downarrow \quad \downarrow \quad \downarrow$
Base ten $2 \times 3 = 6$

The multiplication checks out in this case. However, as in addition, the usual rules do not hold. For example in base $\frac{1}{3}$, $2.2 \times 0.2 = 0.122$.

MATHEMATICAL RECREATIONS

A Numeration System with Base $\frac{1}{3}$

Suppose we agree to write numerals with base $\frac{1}{3}$. The basic symbols that we shall agree to use (as we must do in any base) for base $\frac{1}{3}$ are {0, 1, 2}.

EXAMPLES: $21_{(\frac{1}{3})}$ means $2 \cdot (\frac{1}{3})^1 + 1 \cdot (\frac{1}{3})^0 = \frac{2}{3} + 1 = 1\frac{2}{3} = 1.6\bar{6}_{(ten)}$

$212_{(\frac{1}{3})}$ means $2 \cdot (\frac{1}{3})^2 + 1 \cdot (\frac{1}{3})^1 + 2 \cdot (\frac{1}{3})^0$
$= \frac{2}{9} + \frac{1}{3} + 2 = 2\frac{5}{9} = 2.\bar{5}_{(ten)}$

$0.2_{(\frac{1}{3})}$ means $2 \cdot (\frac{1}{3})^{-1} = 2 \cdot 3^1 = 6_{(ten)}$

$0.12_{(\frac{1}{3})}$ means $1 \cdot (\frac{1}{3})^{-1} + 2 \cdot (\frac{1}{3})^{-2} = 1 \cdot (3) + 2 \cdot (3^2) = 3 + 18 = 21_{(ten)}$

$0.101_{(\frac{1}{3})}$ means $1 \cdot (\frac{1}{3})^{-1} + 0 \cdot (\frac{1}{3})^{-2} + 1 \cdot (\frac{1}{3})^{-3} = 1 \cdot (3) + 0 \cdot (3^2) + 1 \cdot (3^3)$
$= 3 + 0 + 27 = 30_{(ten)}$

Counting with base-$\frac{1}{3}$ numerals is interesting but quite awkward. The table gives the first few counting numbers in base ten and base $\frac{1}{3}$. Try these activities.

Base ten	Base $\frac{1}{3}$
1	1
2	2
3	0.1
4	1.1
5	2.1
6	0.2
7	1.2

1. Continue the base-$\frac{1}{3}$ numerals through $12_{(ten)}$.

2. Write your age in years as a numeral in base $\frac{1}{3}$.

3. Write some addition equations using base-$\frac{1}{3}$ numerals. Check them using base-ten numerals.

base-$\frac{1}{3}$ $\rightarrow 0.1 + 1.1 = 1.2$
base ten $\rightarrow \quad 3 + 4 = 7$

4. Is the base-$\frac{1}{3}$ multiplication correct? $2 \times 0.1 = 0.2$ Check it using base-ten numerals.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A 2; B 4; C $^-1$; D $^-2$ **2.** A tenths; B ten thousandths; C 3 **3.** A 36.069; B 33.92 **4.** A 62.8; B 3.464; C 1.5544 **5.** 7.4×10^{-3} **6.** 0.005×10^5 km or 500 km **7.** A 3.9; B 0.0047 **8.** A 0.438; B 0.604 **9.** A $0.\bar{5}$; B $0.58\bar{3}$; C $0.5\bar{0}$ **10.** D

UNIT F: RATIONAL NUMBERS

Module 3: Ratio and Percent

General Objectives

To strengthen the students' understanding and skill in use of ratios.
To provide relevant problems involving ratios and proportions.
To provide experience with scale drawings.
To strengthen the students' understanding and ability to apply percent concepts.
To provide a wide variety of practical applications of mathematics.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
64	**RED** Given two numbers, the student can use the simplest-terms ratio to compare them.	F-68, 69	**WB** 57		
65	**RED** Given a simple proportion problem, the student can solve it.	F-70, 71	**WB** 58 **DM** 41	**DS** b-2	
66	**RED** Given word problems involving ratio, the student can use proportions to solve them.	F-72, 73, 78 (F-74, 75)	**WB** 59 **DM** 42		**SWM 3** 126-127, 270-271
67	**RED** Given problems involving scale drawings, the student can interpret the drawings to solve the problems.	F-76, 77	**WB** 60 **DM** 43	**SP** f-1	
68	**RED** Given a rational number in fraction or decimal form, the student can express it as a percent, and vice versa.	F-80, 81 F-82, 83	**WB** 61, 62 **DM** 44	**ASC** D-7, 8, 9, 11-14 **PD** h-2	
69	**RED** The student can find the percent of a number.	F-84, 85	**WB** 63 **DM** 45	**PD** h-3 **DS** g-4	
70	**YELLOW** Given word problems involving percent applications such as discounts and sales tax, the student can solve them.	F-86, 87	**WB** 64 **DM** 46	**ASC** AP-20 **PD** h-4	
71	**YELLOW** Given problems involving percents less than 1% or greater than 100%, the student can solve the problems.	F-88, 89	**WB** 65		
72	***GREEN*** *Given problems relating to increases and decreases of measurement, the student can find the percent of increase or decrease.*	F-90, 91	**DM** 47		
	Reviewing the Ideas	F-92	**WB** 66		

MATHEMATICS

The module covers ratio, proportions, and percents and applies these ideas to relevant problem situations.

Formally, ratio may be defined in these terms:

If x and y are numbers and $y \neq 0$, the ratio of x to y is the quotient $\frac{x}{y}$.

The statement that two ratios are equal is called a *proportion*. If $\frac{a}{b}$ and $\frac{c}{d}$ are ratios and $\frac{a}{b} = \frac{c}{d}$, then a and d are called the *extremes* of the proportion and b and c are called the *means*. If $\frac{a}{b} = \frac{c}{d}$, it follows from the properties of rational numbers that $ad = bc$, and conversely.

Percent refers to special ratios in which numbers are compared to 100. The ratio $\frac{n}{100}$ means $n\%$. The work with percent presents no new concepts in mathematics. Percent simply involves new language and notations for rational number concepts that are already familiar to the students.

TEACHING THE MODULE

Materials

Centimetre rulers, spring scales, graph paper, tape measures.

Vocabulary

ammeter
amperes
coefficient of linear expansion
discount
discount rate
electromotive force
equivalent ratios
extremes
means
percent
proportion
ratio
sales tax
scale
scale drawing
simplest-terms ratio

Although few materials are listed for this module, you will want to use familiar objects around the classroom as well as ideas associated with measurement to explain the idea of ratio to the students.

The lessons on Springs and Masses (F-74 and F-75) and Ratios and Electricity (F-78 and F-79) will be enhanced if some models for these experiments could be displayed or demonstrated for the students. Your science teachers might cooperate with you in the presentation of these lessons.

Scale drawing is a familiar application of the concept of ratio. A ratio is set up between the physical model and the drawing so that the drawing can be interpreted more easily. Constructing maps and drawings to scale reinforces for the students the idea of ratio as a comparison. They may enjoy making scale drawings or maps of their neighborhoods or of their school grounds.

Much of the work with percent deals with applications in buying at a discount and figuring sales tax on an item. Most students at this level have had experience with these kinds of applications and will find them quite relevant.

Lesson Schedule

The lesson schedule you follow will depend on the background and abilities of your students. If students have a good background in the concepts of ratio and percent, you may wish to concentrate on the applications and move through the module quickly. However, for students lacking in preparation, additional time in development or remedial work may be necessary. Hence, coverage of the module might require as few as 10 days or as many as 15 days.

Evaluation

Evaluation of the students' work in this module should cover two areas:

1. Language and symbolism of ratio, proportion, and percent.
2. Applications of ratio, proportion, and percent.

Utilize the review pages to prepare your students for an achievement test on this module.

Note that a cumulative review of this learning unit appears at the end of the module. It would be worthwhile to spend a day in review before administering a test covering the complete learning unit.

RESOURCES FOR ACTIVE LEARNING

Activities

Cloudburst, Vol. 5 (science applications), Nos. 5700 series, 8688, 8838–68, Midwest Publications

Developmental Math Cards, K10, Addison-Wesley

ESS: Pendulums, Webster, McGraw-Hill

Experiments in Mathematics, Stage 2 (dissection puzzle), pp. 6–7, Houghton Mifflin

Fractions, "Ratio," Set 7, Herder and Herder

Madison Project: *Discovery*, "A Law of Physics," pp. 264–266 (TT), Addison-Wesley

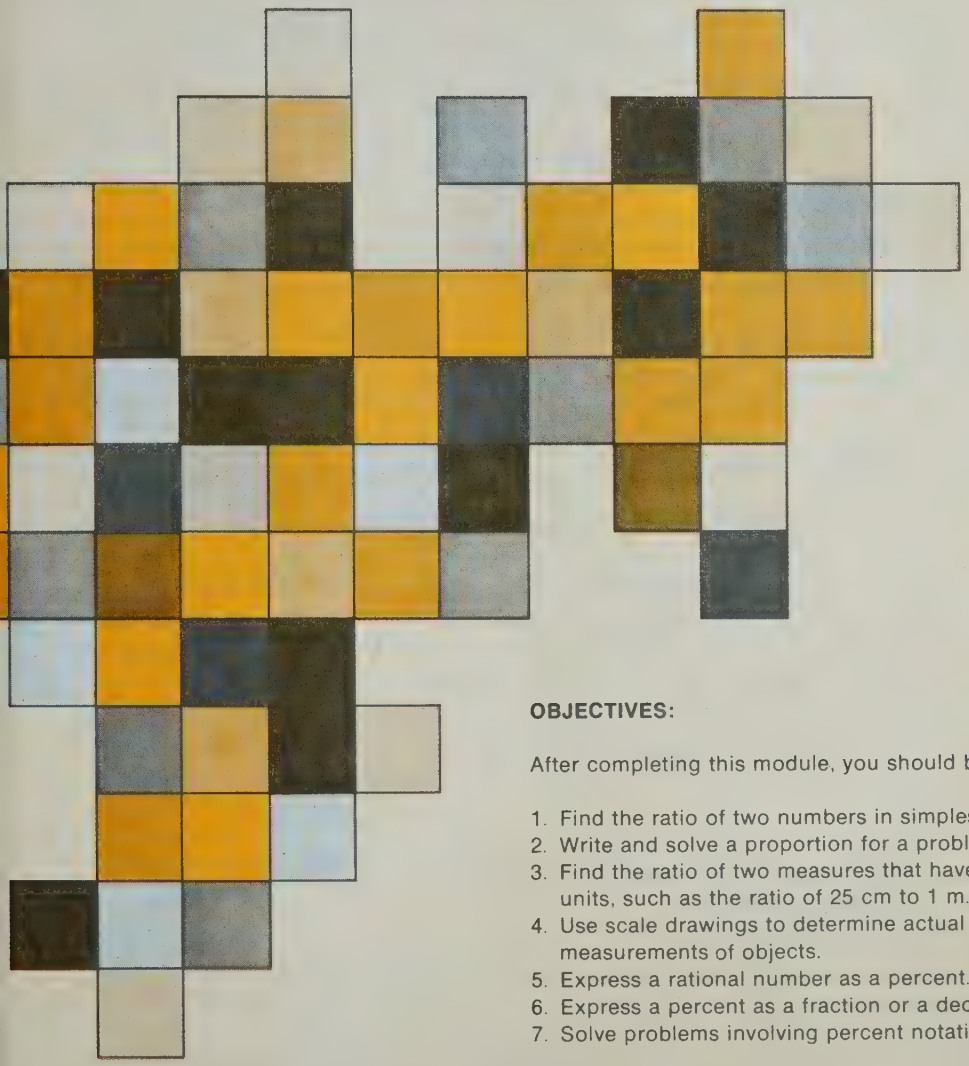
Mathematics: Man's Key to Progress, Book A, pp. 70–71; Book B (gravity and motion), pp. 62–70, Rand McNally

Measure and Find Out, Book 3, Scott, Foresman

For some other activities involving ratio, refer to Resources for Active Learning, Unit G, Module 3

Manipulative Devices

Metric tape measure (Dick Blick)



OBJECTIVES:

After completing this module, you should be able to:

1. Find the ratio of two numbers in simplest form.
2. Write and solve a proportion for a problem.
3. Find the ratio of two measures that have different units, such as the ratio of 25 cm to 1 m.
4. Use scale drawings to determine actual measurements of objects.
5. Express a rational number as a percent.
6. Express a percent as a fraction or a decimal.
7. Solve problems involving percent notation.

Objective

The student can compare two numbers as a ratio in simplest terms.

PREPARATION

As preparation for this section, it might prove helpful, especially for slower students, to review quickly equivalent fractions and how to build sets of equivalent fractions. It might also be helpful to review the property of equality for rational numbers:

$$\frac{a}{b}=\frac{c}{d}, \text{ if and only if } ad = bc.$$

INVESTIGATION

The Investigation is designed to provide a practical application of the concept of ratio. Do not expect students to give exact answers. For example, for a class of 25 students, answers such as 4 pizzas, 35–38 bottles of pop, 40–45 doughnuts, and 4 or 5 bags of potato chips would be reasonable.

DISCUSSION

Students should be able to give intuitive explanations of the results of the Investigation. For example, 1 pizza for 6 students means 4 pizzas would serve about 24 students; 3 bottles of pop for 2 students means 12 bottles would serve 8 students; and so on.

Although ratio and its notation may constitute review topics, you may need to emphasize that the ratio of a number x to a number y is denoted in three ways in this lesson:

A. x to y B. $\frac{x}{y}$ C. $x:y$

You may find it helpful also to discuss ratios such as the one in the example accompanying Exercise 4 on page F-69 before assigning the exercises.

Ratio

Investigating the Ideas

Suppose you are chairman of the Food Committee for a class party. The committee has decided to buy the food shown on the list at the right.

Grocery list:

1 pizza for every 6 students
3 bottles of pop for every 2 students
5 doughnuts for every 3 students
2 bags of potatochips for every 10 students

Can you figure out how many of each item you would have to buy to feed all the students in your mathematics class?

Discussing the Ideas

1. Explain how you decided how much of each item to buy.

2. Suppose a class had 30 students. Explain how to find how many of each item you would need.

3. The **ratio** of pizzas to students is 1 to 6. This ratio can be denoted by the symbols 1:6 or $\frac{1}{6}$.
   Give the ratio of each of the other items on the list to the number of students.

4. Just as the fraction symbols $\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots$ are called equivalent fractions, the ratios 1:4, 2:8, 3:12, . . . are called **equivalent ratios**.
   What are three equivalent ratios that would compare
   - **A** the number of bottles of pop to the number of students?
   - **B** the number of doughnuts to the number of students?

5. The ratio of bags of potato chips to students is 2:10 or $\frac{2}{10}$. Since the lowest-terms fraction for $\frac{2}{10}$ is $\frac{1}{5}$, we say that the **simplest-terms ratio** for 2:10 is 1:5.

   What is the simplest-terms ratio for each of the following?

   A 6:8 **B** 15 to 12 **C** $\frac{6}{15}$ **D** 8:24

Using the Ideas

1. Give three more ratios for each set of equivalent ratios.

A {1 to 2, 2 to 4, 3 to 6, . . .}

B {3:2, 6:4, 9:6, . . .}

C {3 to 4, 6 to 8, 9 to 12, . . .}

D $\left\{\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \ldots\right\}$

E {2:5, 4:10, 6:15, . . .}

F $\left\{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \ldots\right\}$

2. Find the number for n so that each ratio is equivalent to 3:5.

A n:10 **C** 12 to n **E** $\frac{30}{n}$ **G** n:25 **I** $\frac{36}{n}$

B 9:n **D** n to 15 **F** $\frac{n}{35}$ **H** 6 to n **J** n:30

3. Give the simplest-terms ratio for each of the following.

A 4:8 **B** 15:12 **C** 63:36 **D** 42:51 **E** 14:91

4. We can express the ratio of two numbers even though they are not whole numbers. Thus, the ratio $\frac{1}{2}$ to $\frac{3}{4}$ can be written as a ratio involving whole numbers:

$$\frac{1}{2} \text{ to } \frac{3}{4} = \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{4}{6} = \frac{2}{3}$$

Write each ratio as a ratio of whole numbers.

A $\frac{1}{2}$ to $\frac{2}{3}$ **B** $\frac{1}{9}$ to $\frac{1}{2}$ **C** $\frac{3}{10}$ to $\frac{1}{3}$ **D** $1\frac{1}{2}$ to $\frac{7}{4}$ **E** $2\frac{1}{4}$:5

5. Approximate each ratio with a decimal rounded to the nearest hundredth.

EXAMPLE: $\frac{7}{12}$

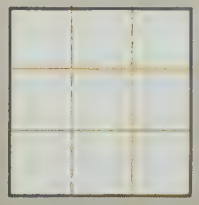
$\frac{7}{12}$ ≈ 0.58

A $\frac{9}{14}$ **B** 15:23 **C** $\frac{7}{24}$ **D** 18:14 **E** 5 to 23

★**6.** In all the basketball games for a season, Jack made 3 out of every 5 free throws that he tried. He missed a total of 26 free throws. How many did he make?

think

Trace the square. Shade the interior and cut it out. Now cut along the dashed lines. Fold along the lines that are not dashed to make a cube that has all faces shaded.



UTILIZATION

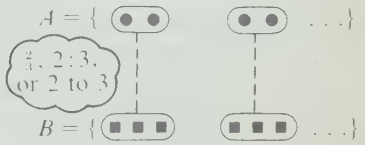
The exercise set gives the students practice in constructing sets of equivalent ratios and solving simple equivalent-ratio problems.

Much of this work will seem like review to your students, as it is very similar to earlier work with fractions and sets of equivalent fractions. You should be sure that students recognize the fact that ratios are closely related to rational numbers.

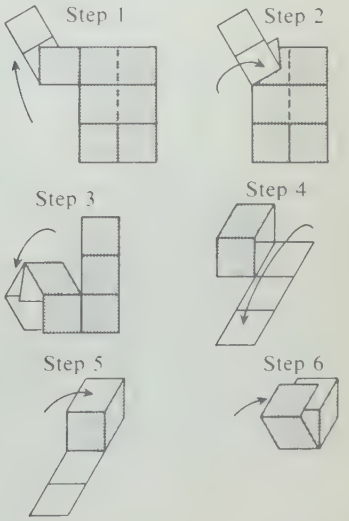
EXTENSION

Additional practice exercises are provided on page 57 of the *Workbook*.

Remedial: Illustrate ratios by means of correspondences such as those shown below. Have students observe that for every 2 objects in set A, there are 3 objects in set B.



Think Solution



Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

Given three of the terms of a proportion $\frac{a}{b}=\frac{c}{d}$, *the student can find the fourth term.*

PREPARATION

Materials: Centimetre rulers.

Once the students have related ratios to fractions, they should have little difficulty understanding equivalent ratios and proportions. Therefore, an important objective for this lesson is to help students gain skill in solving problems involving proportions. The theorem

$$\frac{a}{b}=\frac{c}{d} \text{ if and only if } a \cdot d = b \cdot c$$

is the basic tool used in solving proportions.

INVESTIGATION

There are 27 ratios of $\frac{1}{3}$ in the figure. No student should be expected to find them all.

$\frac{\overline{IB}}{\overline{AB}}, \frac{\overline{HC}}{\overline{AC}}, \frac{\overline{GD}}{\overline{AD}}, \frac{\overline{FE}}{\overline{AE}}, \frac{\overline{CD}}{\overline{AD}}, \frac{\overline{CD}}{\overline{BE}}, \frac{\overline{AB}}{\overline{AD}}, \frac{\overline{AB}}{\overline{BE}}, \frac{\overline{AI}}{\overline{AG}},$

$\frac{\overline{AI}}{\overline{IF}}, \frac{\overline{IH}}{\overline{AG}}, \frac{\overline{IH}}{\overline{IF}}, \frac{\overline{BC}}{\overline{AD}}, \frac{\overline{BC}}{\overline{BE}}, \frac{\overline{DE}}{\overline{AD}}, \frac{\overline{DE}}{\overline{BE}}, \frac{\overline{HG}}{\overline{AG}}, \frac{\overline{HG}}{\overline{IF}},$

$\frac{\overline{GE}}{\overline{AG}}, \frac{\overline{GF}}{\overline{IF}}, \frac{\overline{IB}}{\overline{GD}}, \frac{\overline{IB}}{\overline{BC}}, \frac{\overline{IB}}{\overline{CD}}, \frac{\overline{IB}}{\overline{DE}}, \frac{\overline{HC}}{\overline{BD}}, \frac{\overline{HC}}{\overline{CE}}, \frac{\overline{GD}}{\overline{BE}}$

DISCUSSION

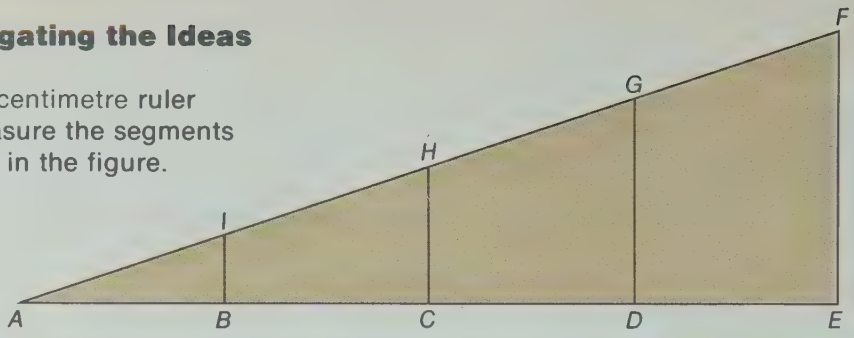
The words "means" and "extremes" are introduced in Exercise 3. Although these words may be new to students, they are related to familiar rational number concepts.

Students have previously solved problems such as the example in Exercise 4 when they studied rational numbers. Therefore, only the language of proportions should be new to the students. Provide additional examples as needed.

Proportions

Investigating the Ideas

Use a centimetre ruler to measure the segments shown in the figure.



How many pairs of segments can you find in which the ratio of their lengths, in simplest terms, is 1:3?

Discussing the Ideas

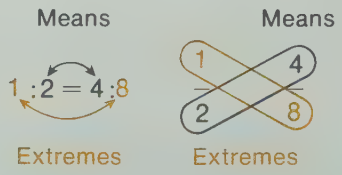
1. A What is the length of $\overline{EF}$? of $\overline{AE}$?
   B What is the ratio of the length $\overline{EF}$ to $\overline{AE}$?
   C What is the ratio in simplest terms?

2. An equation stating that two ratios are equivalent is called a **proportion**. Each of these is a proportion:

   $\frac{1}{3}=\frac{4}{12}$ $\qquad$ 1:3 = 4:12 $\qquad$ 1 to 3 = 4 to 12

   What are some other ratios that could form a proportion when $\frac{1}{3}$ is one of the ratios?

3. In the proportion at the right, the "end" numbers (1 and 8) are called the **extremes**. The "middle" numbers (2 and 4) are called the **means**. What do you think is the relation between the means and the extremes in a proportion?



4. You can use the fact that

   In a proportion, the product of the means equals the product of the extremes.

   to solve problems involving proportions. Study and complete the example.

EXAMPLE:

$$\frac{2}{3}=\frac{8}{x}$$

$2 \cdot x = 3 \cdot 8$

$x = 24 \div 2 =$

Using the Ideas

1. In each proportion, show that the product of the means is equal to the product of the extremes.

A $1:3 = 3:9$ **C** $4:6 = 10:15$ **E** $3:5 = 60:100$

B $\frac{3}{5} = \frac{12}{20}$ **D** $\frac{7}{3} = \frac{28}{12}$ **F** $30:90 = 27:81$

2. Solve the proportions.

A $\frac{5}{6} = \frac{15}{s}$ **D** $4:6 = y:15$ **G** 10 to 20 = 5 to b **J** $\frac{14}{16} = \frac{x}{8}$

B $\frac{t}{40} = \frac{6}{8}$ **E** $d:12 = 12:16$ **H** c to 6 = 5 to 30 **K** $\frac{15}{r} = \frac{5}{8}$

C $\frac{4}{10} = \frac{w}{40}$ **F** $10:12 = n:6$ **I** 7 to 10 = h to 10 **L** $\frac{14}{16} = \frac{a}{24}$

3. Write and solve a proportion for each problem.

EXAMPLE: Two of every 5 students in class are boys. There are 14 boys in the class. How many students are in the class?

SOLUTION: Let x represent the number of students in the class.

$$\begin{matrix}\text{Boys} \\ \textbf{Students}\end{matrix} \begin{matrix}\rightarrow 2 \\ \rightarrow 5\end{matrix} = \frac{14}{x} \Rightarrow 2 \cdot x = 5 \cdot 14$$
$$x = 35$$

A Oranges sell at 5 for 39 cents. How much do 15 oranges cost?

B In a group, the ratio of girls to boys is 3 to 4. If there are 15 girls, how many boys are there?

C On a map, 2 cm represents 50 km. What distance is represented by 7 cm?

D Two of every 100 people at a concert are over 80 years old. If there are 15 people over 80 years old, how many people are at the concert?

4. On a map, 0.7 cm represents 14 km. What distance would be represented by 1.3 cm?

★ **5.** Solve the proportions.

A $\frac{x}{1\frac{1}{2}} = \frac{3}{4}$ **B** $\frac{1}{x} = \frac{2\frac{1}{4}}{7}$ **C** $\frac{2}{3} = \frac{x}{7}$ **D** $\frac{2\frac{1}{2}}{3\frac{1}{4}} = \frac{6}{x}$

★ **6.** Solve the proportions.

A $\frac{2}{3} = \frac{x}{x+12}$ **B** $\frac{3}{4} = \frac{n}{28-n}$ **C** $\frac{s+9}{s} = \frac{5}{2}$ **D** $\frac{24-b}{b} = \frac{3}{5}$

More practice, page S-19, Set 36

UTILIZATION

Before assigning the exercises you may want to discuss the methods of solving word problems such as those presented in Exercises 3 and 4.

Exercises 5 and 6 are more algebraic in nature and should be assigned as optional problems.

Solutions, Exercise 6

The following are algebraic solutions, but encourage other reasoning as well.

A.
$$\frac{2}{3} = \frac{x}{x+12}$$
$$3x = 2(x+12)$$
$$3x = 2x + 24$$
$$2x + x = 2x + 24$$
$$x = 24$$

D.
$$\frac{24-b}{b} = \frac{3}{5}$$
$$3b = 5(24-b)$$
$$3b = 120 - 5b$$
$$3b + 5b = 120$$
$$8b = 120$$
$$b = 15$$

EXTENSION

To provide practice, make assignments from Supplementary Exercise Set 36 on page S-19, *Workbook* page 58, and *Duplicator Masters* page 41.

Enrichment: Have students choose a ball (golfball, tennis ball, handball, etc.) and record the height of its bounce when dropped from a height of one metre. They may need to do this several times to obtain a good measure of the height of the bounce. Using the ratio $\frac{\text{height of bounce}}{\text{height of drop}}$, they can predict how high the ball will bounce when dropped from other heights. Then, have them experiment to check their prediction.

Assignments

Minimum 1, 2A–F, 3. Average 1–4.
Maximum 1, 2G–L, 3–6.

Objective

The student can express two measures with a common unit and give the ratio of the measures.

UTILIZATION

The ability to convert from one unit of measure to another is important both in many areas of the scientific world and in everyday experiences. The number of existing units of measure is much too large to be considered here. Therefore, in planning this section, emphasize the general ideas involved in converting from one unit to another, so that the units can be applied in a wide variety of situations. Do not expect the students to remember how many of one unit is contained in another or to become skilled at particular conversions. They should, however, know the more common basic unit relationships, such as the number of seconds in a minute, minutes in an hour, hours in a day, centimetres in a metre, grams in a kilogram, and so on.

EXTENSION

Have students place one foot on a sheet of centimetre graph paper (*Duplicator Masters*, page 83) and trace the outline of their foot. Then they should try to determine the area of their foot by counting the number of squares enclosed by the outline. If the students then divide their mass in kilograms by the area in cm^2 of their foot, they can determine the pressure (kg/cm^2) exerted on the floor where they stand on one foot.

Assignments

Minimum 1A–F, 2. Average 1A–L, 2, 3. Maximum 1–3.

Ratio and Units of Measure

It is often necessary to convert measurements to different units. For example, a speed in kilometres per hour (km/h) must be changed to metres per minute or, perhaps, a measurement in millilitres must be changed to litres. Ideas of ratio and proportion are helpful in such calculations.

Highest speed by man: 39 889 km/h by the Apollo X astronauts

Thus, $\frac{48\text{ km}}{1\text{ h}} = \frac{48\text{ km}}{60\text{ min}} = \frac{48\,000\text{ m}}{60\text{ min}} = \frac{800\text{ m}}{1\text{ min}} = 800$ metres per minute

and $780\text{ ml} = 780\text{ ml} \cdot \frac{1\,\ell}{1000\text{ ml}} = 0.780\,\ell$

1. Find the correct number for each ▒.

A $7\text{ m} = ▒\text{ cm}$
B $7\text{ g} = ▒\text{ mg}$
C $19\text{ m} = ▒\text{ mm}$
D $180\text{ min} = ▒\text{ h}$
E $48\text{ h} = ▒\text{ day}$
F $3000\,\ell = ▒\text{ kl}$
G $▒\text{ cm} = 32\text{ mm}$
H $36\text{ mo} = ▒\text{ year}$
I $▒\text{ mg} = 20\text{ g}$
J $▒\text{ kg} = 4000\text{ g}$
K $2\text{ m} = ▒\text{ cm}$
L $▒\text{ m} = 450\text{ cm}$
M $▒\text{ m} = 0.9\text{ km}$
N $0.3\text{ m} = ▒\text{ cm}$
O $▒\text{ mm} = 30\text{ cm}$
P $600\text{ g} = ▒\text{ kg}$
Q $▒\text{ cm} = 2\text{ m}$
R $7\text{ h} = ▒\text{ s}$

2. Copy the equations and give the numbers for ***a*** and ***b***.

A $\frac{120\text{ km}}{1\text{ h}} = \frac{120\text{ km}}{a\text{ min}} = \frac{b\text{ km}}{1\text{ min}} = b\text{ km/min}$

B $\frac{1400\text{ km}}{1\text{ wk}} = \frac{1400\text{ km}}{a\text{ d}} = \frac{b\text{ km}}{1\text{ d}} = b\text{ km/d}$

C $\frac{3000\,\ell}{1\text{ min}} = \frac{3000\,\ell}{a\text{ s}} = \frac{b\,\ell}{1\text{ s}} = b\,\ell/\text{s}$

3. Sometimes it is helpful to give the ratio of two measurements without giving the units. Both measurements must first be expressed with a common unit.

Thus, $\frac{20\text{ cm}}{3\text{ m}} = \frac{20\text{ cm}}{300\text{ cm}} = \frac{20}{300} = \frac{1}{15}$,

and $12\text{ min} : 1\text{ h } 20\text{ min} = 12\text{ min} : 80\text{ min} = 12:80 = 3:20$.

Express these ratios without units.

A $\frac{8\text{ mm}}{4\text{ cm}}$ **B** $\frac{25\text{ min}}{1\text{ h}}$ **C** $\frac{1\text{ cm}}{2\text{ km}}$ **D** $\frac{40\text{ s}}{1\text{ min }40\text{ s}}$ **E** $\frac{680\,\ell}{2\text{ kl}}$

SPORTS RATIOS

1. Jim Plunkett completed 158 passes in 328 attempts during one football season.
   - A What is the ratio of completed passes to attempts in simplest form?
   - B Find a decimal to the nearest thousandth for that ratio.

2. Billie Jean King won the Wimbledon Tennis Championship five times in eight years. Express this ratio as a decimal to the nearest thousandth.

3. George Blanda once completed 37 passes in 68 attempts in a single football game. Find a 3-place decimal to express the ratio of completed passes to attempts.

4. Phil Esposito's NHL scoring record: 152 points, 78 games. What is the ratio of points to games?

5. In 1972, Ken Dryden played 54 hockey games, conceding 119 goals. Express his goals-against average as a decimal of three significant digits.

6. In a certain year the Los Angeles Lakers lost only 13 games out of 82 games that they played. What is the ratio of the games won to the games played? the games lost to the games played?

7. The leading scorer in the NBA in a certain year was Kareem Abdul-Jabbar. In that year, he made 1159 field goals in 2019 attempts. Find the ratio of field goals to attempts as a decimal to the nearest thousandth.

8. Steve Carlton was the leading pitcher in the National League one year when he won 27 games and lost 10. What part of the games did he win?

9. Rod Carew led the American League in hitting with 170 hits in 535 times at bat. Find the ratio of hits to times at bat as a decimal to the nearest thousandth.

10. The Toronto Maple Leafs hockey team had a record of 33 wins, 31 losses, and 14 ties for a season. What part of their games did they win? lose? tie?

Objective

The student can interpret and solve word problems involving ratios.

UTILIZATION

This special set of problems related to various sports should prove interesting and motivating to many students. Since ratios occurring in sports statistics are often given as decimals, students are asked to give a decimal for the ratio in several of these exercises.

EXTENSION

Suitable further practice assignments may be made from *Workbook* page 59 and *Duplicator Masters* page 42.

Enrichment: Some students might be interested in doing some research into current sports records which involve ratios. Have the students create their own story problems similar to the ones on this page. Duplicate the problems and distribute them to other students to solve.

Assignments

Minimum Odd- or even-numbered problems. Average 1–8. Maximum 1–10.

Objective

The student can solve problems involving springs and masses by writing and solving proportions.

PREPARATION

Obtain a spring balance and some standard masses before introducing this lesson. Demonstrate that when the mass applied to the string is doubled, the spring is stretched twice as far; when the mass is tripled, the spring is stretched 3 times as far; and so on. This principle of physics is called Hooke's Law (after Robert Hooke, 1635–1703). This law asserts that, if a force is applied to an elastic object (such as a spring), the amount of stretch of the spring is directly proportional to the force applied, provided that the elastic limit of the spring is not exceeded. (The elastic limit is the maximum stretch that the spring can withstand and still return to its original state.)

INVESTIGATION

To solve the Investigation problem, the students may write the proportion

$$\frac{3 \text{ cm}}{x \text{ cm}} = \frac{10 \text{ kg}}{36 \text{ kg}}$$

Then $10x = 108$ and $x = 10.8$. Thus, the 36 kg mass would stretch the spring about 10.8 cm.

DISCUSSION

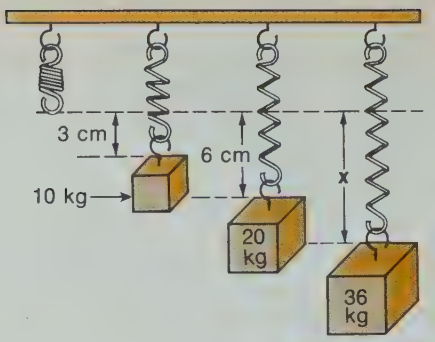
In discussing the exercises, emphasize the importance of writing the proportions correctly. If masses m_1 and m_2 stretch a spring by distances d_1 and d_2 respectively, then a proportion is $\frac{m_1}{m_2} = \frac{d_1}{d_2}$. Variations of this proportion are possible, such as $\frac{m_1}{d_1} = \frac{m_2}{d_2}$.

You may want to introduce the term "elastic limit" when discussing Exercise 4 (see Preparation section). You may demonstrate this with a rubber band. When a rubber band is stretched beyond its elastic limit, it either will break or will fail to return to its unstretched state.

Investigating the Ideas

Suppose that a 10-kilogram mass stretches a spring 3 centimetres. Then a 20-kilogram mass will stretch the spring twice as far, or 6 centimetres. The stretching of the spring is **proportional** to the masses:

that is, $\frac{3 \text{ cm}}{6 \text{ cm}} = \frac{10 \text{ kg}}{20 \text{ kg}}$.



Can you find how far a 36-kilogram mass would stretch the same spring?

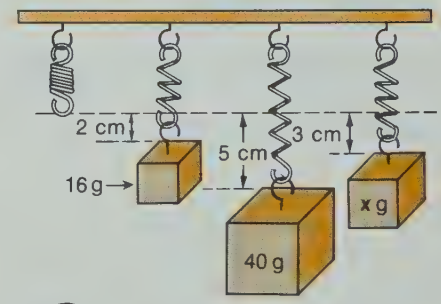
Discussing the Ideas

1. A What proportion involving distance **x** above can be written for the Investigation question?
   B Explain how the proportion can be solved for **x**.

2. A Is the ratio $\frac{2 \text{ cm}}{5 \text{ cm}}$ the same as the ratio $\frac{16 \text{ g}}{40 \text{ g}}$?

   B What proportion can you write to find the mass **x**?

   C Show how you would solve the proportion.



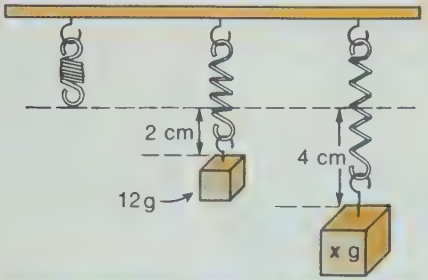
3. How is a **spring scale** like the one shown at the right related to the problems above?

4. If you were conducting an experiment with a spring and some masses, would you need to be careful about how large the masses are? Explain.

Using the Ideas

For Exercises 1 through 6, find the missing measure.

1.



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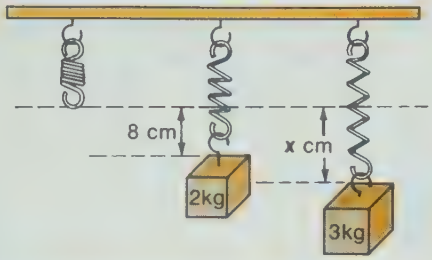

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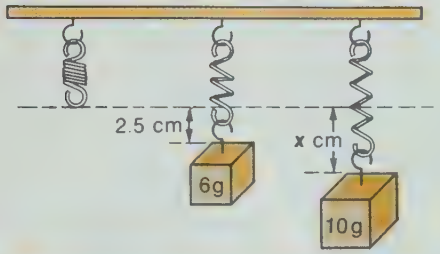

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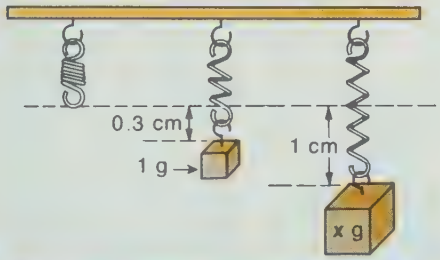
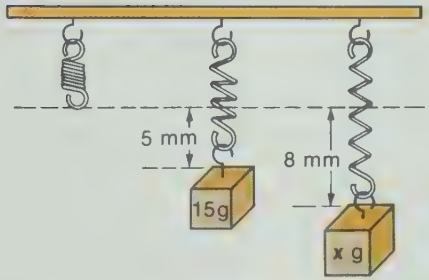
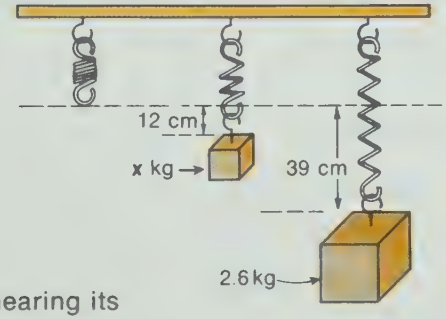


5.



6.

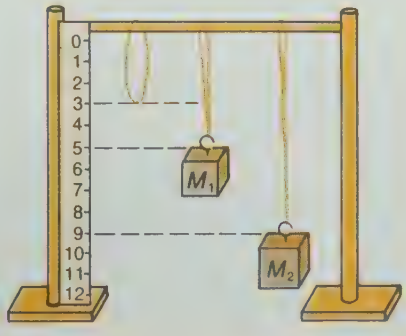


7. A certain spring will stretch 12 cm before nearing its breaking point. If a 2-kg mass stretches the spring 5 mm, how much will the spring be stretched by

   A 10 kg? C 40 kg?

   B 1 kg? D 12 kg?

★8. Two masses M_1 and M_2 were suspended on the same rubber band as shown in the diagram. How many times greater is the mass M_2 than M_1?



UTILIZATION

Exercises 1 through 6 provide pictorial representations of spring problems. The proportions for the problems can be derived from the pictures.

Solution, Exercise 8

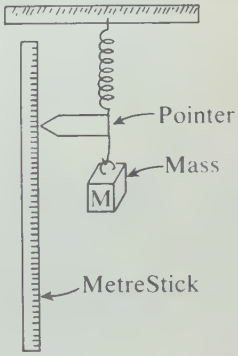
Mass M_1 stretches the rubber band 2 units while mass M_2 stretches the band 6 units. Therefore,

$$\frac{2}{6} = \frac{M_1}{M_2} = \frac{1}{3}.$$

Hence, M_2 has a mass three times that of mass M_1.

EXTENSION

A great deal of interest might be generated if some spring experiments could be set up in the classroom. The science department in your school may be able to provide the necessary equipment or you may improvise with some homemade equipment. Place a metre stick beside a spring to which a cardboard pointer has been attached. Attach a hook to the spring and load the spring with a known mass.



Record the amount of stretch of the spring. Then attach an unknown mass to the spring and note the amount of stretch. Determine the unknown mass by using a proportion.

Assignments

Minimum 1–6. Average 1–7. Maximum 1–8.

Objective

The student can solve problems concerning scale drawings.

PREPARATION

Materials: Centimetre rulers.

As preparation, you might ask your students to bring to class examples of scale drawings, such as maps, blueprints, dress patterns, or science book illustrations. Dictionaries and encyclopedias are also sources for scale drawings. Use the scale drawings brought in by your students to introduce the lesson.

INVESTIGATION

Since the scale in the Investigation drawings is very simple (1 cm = 50 m), students should encounter little difficulty in determining the actual heights of the objects represented:

Great Pyramid 140 m	Grand Dixence Dam 285 m
Gateway Arch 190 m	Eiffel Tower 320 m

Some students may be interested in verifying their results by checking the heights of these objects in encyclopedias or other reference books. If so, you may need to caution them that such figures may vary somewhat from one source to another, depending on different interpretations of what is considered the "base" and the "top" of a given structure.

DISCUSSION

Scale drawings are one of the most important uses of ratios that your students will meet in everyday life. Students will frequently need to interpret such things as maps, directions for assembling objects, blueprints for houses, and so forth.

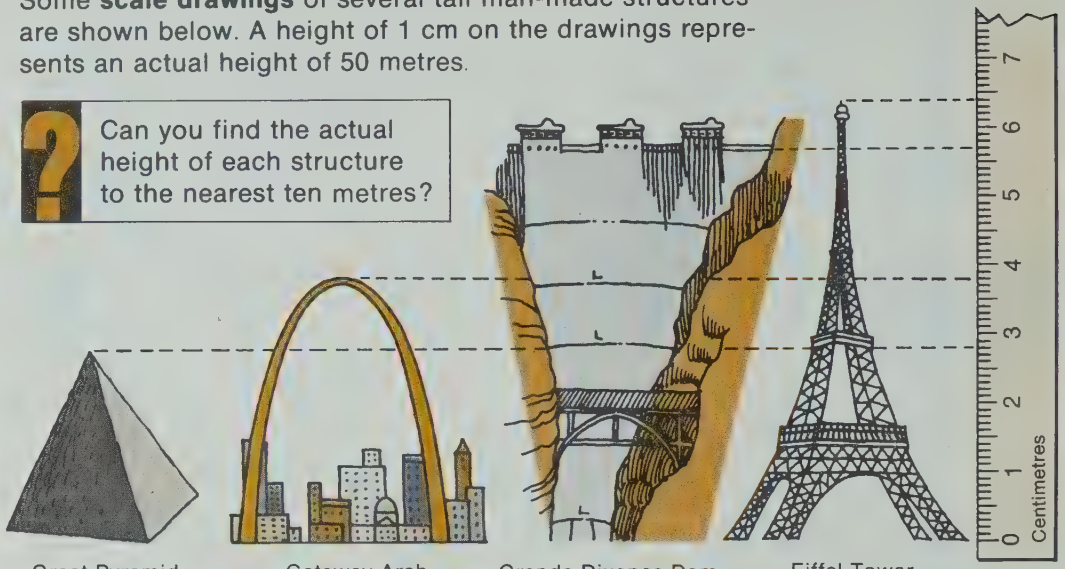
The three methods of showing a scale for a drawing in Exercise 2 are equivalent. If a scale is indicated as 1 cm = 50 m, then the ratio of the drawing to the actual object is $\frac{1\text{ cm}}{50\text{ m}} = \frac{1\text{ cm}}{5000\text{ cm}} = 1:5000$.

Scale Drawings

Investigating the Ideas

Some **scale drawings** of several tall man-made structures are shown below. A height of 1 cm on the drawings represents an actual height of 50 metres.

Can you find the actual height of each structure to the nearest ten metres?



Great Pyramid of Egypt — Gateway Arch St. Louis — Grande Dixence Dam Switzerland — Eiffel Tower Paris

Discussing the Ideas

1. Pictures of large or small objects are often drawn to scale. The scale of a drawing is the ratio of the distance measured on the drawing to the corresponding actual distance. What is the scale for the drawings in the Investigation?

2. A scale for a drawing might be indicated in several ways. Explain how each way expresses the same scale.

   A 1:5000 **B** 1 cm = 50 metres **C**

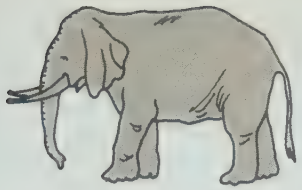
3. The drawing of the ant is four times as large as an actual ant. Is the scale for the drawing 1:4 or 4:1?

4. The drawing of the ant is about 24 mm long. Explain how to find the actual length of the ant.

5. A scale on a map is 1 cm = 80 km. What is the actual distance between cities whose distance apart on the map measures 2.8 cm?

Using the Ideas

1. Measure the height of the elephant drawing with a centimetre ruler. What is the actual height of the elephant?



SCALE: 1 cm = 1.2 m

2. Measure the length of the elephant from the tip of his tusks to his tail. Find the actual length of the elephant in metres

3. Use the scale drawing to estimate the Statue of Liberty's height in metres. The scale is 1 : 1000.

Statue of Liberty

4. A Estimate the diameter of a white blood cell in millimetres.
   B Estimate the diameter of a red blood cell in millimetres.

SCALE: 1000 to 1

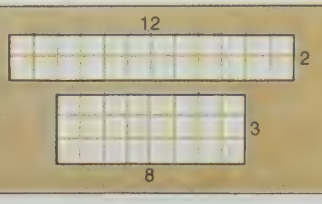
5. On a map, the distance between Paris and Rome measures 4.4 centimetres. The airline distance between these two cities is about 1100 kilometres. What is the scale for this map?

6. Suppose you drew a map of your province using the scale 1 cm = 100 km.
   A About how long would the north-south distance be on your drawing?
   B About how long would the east-west distance be?

7. A swimming pool is 50 metres long and 20 metres wide.
   A Make a scale drawing using the scale 1 : 500.
   B Make a scale drawing using the scale 1 cm = 10 m.

think

Make a copy of each rectangle on graph paper. Then show how to divide the 3 by 8 rectangle into two identical parts that will exactly cover the 2 by 12 rectangle.



More practice, page S-20, Set 37

UTILIZATION

Students will need centimetre rulers for the exercises. Measurements should be made to the nearest millimetre.

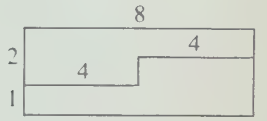
If Exercise 6 is assigned, the needed information should be provided by you or appropriate reference books should be available. If possible, bring maps of your state or province to class and let students use them to determine the distances in question.

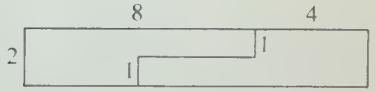
EXTENSION

Workbook page 60 and *Duplicator Masters* page 43 offer problems that would be suitable for extra practice assignments.

Enrichment: Have your students select some object such as a classroom, building, or school ground to measure and then draw to scale. Such experiences will provide work with measuring equipment and involves choosing a scale such that the drawing will fit on a sheet of paper of a given size.

Think Solution





Assignments

Minimum 1–5, 6 optional.
Average 1–7, 6 optional. Maximum 1–7.

Objective

The student can use ratios and proportions to solve problems related to electrical circuits.

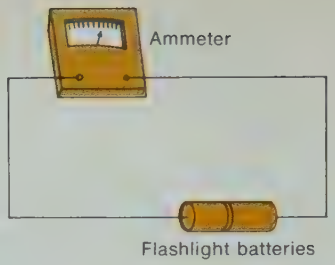
UTILIZATION

This is an optional lesson. One of its chief purposes is to stimulate interest in mathematics and its applications. If this lesson gives students more practice with ratio and proportion and, in addition, encourages a few students to explore these ideas further, then its purpose has been accomplished. Therefore, as you present this section, encourage students to develop, on their own, special projects related to the ideas introduced here. The basic idea presented is that the amount of current in a given electrical circuit is proportional to the voltage. A short exercise set is comprised of some simple problems based on this idea.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–5.

Ratios and Electricity



An ordinary dry cell (flashlight) battery produces an electromotive force E of about 1.5 volts. Two such cells connected as in the figure will produce 3 volts. An **ammeter** measures the rate of flow of current, I, in **amperes** (A). If an electromotive force of 3 volts produces a current of 5 A in a circuit, then an electromotive force of 6 volts in the same circuit will produce 10 A, 9 volts will produce 15 A, and so on. Notice that we have the proportions:

$$\frac{3}{6}=\frac{5}{10}, \qquad \frac{3}{9}=\frac{5}{15} \rightarrow \frac{E_1}{E_2}=\frac{I_1}{I_2}$$

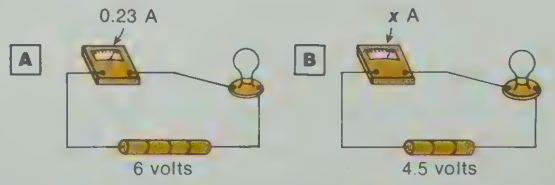
1. Hooking up one dry cell (1.5 volts) to a certain electrical circuit induces a current of 1.2 A. What will be the current using three dry cells (4.5 volts)?

2. An electromotive force of 10 volts in a certain electrical circuit induces a current of 3.2 A. Give the current induced in the same circuit by an electromotive force of

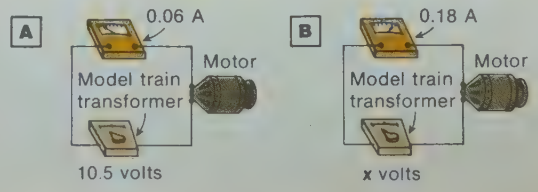
   A 5 volts **B** 20 volts **C** 16 volts **D** 4.5 volts

3. For a certain electrical circuit, when $E_1 = 3.6$, then $I_1 = 5.4$. Use the proportion $\frac{E_1}{E_2}=\frac{I_1}{I_2}$ to find the amperage I_2 induced by an electromotive force $E_2 = 9.6$ volts in the circuit.

4. Figures **A** and **B** show the same circuit but with different electromotive forces. Find the amperage **x** for figure **B**.



5. The model train transformer can vary the electromotive force which changes the current in the circuit shown. Write and solve a proportion to find the electromotive force **x**.



Expansion of Metals

It has been found in experiments that a given material expands the same amount for each degree of increase in temperature. For example, a rod of aluminum will increase about 2.4×10^{-5} cm (0.000024 cm) for each centimetre of its length when the temperature is raised 1° C. The number 2.4×10^{-5} is called the **coefficient of linear expansion**.

EXAMPLE: Find the increase in length of an aluminum rod 100 cm long when the temperature is increased 20° C. The flow chart below gives instructions for finding the increase in length.

Length of material
100 cm
Multiply by coefficient of expansion
100 × 2.4 × 10⁻⁵ = 240 × 10⁻⁵
Multiply by increase in temperature
20 × 240 × 10⁻⁵
Increase in length
4800 × 10⁻⁵ = 0.048 cm

1. A copper wire is 100 metres long at 0° C. How much does the length increase when the wire is heated to 40° C? Express your answer in centimetres.

2. At 0° C an aluminum bar is 65 cm long. How much will its length increase if it is placed in boiling water (100° C)?

3. A steel beam is 30 metres in length at 0° C. How much longer will it be (in cm) when the temperature rises to 20° C?

4. Tungsten does not expand readily. What would be the increase in length of a tungsten wire 10 cm long at 0° C if the temperature of the wire is raised to 500° C?

5. The tallest television tower in the world is in Fargo, North Dakota. It is about 628.8 metres tall. If the tower is made of steel, how many centimetres will it expand when the temperature rises from 0° C to 30° C?

Material	Coefficient of linear expansion
Aluminum	2.4×10^{-5}
Copper	1.4×10^{-5}
Gold	1.3×10^{-5}
Steel	1.1×10^{-5}
Silver	1.7×10^{-5}
Tin	2.3×10^{-5}
Tungsten	4.2×10^{-6}

Objective

The student can solve problems which apply ratio concepts to expansion of metals.

UTILIZATION

The most familiar application of the relation between temperature and expansion or contraction of materials is the thermometer. The height of the column of mercury increases a constant amount for each 1 °C rise in temperature. More generally, the change in length of 1 cm of material per 1 ° C change in temperature is called the *coefficient of linear expansion* of the material. The coefficient of linear expansion varies with different materials.

Have the students read the introductory material and study the flow chart and the accompanying example before you assign the exercises.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can express a fraction in hundredths or a decimal as a percent.

PREPARATION

Since most students will have had previous experiences with percents some of the material will be review. You should adjust the time spent in reviewing the basic ideas of percent according to the ability of your students.

INVESTIGATION

The Latin phrase *per centum* means "out of a hundred." Thus, a percent is always a ratio of a number to 100. The Investigation contains the gist of this idea in that it poses the problem of writing a given ratio with denominator 100. Students should be free to use any method they choose for this activity.

DISCUSSION

It is important not only that students be able to find percents for fractions and decimals but that they have a good intuitive understanding of the meaning of percent. For example, they should understand that 1% means 1 out of 100, 2% means 2 out of 100, 5% means 5 out of 100, and so on. This understanding will help them reason correctly about such problems as, "What is 6% of 300?" Since 6% means 6 out of each 100, 6% of 300 means 6×3, or 18.

In the example in Exercise 5, the first step produces a fraction with a denominator which is a multiple of 100. The second and third steps change the fraction to a fraction with a denominator of 100. The next step changes the fraction to a mixed decimal numeral, and the last step shows the percent.

The Language and Symbols of Percent

Investigating the Ideas

Each card below contains a symbol for a rational number.

A: 8 to 400 B: 0.76 C: $\frac{3}{5}$

D: $0.3\frac{1}{3}$ E: 7:20 F: $\frac{11}{8}$

Can you write a fraction with a denominator of 100 for each rational number shown on the cards?

Discussing the Ideas

1. Explain how you found the fraction with denominator 100 for each numeral shown on the cards.

2. Rational numbers may be represented by fractions or decimals. Another way of representing rational numbers is by **percents** (%). When we use the language of percent we are thinking of **hundredths**.

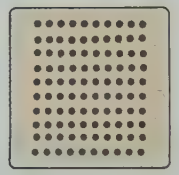
   For example, $0.25 = \frac{25}{100} = 25\%$; $2.37 = \frac{237}{100} = 237\%$; $0.33\frac{1}{3} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3}\%$

   What percent can be written for each of the fractions you found in the Investigation?

3. What percent can be written for each decimal?

   A 0.43 B 1.25 C $0.66\frac{2}{3}$ D 0.5

4. When we say, "The number of objects in set A is seven percent (7%) of the number of objects in set B," we mean that there are 7 objects in A for every 100 objects in B.

   $A =$ ⬭ $B =$ 

   A If there are 7% as many objects in set A as in set B and there are 500 objects in set B, how many objects are in set A?

   B 28 out of 400 students are absent from school. How many are absent from each 100? What percent were absent?

5. Explain the steps for representing the fraction as a percent.

   $$\frac{1}{6} = \frac{100}{600} = \frac{100 \div 6}{600 \div 6} = \frac{16\frac{2}{3}}{100} = 0.16\frac{2}{3} = 16\frac{2}{3}\%$$

Using the Ideas

1. Write each ratio as a percent.

A 19 to 100 **C** 16 to 400 **E** $\frac{1}{2}$ to 100 **G** 13 to 20

B 13 to 50 **D** 150 to 100 **F** 50 to 50 **H** 240 to 1000

2. Express each fraction as a percent. EXAMPLE: $\frac{22}{300} = \frac{22 \div 3}{300 \div 3} = \frac{7\frac{1}{3}}{100} = 7\frac{1}{3}\%$

A $\frac{15}{300}$ **C** $\frac{5}{200}$ **E** $\frac{42}{1200}$ **G** $\frac{23}{1000}$ **I** $\frac{9}{1800}$

B $\frac{24}{600}$ **D** $\frac{25}{800}$ **F** $\frac{17}{600}$ **H** $\frac{371}{1000}$ **J** $\frac{33}{4400}$

3. Copy each exercise, replacing ***a*** and ***b*** by the correct number.

A $\frac{4}{30} = \frac{a}{300} = \frac{b}{100} = b\%$ **C** $\frac{5}{12} = \frac{a}{1200} = \frac{b}{100} = b\%$

B $\frac{2}{3} = \frac{a}{300} = \frac{b}{100} = b\%$ **D** $\frac{1}{8} = \frac{a}{800} = \frac{b}{100} = b\%$

4. Carry out each indicated division to two decimal places and then express your answer to the nearest whole percent.

EXAMPLE: $\frac{31}{47} \rightarrow 31 \div 47 = 0.65\frac{45}{47} \rightarrow 65\frac{45}{47}\% \rightarrow 66\%$

A $\frac{7}{12}$ **C** $\frac{5}{6}$ **E** $\frac{24}{35}$ **G** $\frac{131}{85}$ **I** $\frac{4}{7}$

B $\frac{13}{18}$ **D** $\frac{18}{24}$ **F** $\frac{7}{111}$ **H** $\frac{7}{8}$ ★ **J** $\frac{22}{23}$

5. Each quotient is given to four decimal places. Express each fraction to the nearest tenth of a percent.

A $\frac{7}{12} = 0.5833\ldots$ **C** $\frac{45}{187} = 0.2406\ldots$ **E** $\frac{37\,000}{155\,000} = 0.2386\ldots$

B $\frac{5}{64} = 0.0781\ldots$ **D** $\frac{257}{71} = 3.6197\ldots$ **F** $\frac{4256}{654\,000} = 0.0065\ldots$

6. Carla was absent 9 days out of 180 school days during a certain year. What percent of the days was she absent?

7. In a free-throw shooting contest, Dick made 29 out of 40 free throws. Dick said, "If I make the next free throw, I will have made 75% of my shots."

A Show that Dick's statement is incorrect.

B What is the fewest number of free throws that Dick must make in a row after his first 40 shots in order to make at least 75% of his shots?

Hmk

Arrange the eight digits below so that a numeral will be formed in which the 1's are separated by one digit, the 2's by two digits, the 3's by three digits, and the 4's by four digits.

1 2 3 4 4 3 2 1

UTILIZATION

Solution, Exercise 7

A. If Dick makes the next free throw, he will have made 30 out of 41 shots. Since $\frac{30}{41} \approx 0.73$, $\frac{30}{41} < 75\%$.

B. By increasing the numerator and denominator of the fraction $\frac{29}{40}$ by 1 successively, the fraction $\frac{33}{44} = 75\%$ can be found. Hence 4 free throws in a row are needed.

Algebraically, let x = number of consecutive free throws needed.

$$\text{Then } \frac{29 + x}{40 + x} = \frac{75}{100}$$
$$100\ (29 + x) = 75\ (40 + x)$$
$$2900 + 100x = 3000 + 75x$$
$$25x = 100$$
$$x = 4$$

EXTENSION

Additional practice exercises are provided on page 61 of the *Workbook*.

Remedial: The following activity will provide an intuitive approach to percent ideas.

Provide each student with graph paper. Have the students outline a 10×10 square region. Since there are 100 small squares in the region, each small square represents 1% of the whole region. Name a percent such as 23% and then have the students shade that portion of the 10×10 square. Ask the students to name the percent that is unshaded, also.

Assignments

Minimum 1–3, 4A–F. Average 1–6.
Maximum 1–7.

Objective

The student can express decimals and fractions as percents and can express percents as decimals or fractions.

PREPARATION

The basic concept of percent and the percent symbols were introduced in the preceding lesson. Review these concepts briefly. This lesson extends the ideas of that lesson.

DISCUSSION

If your students are well prepared, you may proceed through the Discussion Exercises rapidly on an oral basis.

Exercise 4 reviews the division method for finding a percent for a fraction, that is, finding a decimal approximation for a fraction and then expressing it as a percent. Provide extra examples as needed and have students work them out at the chalkboard or at their seats. Such additional examples may be selected from Supplementary Exercise Set 28 on page S-20 or from *Arithmetic Skill Cards* D-7 through D-9 and D-11 through D-14.

Discussing the Ideas

Percent language is useful in presenting information. For computing purposes, however, it is usually easier to use decimals or fractions for percents. You should be able to express decimals and fractions as percents, and percents as decimals or fractions.

1. Study the examples for changing percents to decimals.

EXAMPLES: [A] $17\% = 0.17$ [B] $23\% = 0.23$ [C] $3.8\% = 0.038$

Change each percent to a decimal.

A 5% B 44% C 125% D 16.4% E 1%

2. Give the lowest-terms fraction for each percent.

EXAMPLES: [A] $48\% = \frac{48}{100} = \frac{12}{25}$ [B] $37.5\% = \frac{37.5}{100} = \frac{375}{1000} = \frac{3}{8}$

A 20% B 4% C 12.5% D 0.1% E 1000%

3. You should be able to find the lowest-terms fraction for each of these percents mentally. Can you do it?

A 10% C 50% E 90% G 100% I 75%

B 25% D 30% F 40% H 80% J $33\frac{1}{3}\%$

4. The example shows how to find a percent for a fraction to the nearest whole percent. What percent is $\frac{11}{23}$ to the nearest whole percent?

EXAMPLE:

$$\frac{17}{26} \rightarrow 26\overline{)17.00} \rightarrow \frac{17}{26} = 0.65\frac{10}{26}$$

$$\begin{array}{r} 0.65 \\ 26\overline{)17.00} \\ 15\,6 \\ \hline 1\,40 \\ 1\,30 \\ \hline 10 \end{array}$$

5. Round each decimal to the nearest hundredth. Then express the rounded decimal as a percent.

A 0.2199 . . . B 0.36248 . . . C 2.144 D 0.009617

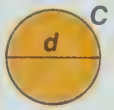
More practice, page S-20, Set 38

Using the Ideas

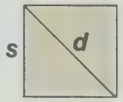
1. About 73.5% of the people in the United States live on 1.5% of the land. Express each percent as a decimal.

2. Nearly 87% of Quebec's population speak French. Express this percent as a fraction and as a decimal.

3. The diameter ***d*** of any circle is 0.3183 . . . as long as the circumference ***C*** of the circle. Express this decimal as a percent to the nearest whole percent.



4. The side ***s*** of any square is about 0.7071 as long as the diagonal ***d*** of the square. What percent of the length of the diagonal is the side of the square?



5. A baseball player's batting average is reported as 0.329. What percent of the times at bat are hits according to this decimal?

6. In some places you must pay 4¢ sales tax on each dollar of sales. What is the sales tax rate as a percent?

7. A picture of an animal in a book is $\frac{1}{50}$ the actual size of the animal. What percent of the actual size is the picture?

8. If 74 out of every 140 registered voters voted in an election, what percent of the registered voters voted, to the nearest whole percent?

9. A carton of milk holds 0.946 litres. What percent of a litre is the carton, to the nearest whole percent?

10. A student answered 39 out of 48 questions correctly. What percent were answered correctly, to the nearest whole percent?

11. The inflation rate in the United States in 1973 was 8.5%. Express this percent as a decimal.

★12. The total amount of water on Earth is estimated to be 1.3×10^9 km^3. About 2.9×10^7 km^3 of the water is in ice caps and glaciers. About what percent of the Earth's water is in ice caps and glaciers?

UTILIZATION

The exercises provide experiences with a variety of applications of percent. Allow time for discussing the exercises after students have completed them.

Solution, Exercise 12

$$\frac{2.9 \times 10^7 \text{ km}^3}{1.3 \times 10^9 \text{ km}^3} \times \frac{2.23}{10^2} = \frac{2.23}{100} \approx 2.2\%$$

EXTENSION

Workbook page 62 and *Duplicator Masters* page 44 provide further practice exercises.

Enrichment: Many students carry small pictures of themselves or their friends. Have students measure the distance from the top of the head to the chin on a picture and then measure the actual distance on the person from the top of the head to the chin. Students can then write the ratio of the two measurements and express it as a percent to describe what percent of the actual size of the person the photograph is. For example, if on the photograph the distance measured is 3.8 cm while the actual distance is 22 cm, then $\frac{3.8}{22} \approx 17\%$. Hence, the photograph is about 17% of the actual size of the person.

Assignments

Minimum 1–6. Average 1–10. Maximum 1–12.

Objective

The student can find what percent one number is of another number.

PREPARATION

Present an oral drill which involves students in naming percents for simple fractions such as $\frac{1}{2}$, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{1}{4}$, $\frac{3}{4}$, etc.

INVESTIGATION

The Investigation will introduce the student to the basic problem of this lesson: finding what percent one number is of another. The Investigation presents a variety of examples of this kind of problem.

A. 8 out of 16 $\rightarrow \frac{8}{16} = \frac{1}{2} = \frac{50}{100} = 50\%$

B. 3:8 $\rightarrow \frac{3}{8} = \frac{300}{800} = \frac{37\frac{1}{2}}{100} = 37\frac{1}{2}\%$

C. 117 to 100 $\rightarrow \frac{117}{100} = 117\%$

D. $\frac{3}{11} \approx \frac{27}{99} \approx \frac{27}{100} = 27\%$

E. 48 is ||||| % of 240 $\rightarrow \frac{48}{240} = \frac{480}{2400} = \frac{20}{100} = 20\%$

F. 4¢ out of each dollar $\rightarrow \frac{4}{100} = 4\%$

DISCUSSION

As you work through the Discussion Exercises, have the problems placed on the board and the solutions explained. It will be especially helpful for your slower students to watch some of these problems being solved.

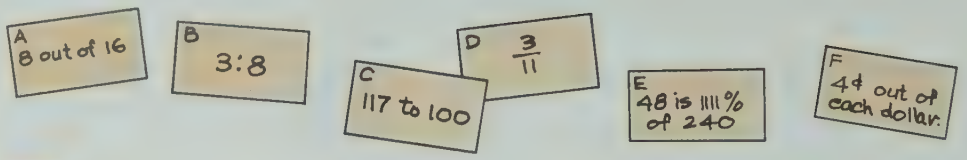
Special attention should be given to the symbol $x\%$, which is used in Exercise 3. It is important to stress that in equations we can replace this symbol with $x \cdot \frac{1}{100}$, or $\frac{x}{100}$, or $x \cdot (0.01)$.

Help students translate sentences like "48 is what percent of 240?" into the algebraic equation $48 = x\% \cdot 240$.

Finding Percents

Investigating the Ideas

Study the cards below.



Can you give a percent for the number suggested on each card?	First write the number as a fraction with denominator 100.

Discussing the Ideas

1. Explain your method for finding some of the percents.

2. Each card suggests the problem of finding what percent one number is of another number. For example, for card D, $\frac{3}{11}$ suggests finding what percent 3 is of 11.

$$\text{If } \frac{3}{11} = x\%, \text{ then } \frac{3}{11} = \frac{x}{100} \text{ (Why?)}$$
$$\text{and } 11x = 300 \text{ (Why?)}$$
$$\text{and } x = 27\frac{3}{11}$$
$$\text{Therefore, } \frac{3}{11} = 27\frac{3}{11}\%$$

Use this method to find the percent for card B.

3. Notice in the example above that $x\%$ is written $\frac{x}{100}$. We can also represent $x\%$ as $x \cdot \frac{1}{100}$ or $x \cdot 0.01$. Which numerals below represent the same number as 8%?

A $8 \cdot \frac{1}{100}$	C $\frac{80}{100}$	E $8 \cdot 0.01$	G $\frac{8}{100}$
B 0.08	D $\frac{8}{10}$	F $\frac{2}{25}$	H 0.8

4. Card E suggests the equations shown at the right.
   - A Explain each step and then solve the equation for x.
   - B Solve: 8 is what percent of 40?

$$48 = x\% \text{ of } 240$$
$$48 = x\% \cdot 240$$
$$\frac{48}{240} = x\%$$
$$\frac{48}{240} = \frac{x}{100}$$

Short Stories . . . PERCENT

Using the Ideas

1. Math test: 20 problems.
   Had 18 correct.
   What percent correct?

2. Class: 30 students.
   Today: 24 present.
   What percent present?

3. Played 10 games.
   Lost 4 games.
   What percent lost?

4. True-False quiz: 24 questions.
   Had 6 incorrect.
   What percent correct?

5. Regular price: $40.
   Sale price: $10 off.
   What percent off?

6. Made 12 out of 16 free throws.
   What percent were made?

7. Earnings: $120.
   Tax: $12.
   What percent tax?

8. Rent: $150 per month.
   Salary: $600 per month.
   What percent for rent?

9. Attendance: $\frac{4}{5}$ of students.
   What percent attended?

10. If 16 out of 24 students voted for Denise, what percent is this?

11. Played 24 games. Won 10 games.
    Lost what percent?

12. Test: 50 questions. Had 43 correct.
    What percent incorrect?

★13. 60 students. 12 wear glasses.
    $\frac{1}{3}$ of those with glasses have red hair.
    - A What percent wear glasses?
    - B What percent wearing glasses have red hair?
    - C What percent of the students wear glasses and have red hair?

think

A school club composed of 99 students planned a drawing for three prizes. Each student held a different number from 1 through 99.

Each of the three students whose numbers were drawn would receive as a prize as many free movie passes as the sum of the digits of his number. For example, if number 17 were drawn, the holder would get 8 passes.

John was absent during the drawing and the next day he asked the other members who had won. The club secretary wouldn't tell him.

"Well, then, tell me the total number of free passes you gave out," John said.

"If I do, I'll give away the numbers drawn," the secretary replied.

Then John said, "Well, never mind. You've already told me."

What were the winning numbers?

UTILIZATION

The problems in this exercise set are presented in short-story form so that reading and interpretation difficulties are minimized. Most students will benefit if some of the solutions are placed on the chalkboard and discussed.

EXTENSION

For further practice exercises, see *Workbook* page 63 and *Duplicator Masters* page 45.

Think Solution

First, consider the number of movie tickets and the numbers whose digits when added equal the number of movie tickets.

Tickets	*Numbers*
1	= 1, 10
2	= 2, 11, 20
3	= 3, 12, 21, 30
4	= 4, 13, 22, 31, 40
⋮	
15	= 69, 78, 87, 96
16	= 79, 88, 97
17	= 89, 98
18	= 99

The only ticket numbers which could give John the answer are three numbers whose sum is unique. Using the table, you might argue that the ticket numbers 1, 10, and 99 give a unique sum of digits. Note, however, that their digit sum ($1 + 1 + 18 = 20$) is not unique since it can also be obtained from the tickets 11, 12, and 78 (sum of digits: $2 + 3 + 15 = 20$). The sum 20 can further be obtained from many other groups of ticket numbers. Any other sums from 3 to 51 are also not uniquely determined.

The unique sum must, therefore, be the largest sum obtained from the theatre ticket numbers. This sum is 52, uniquely determined from the ticket numbers 89, 98, and 99 (digit sum $17 + 17 + 18 = 52$).

Assignments

Minimum 1–7. Average 1–10. Maximum 1–13.

Objective

The student can solve problems involving discount, sale price, and sales tax.

PREPARATION

Select some newspaper advertisements of sales which show price reductions in percents or fractions. Discuss the meaning of these reductions. Also discuss the local sales tax rate and how sales tax may be computed without using a table.

INVESTIGATION

Even though all of the articles pictured have prices above $50, when the discount is subtracted from the original price and the amount of sales tax is added, the total amount for the dress and the watch will be less than $50 for each. (See the table for Exercise 5.)

DISCUSSION

Discount rate is usually expressed as a percent or a fraction. The *discount* is the product of the original price and the discount rate. The *sale price* is the original price minus the discount. When discussing Exercise 4, emphasize that sales tax is computed on the sale price, not the original price.

Discounts and Sale Prices

Investigating the Ideas

Suppose a store advertises "25% off" the marked price on each article pictured. The sales tax rate is 6%.

Which of the articles shown could you buy if you had $50?

Discussing the Ideas

1. What does "25% off" the marked price mean?

2. The percent "off" is called the **discount rate**. The amount that is "marked off" is the **discount**. How can you find the sale price of this item?

Marked price →	$60
Discount rate →	× 0.25
	300
	120
Discount →	$15.00

3. Since $25\% = \frac{1}{4}$, how could you use this fraction to compute the amount of the discount?

4. Should the original price or the sale price be used in computing the sales tax?

A		B	
Original price →	$60	Sale price →	$45
Tax rate →	× 0.06	Tax rate →	× 0.06
Sales tax →	$3.60	Sales tax →	$2.70

5. Can you make a chart like this and complete it?

Article	Original price	Discount	Sale price	Sales tax (6%)	Total amount
Dress					
Suit					
Cassette deck					
Watch					

Using the Ideas

1. An $18 sweater is on sale. It is marked "25% off."
   - A What is the amount of discount?
   - B What is the sale price of the sweater?

2. Dick bought for half price a shirt which was originally priced at $11.00. He had to pay a 5% sales tax.
   - A What was the sale price of the shirt?
   - B What was the total cost of the shirt including sales tax?

3. Maria saw a pair of shoes marked $21.95. They were on sale for 10% off the marked price. Maria had only $20 with her.

   - A Did she have enough money to pay for the shoes?
   - B If the sales tax rate was 4%, would Maria have enough money?

4. A 10-speed bicycle regularly priced $125 is marked down 15% for a sale. Find the total amount needed to purchase the bicycle on sale including a sales tax of 6%.

5. Which is less expensive?
   - A A Super 174XC turntable on sale for $69.95.
   - B A Super 174XC turntable regularly priced $79.95 but on sale at 15% off of the regular price.

6. Nicole saw the same brand and style of sweater on sale in two different stores. At Erton's Department Store the price was $15 with 20% off. At Sampson's the price was $18 with $\frac{1}{3}$ off. What is the sale price of the sweater at each store?

7. The price of a $180 lounge chair was reduced 20%. Later the sale price was reduced by 10%. What was the final sale price?

8. Suppose the price of the lounge chair originally marked $180 was reduced 30%.
   - A What is the sale price of the lounge chair?
   - B How much more or less is this price than the final sale price of the lounge chair in Exercise 7?

More practice, page S-21, Set 39

UTILIZATION

The exercises present realistic applications involving the use of discount and sales tax rates. After students have completed the exercises, have selected students place their solutions on the chalkboard and explain them to the class.

EXTENSION

Additional practice assignments may be chosen from Supplementary Exercise Set 39 on page S-21, *Workbook* page 64, *Duplicator Masters* page 46, and *Arithmetic Skill Card* AP-20.

Enrichment: Have students make up problems based on advertisements involving sales tax, discounts, and sale prices. Tape the advertisement displays to a sheet of paper, then have students write related problems below the picture. Students can exchange papers and solve each other's problems.

Assignments

Minimum Odd-numbered exercises.
Average 1–6. Maximum 1–8.

Objective

The student can solve problems that involve percents either greater than 100% or less than 1%.

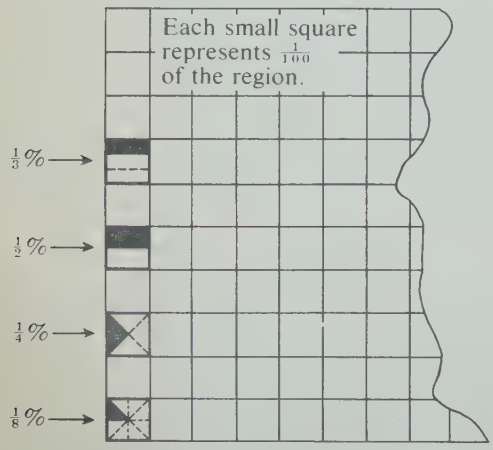
PREPARATION

Materials: Centimetre graph paper (*Duplicator Masters*, page 83).

Prepare a 10×10 grid for the overhead projector or draw such a grid on the chalkboard for use in discussing the Investigation.

INVESTIGATION

Students can use graph paper for the Investigation, or they can draw a 10×10 grid on their paper. Since responses to the Investigation question will vary, allow several students to show their illustrations of percents less than 1% on the overhead projector or chalkboard. Some sample responses are shown below.



DISCUSSION

You may need to demonstrate why $\frac{1}{2}\%$ can be written as 0.005. Thus $\frac{1}{2}\% = \frac{1}{2}$ of $1\% = \frac{1}{2} \times 0.01 = 0.5 \times 0.01 = 0.005$, or

$$\frac{1}{2}\% = \frac{\frac{1}{2}}{100} = \frac{1}{200} = 0.005.$$

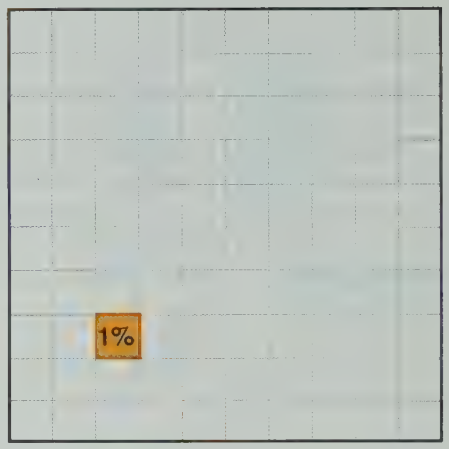
For larger percents, point out that 100% means $\frac{100}{100} = 1$, 200% means $\frac{200}{100} = 2$, 357% means $\frac{357}{100} = 3.57$, and so on.

Large and Small Percents

Investigating the Ideas

The large square contains 100 small squares. Each small square is 1% of the large square.

Draw a large square region containing 100 small squares on a piece of graph paper.



Can you shade and label three regions within your square so that each one is less than 1% of the large region?

Discussing the Ideas

1. **A** What part of the small square region is shaded?
   B What fractional percent of the large region is this?

2. Study the example. Then explain how to find the amount for each part.

> **EXAMPLE:** Find $\frac{1}{2}\%$ of \$1000.
> **SOLUTION:** $\frac{1}{2}\% = 0.005$
> $\$1000 \times 0.005 = \$5.00$

   A $\frac{1}{2}\%$ of \$840 **B** $\frac{1}{4}\%$ of \$70 000 **C** 0.1% of \$10 000

3. If you wanted to cut out a region from a sheet of graph paper that was 250% the size of the large square region shown in the Investigation, how many of the small squares should your new region contain?

4. You can write decimals for percents greater than 100%.

$$250\% = \frac{250}{100} = 2.50$$

   What decimal represents each of these percents?

   A 300% **B** 120% **C** 1000% **D** 105%

Using the Ideas

1. Write a decimal for each percent.

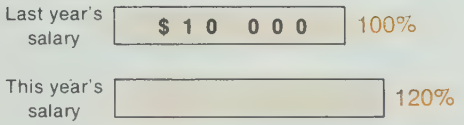
A $\frac{1}{2}\%$	**D** 800%	**G** $\frac{1}{100}\%$	**J** 0.2%
B 274%	**E** 0.7%	**H** 210%	**K** 0.05%
C $\frac{3}{10}\%$	**F** $\frac{1}{4}\%$	**I** 350%	**L** $\frac{1}{8}\%$

2. A piece of material is guaranteed not to shrink more than $\frac{7}{10}\%$ when washed. What is the greatest amount of shrinkage expected from a piece one metre long?

3. An aluminum rod 500 centimetres long is found to expand 0.12% of its length when heated. How many centimetres does the rod expand?

4. Fredrick read that the cost of living today is 160% of what it was ten years ago. Is this *160 times*, *16 times*, or *1.6 times* the cost of living of ten years ago?

5. Joan earned $10 000 last year. This year Joan earned 120% of last year's salary. How much did she earn this year?



6. The population of Sunnyvale in 1960 was 53 000. In 1970, it was 180% of that. What was the population in 1970?

7. Ralph borrowed $1200 to buy an automobile. He had to pay back 109% of what he had borrowed. How much did he have to pay back?

8. Sometimes you may hear this expression: "There is only one chance in a million!" Express this ratio as a percent.

More practice, page S-21, Set 40

UTILIZATION

Exercises 2 through 7 involve applications of both large and small percents. Students can place their solutions on the chalkboard or overhead projector and explain them to the class. Slower students will especially benefit from these explanations.

EXTENSION

See Supplementary Exercise Set 40 on page S-21 and *Workbook* page 65 for further practice exercises.

Remedial: Have students write each percent as a decimal.

1. $\frac{1}{10}\%$	2. $\frac{3}{4}\%$	3. 0.6%
4. $\frac{4}{5}\%$	5. 0.02%	6. $\frac{1}{8}\%$
7. 160%	8. 1000%	9. 275%
10. 101%	11. 500%	12. 10 000%

Have students express these decimals or fractions as percents.

13. 0.002	14. $\frac{1}{200}$	15. 6.0
16. 0.0006	17. $\frac{1}{1000}$	18. 20

Answers:

1. 0.001	2. 0.0075	3. 0.006
4. 0.008	5. 0.0002	6. 0.00125
7. 1.60	8. 10.0	9. 2.75
10. 1.01	11. 5.00	12. 100.00
13. 0.2%	11. $\frac{1}{2}\%$	15. 600%
16. 0.06%	17. $\frac{1}{10}\%$	18. 2000%

Assignments

Minimum 1–5. Average 1–7. Maximum 1–8.

Objective

The student can solve problems involving percent of increase or decrease.

PREPARATION

Materials: Metric tape measures (*Duplicator Masters*, page 88) or metre sticks and string.

INVESTIGATION

If metric measuring tapes are not available, supply students with pieces of string about one metre long. The string can be wrapped around a student's chest and then laid along a metre stick to find the chest measurement.

As an alternative Investigation, you might have students measure their biceps, first with the arm outstretched and the muscle relaxed, then with the elbow bent and fist clenched so that the muscle is flexed. The increase in size from one measure to the other could then be determined.

DISCUSSION

Work through Discussion Exercises 1 and 2 with the class, having each student use his or her own measurements. If you prefer, you might suggest that students round all measurements to the nearest whole centimetre. This will enable students to concentrate on the main idea of the lesson by freeing them from the burden of unnecessarily awkward ratios and harder computations.

In discussing Exercise 3, note that the percent of decrease is not the same as the percent of increase because the denominators of the two ratios are different; the percent of decrease is smaller because the change in measurement is compared to a larger number.

Percent of Increase or Decrease

Investigating the Ideas

Carry out this measurement activity:

1. Exhale. Then find your chest measurement in centimetres.
2. Inhale as much as possible. Measure your chest again.

About how much does your chest measurement increase when you inhale?

Discussing the Ideas

1. When a measurement increases, it is often convenient to compare the increase to the original measurement by means of a percent. This is called the **percent of increase** in the measurement.

   A What was your chest measurement when you exhaled?

   B How many centimetres did it increase when you inhaled?

   C Find the ratio: $\frac{\text{Increase}}{\text{Exhaled measurement}}$

   D Express the ratio as a percent to the nearest tenth.

Exhaled measurement:	80 cm
Inhaled measurement:	88 cm
Increase:	8 cm
Percent of increase:	$\frac{8}{80} = \frac{1}{10} = 10\%$

2. If a measurement decreases, you can find the **percent of decrease** as follows:

   A How much did your chest measurement decrease when you exhaled?

   B Find the ratio: $\frac{\text{Decrease}}{\text{Inhaled measurement}}$

   C Express the ratio as a percent to the nearest tenth.

Inhaled measurement:	88 cm
Exhaled measurement:	80 cm
Decrease:	8 cm
Percent of decrease:	$\frac{8}{88} = \frac{1}{11} \approx 9\%$

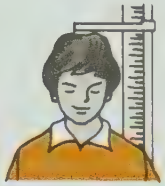
3. Can you explain why the percent of increase (Exercise 1**D**) is slightly different from the percent of decrease (Exercise 2**C**)?

Using the Ideas

1. Copy and complete the table.

	Chest Measurement (cm) Exhaled	Inhaled	Increase	Increase / Exhaled measurement	Percent of increase (nearest tenth)
	89	97	8	$\frac{8}{89}$	$\frac{8}{89} \approx 9\%$
A	84	90			
B	78	86			
C	80	89			
D	70	75			

2. Margaret was 140 centimetres tall on her 13th birthday. On her 14th birthday, she was 146 centimetres tall.



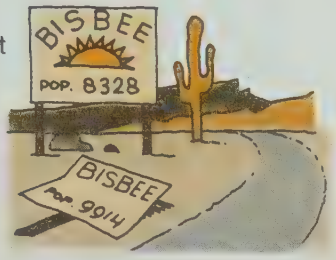
- A What was her increase in height?
- B What was her percent of increase in height to the nearest tenth of a percent?

3. On July 12, Mr. Marsh weighed 142 kilograms. Then he went on a diet and on December 15, he weighed 110 kilograms.

- A How much had his weight decreased?
- B What percent had his weight decreased to the nearest tenth?

4. Linda earned $122 per week. She got a raise and now earns $130 per week. What percent raise did she get?

5. In 1960, the population of a B.C. town was 12 230. By 1970, the population had increased to 19 366. What was the percent of increase in population to the nearest whole percent?



6. In 1960, Bisbee, Arizona, had a population of 9914. By 1970, the population had decreased to 8328. What was the percent of decrease in the population?

UTILIZATION

Most students will benefit from a discussion of the exercises. Have selected students place their solutions on the chalkboard and explain the problem to the rest of the class.

EXTENSION

Additional suitable practice exercises are provided on page 47 of the *Duplicator Masters*.

Enrichment: Pairs of students can work together on this activity which involves finding percent of increase. The students check their pulse rate per minute while sitting still. Then they perform 1 minute of quick exercises such as running in place, jumping jacks, or deep-knee-bends. Then they check their pulse rate again. The pulse rate increases with exercise. The students then can compute the percent of increase in pulse rate. For example:

Pulse rate sitting still – 72 beats per minute
Pulse rate after exercise – 90 beats per minute
Increase – 18 beats per minute
% of increase – $\frac{18}{72} = \frac{1}{4} = 25\%$

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can demonstrate the ability to work with the major concepts and skills developed in this module.

UTILIZATION

Encourage your students to ask questions about those exercises with which they have difficulty. Students can display solutions and discuss their methods of solution for some of the problems.

Workbook page 66 provides further exercises suitable for use in reviewing the ideas presented in this module.

REVIEWING THE IDEAS

1. Give three ratios equivalent to 2:5.

2. Give the simplest-terms ratio for each.

A 7:28 B $1\frac{1}{2}$ to 4 C $\frac{1}{2}$ to $\frac{3}{4}$

3. What numbers are the means in the proportion $6:16 = 9:24$?

4. Write a proportion which has 4 and 9 as the means and 3 and 12 as the extremes.

5. Complete: In any proportion, the product of the means equals ? ? ? ? ?.

6. Solve each proportion for x.

A $\frac{4}{6} = \frac{x}{12}$ D $\frac{9}{15} = \frac{15}{x}$

B $\frac{x}{3} = \frac{10}{6}$ E $\frac{6}{75} = \frac{x}{100}$

C $\frac{4}{9} = \frac{x}{36}$ F $\frac{x}{30} = \frac{7}{35}$

7. A distance of 2 cm on a map represents 120 km. What distance is represented by 1 cm on the map?

8. An enlargement of a photograph of an insect shows the insect 100 times its actual size. If an eye of the insect in the photograph measures 9 mm, what is the actual size of the eye?

9. A mass of 400 grams stretches a spring 3.2 cm. How far would a mass of 500 grams stretch the spring?

10. Which is the ratio of 36 mm to 10 cm?

A $\frac{36}{10}$ B $\frac{10}{36}$ C $\frac{9}{25}$ D $\frac{18}{5}$

11. Express 18 km/h in terms of metres per second.

12. Express each ratio as a percent.

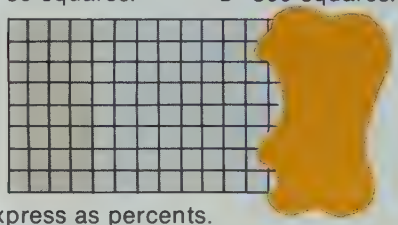
A 11 to 100 C 3:25

B 4:50 D 8 to 200

13. None of the covered squares in the figure is shaded. Tell what percent of the squares are shaded if each row contains

A 100 squares. C 20 squares.

B 50 squares. D 800 squares.



14. Express as percents.

A $\frac{3}{5}$ C $\frac{2}{3}$ E 0.002

B 0.17 D 1.25 F $\frac{9}{4}$

15. Barbara got 17 out of 20 problems correct. What percent of her problems were correct?

16. A sport coat regularly priced $55 is on sale for 20% off. What is the sale price?

17. A province has a sales tax rate of 5%. How much tax must be paid on a purchase of $19.98?

18. 1960 population: 4 million.
1970 population: 5 million.
What is the percent of increase?

19. If you bought a car for $800 and later sold it for 120% of what you paid for it, what was the selling price?

20. In a school of 400 students only 12 were absent one day. What percent were present?

TEST YOURSELF

1. What ratio compares the number of elements in set A with the number of elements in set B?

   A: {□ □ □ □ . . .}
   ↕ ↕ ↕ ↕
   B: {○○ ○○ ○○ ○○ . . .}

2. What is the simplest-terms ratio for 12:20?

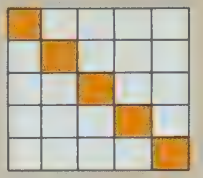
3. Solve the proportions.
   A $\frac{3}{8} = \frac{12}{x}$ B $\frac{x}{9} = \frac{4}{12}$

4. How many kilometres per hour is 5 m/s?

5. Which is the ratio of 1 mm to 1 dm?
   A $\frac{1}{10}$ B $\frac{1}{100}$ C $\frac{1}{1000}$ D $\frac{1}{10\,000}$

6. A scale drawing for a room has the scale 1 cm = 2.5 m. The length of the room shown in the drawing is 3.2 cm. What is the actual length of the room?

7. What percent of the squares are shaded?



8. Express each symbol as a percent.
   A 6:10 B 0.37 C $\frac{5}{4}$

9. A Express $12\frac{1}{2}\%$ as a fraction.
   B Express 0.3% as a decimal.

10. Solve these short story problems.
    A Regular price: $85.
    Sale price: 20% off.
    What is the sale price?
    B 1965 population: 450 000.
    1970 population: 1.5% less.
    What is the 1970 population?

RESEARCH PROJECTS

A Look up the topic "The Golden Section" in a mathematics reference book. Illustrate this famous ratio with posters or drawings. (See *Mathematical Snapshots* by H. Steinhaus; New York: Oxford University Press, 1969, pp. 45–51.)

B Follow the Dow Jones Industrial Average in a newspaper for a week, noting its increases and decreases. Make a chart showing each day's change. Express the increase or decrease in terms of percent for each day.

C Many people have possessed extraordinary powers of mental calculation. Find out about some of the calculating prodigies and write a report on them. (See *Mathematical Recreations and Essays* by W. Rouse Ball; New York; Macmillan, 1960, pp. 350–378.)

TEST YOURSELF

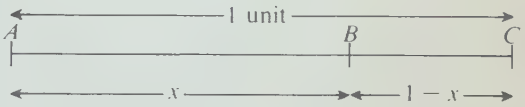
This self-evaluation test will give students a measure of their understanding of the major concepts of the module as set forth in the objectives listed at the beginning of the module. Note that the related objectives are annotated beside the individual test items on the student test page at the left.

Encourage students to use the answer key on the next page to check their own work and to review the lesson pertinent to any test items which may have proved troublesome.

RESEARCH PROJECTS

The "Golden Section," the subject of Research Project A, is a ratio that can be represented as follows:

Let $\overline{AC}$ be a segment of unit length and let B be a point between A and C such that $\ell(\overline{AB}) = x$.



If point B is located so that

$$\frac{\ell(\overline{AC})}{\ell(\overline{AB})} = \frac{\ell(\overline{AB})}{\ell(\overline{AC})} \text{ or } \frac{1}{x} = \frac{x}{(1-x)}$$

then point B divides $\overline{AC}$ so that either ratio is the "golden ratio." Thus the golden ratio is

$$\frac{1}{x} = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Research Project B could be extended into a long-range project. Students might begin with a specific amount of "imaginary" money which they invest in stocks. Then, after the stocks are "sold," they could compute the loss or gain in the investment. Buying and selling commissions might also be considered.

Students of all ability levels might be intrigued by the information they will find if they choose to undertake Research Project C.

Objective

The student can demonstrate the ability to work with the significant concepts and skills developed in this unit.

UTILIZATION

Since there are a large number of exercises in the Cumulative Review, you can assign them on a selective basis according to the amount of review necessary for your students. The exercises should provide adequate review in preparation for a unit achievement test.

Exercises 1–14 are related to the topics presented in Module 1 of this unit; Exercises 15–25, to Module 2; and Exercises 26–36, to Module 3.

CUMULATIVE REVIEW

1. What part of the rectangle is shaded?

2. Name five fractions equivalent to $\frac{5}{6}$.

3. A What is the opposite of $-\frac{2}{3}$?

B What is the opposite of $\frac{5}{4}$?

4. Give the lowest-terms fraction.

A $\frac{63}{81}$ B $\frac{2^3}{2^4}$ C $\frac{110}{345}$

5. Solve the equations.

A $\frac{3}{n} = \frac{18}{12}$ C $\frac{9}{10} = \frac{m}{90}$

B $\frac{r}{48} = \frac{3}{16}$ D $\frac{1}{8} = \frac{8}{v}$

6. Copy each problem and give the correct symbol ($<$, $>$, or $=$) for each ◍.

A $\frac{3}{4}$ ◍ $\frac{7}{8}$ C $\frac{5}{2}$ ◍ $\frac{8}{3}$

B $\frac{6}{48}$ ◍ $\frac{7}{56}$ D $\frac{11}{16}$ ◍ $\frac{11}{17}$

7. Compute.

A $8\frac{1}{3} + 6\frac{2}{5}$ C $2\frac{1}{7} + 3\frac{3}{4}$

B $8\frac{1}{3} - 6\frac{2}{5}$ D $\frac{7}{8} - \frac{1}{9}$

8. Give the principle illustrated in each problem.

A $\frac{2}{3} + \frac{4}{7} = \frac{4}{7} + \frac{2}{3}$

B $\left(\frac{2}{3} + \frac{4}{7}\right) + \frac{3}{4} = \frac{3}{4} + \left(\frac{2}{3} + \frac{4}{7}\right)$

C $\frac{3}{5} \cdot \left(\frac{2}{3} + \frac{1}{3}\right) = \frac{3}{5}$

D $\frac{4}{5} \cdot \left(\frac{5}{9} + \frac{10}{11}\right) = \left(\frac{4}{5} \cdot \frac{5}{9}\right) + \left(\frac{4}{5} \cdot \frac{10}{11}\right)$

9. Compute.

A $\frac{7}{9} \cdot \frac{3}{8}$ B $\frac{7}{9} \div \frac{3}{8}$ C $\frac{16}{47} \div \frac{16}{47}$

10. Give the reciprocal of each number.

A $\frac{2}{3}$ B 4 C $-\frac{1}{9}$

D Does zero have a reciprocal? Explain.

11. Find the sum or difference.

A $\frac{3}{4} + -\frac{2}{3}$ C $1 - -\frac{2}{3}$

B $-\frac{1}{8} + -\frac{1}{2}$ D $-\frac{4}{5} - \frac{1}{8}$

12. Find the product or quotient.

A $\frac{1}{3} \cdot -\frac{3}{4}$ C $-\frac{6}{7} \div -\frac{14}{15}$

B $-\frac{1}{5} \cdot -\frac{3}{2}$ D $-\frac{3}{8} \div \frac{11}{16}$

13. A music store is selling some LP records at $\frac{1}{3}$ off the regular price. What is the sale price of a $4.98 record?

14. Mr. Wood bought a house costing $26 000. He paid $\frac{1}{5}$ of the cost as a down payment. How much was the down payment?

15. Express each fraction as a decimal.

A $\frac{47}{100}$ C $\frac{2}{5}$ E $\frac{1}{6}$

B $\frac{8}{25}$ D $\frac{3}{8}$ F $\frac{2}{7}$

16. Express each decimal as a fraction in simplest terms.

A 0.08 C 0.33 E 16.11

B 0.3 D 2.4 F 8.35

17. Find n.

A $0.2 + 73.7 + 4 + 0.006 = n$

B $35 + 0.421 + 0.3 + 8.1 = n$

18. Write each number in scientific notation.

A 0.0073

B 58 000

C 0.000082

D 98 000 000 000

19. Solve the equations.

A $4.76 - 3.128 = d$ C $3.2 + a = 9.6$

B $3 - 0.1962 = v$ D $s + 2.2 = 6.1$

20. Find the products.

A $10 \cdot 7.1$ D $0.1 \cdot 2.17$

B $100 \cdot 0.082$ E $0.01 \cdot 639$

C $100 \cdot 2.16$ F $0.001 \cdot 55.4$

21. Find the products.

A $6.2 \cdot 14$ C $3.4 \cdot 6.17$

B $0.014 \cdot 395$ D $3 \cdot 0.0072$

22. Find the quotients.

A $81.4 \div 10$ C $96 \div 0.01$

B $6.28 \div 0.1$ D $2.83 \div 100$

23. Find the quotients.

A $38.46 \div 0.002$ C $10.26 \div 0.38$

B $783.123 \div 0.3$ D $3.9545 \div 55$

24. Round each number to one significant digit. Then estimate the product.

A $3.9 \cdot 7.2$ C $624 \cdot 0.1934$

B $0.301 \cdot 5.88$ D $88\,947 \cdot 0.29$

25. Round each number to one significant digit. Then estimate the quotient.

A $0.0842 \div 0.0217$

B $17.92 \div 3.14$

C $5.36 \div 0.00984$

26. What is the simplest ratio that compares the number of circles to the number of triangles?

27. Solve each proportion.

A $\frac{7}{x} = \frac{14}{32}$ C $\frac{x}{11} = \frac{28}{44}$ E $\frac{6}{x} = \frac{24}{32}$

B $\frac{2}{3} = \frac{x}{27}$ D $\frac{1}{8} = \frac{8}{x}$ F $\frac{3}{x} = \frac{15}{45}$

28. Which is the larger of the two ratios, 3 to 9 or 5 to 16?

29. Express each number as a percent.

A $\frac{3}{4}$ C $\frac{33}{100}$ E $\frac{11}{10}$

B $\frac{7}{10}$ D 0.53 F $\frac{4}{25}$

30. Ellen got 21 problems correct on a test of 25 problems. What percent did she have correct?

31. Virginia was 150 cm tall. During the summer, she grew 3 cm. By what percent did her height increase?

32. A dress was originally marked \$14.70. If there was a discount of $33\frac{1}{3}\%$, what was the sale price of the dress?

33. A province has a 6% sales tax rate.

A How much tax would be charged on a suit costing \$75.95?

B How much would the suit cost including the sales tax?

34. What percent of 98 is 7 to the nearest tenth of a percent?

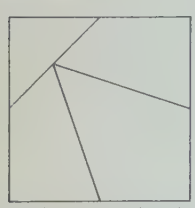
35. Sam sold a used car for \$600. He made a profit of 20%. How much did he originally pay for the car?

36. What is 0.4% of 15?

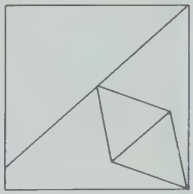
MATHEMATICAL RECREATION

You might encourage some students to make more durable models of these puzzles by using their tracings as patterns and cutting the pieces from posterboard or thin pieces of wood.

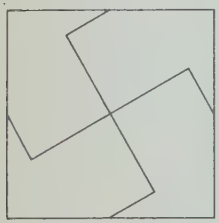
Solutions



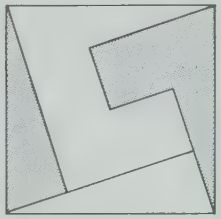
Equilateral Triangle Pieces



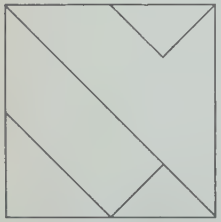
Star of David Pieces



Greek Cross Pieces



Letter E Pieces (The shaded regions are flipped.)

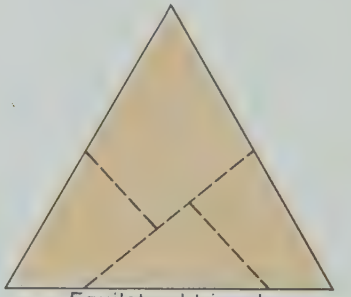


Letter F Pieces

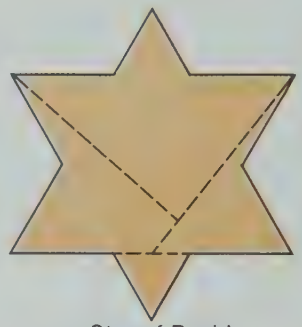
MATHEMATICAL RECREATIONS

DISSECTION PUZZLES

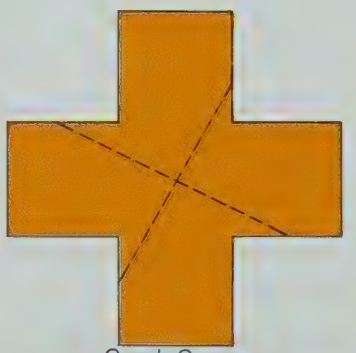
The pieces in each region can be arranged to make a square region which has the same area as the original region. Trace each region, cut it out, and cut it into the pieces along the dashed lines. Can you make a square from each set of pieces?



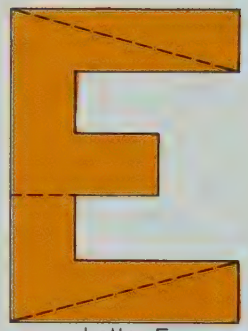
Equilateral triangle



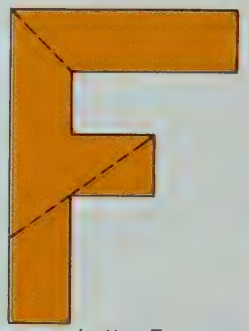
Star of David



Greek Cross



Letter E



Letter F

RECREATIONS MATHEMATICAL

TEST YOURSELF Answers

1. 1 to 2 **2.** 3:5 **3.** **A** 32; **B** 3 **4.** 18 km/h

5. **B** **6.** 8 m **7.** 20% **8.** **A** 60%; **B** 37%; **C** 125%

9. **A** $\frac{1}{8}$; **B** 0.003 **10.** **A** $68; **B** 443 250

UNIT G: GEOMETRY

Module 1: Rigid Motions and Congruence

General Objectives

To develop students' understanding of translations, rotations, and reflections in geometry.

To develop congruence concepts using the properties of the rigid motions.

To focus attention on the basic congruence theorems (SSS, SAS, and ASA) for triangles.

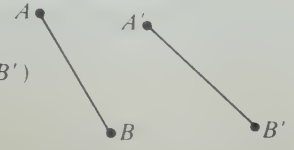
To review ideas of perpendicularity and parallelism of lines and planes.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
73	**RED** Given a geometric figure in a co-ordinate plane, the student can find a specified translation image of the figure.	G-2, 3 G-4, 5	**WB** 67 **DM** 48	**SWM 3** 164-165
74	**RED** Given a geometric figure in a co-ordinate plane, the student can identify or find a specified rotation image of the figure.	G-6, 7	**WB** 68 **DM** 49	**SWM 3** 160-161
75	**RED** Given a geometric figure in a co-ordinate plane, the student can find the reflection image of the figure as well as identify lines of symmetry in the figure.	G-8, 9	**WB** 69 **DM** 50	**SWM 3** 156-159
76	**RED** Given two congruent figures, the student can identify corresponding parts of the figures.	G-12, 13 G-14, 15	**WB** 70 **DM** 51	**PD** b-2
77	**RED** Given appropriate examples, the student can identify parallel and perpendicular lines and planes, and prisms.	G-22, 23 G-24, 25	**WB** 74, 75 **DM** 53	
78	**YELLOW** Given appropriate examples, the student can identify and apply the S-S-S, S-A-S, and A-S-A triangle congruence theorems.	G-16 through 21	**WB** 71-3 **DM** 52	
79	***GREEN*** *Given a geometric figure in a coordinate plane, the student can find its image after a combination of rigid motions.*	G-10, 11		
	Reviewing the Ideas	G-26	**WB** 76	

MATHEMATICS

The three rigid motions studied in this module are sometimes called *isometries* because each motion is a "distance preserving" function. If M denotes one of the rigid motions, we can think of M as a function whose domain is the set of points in the real plane. Then if A and B are any two points in the plane and their corresponding images for the rigid motion are A' and B' respectively, we must have

$M(A) = A'$
$M(B) = B'$
and $\ell(AB) = \ell(A'B')$



Three rigid motions are studied in the module: *translations*, *rotations*, and *reflections*. A complete study of the isometries would include a fourth motion, called a glide-reflection, which is a combination of a translation and a reflection. These four rigid motions are closed with respect to composition (combination). That is, a combination of any two of the motions is equivalent to one of the four motions. In a formal course of transformational geometry, it can be shown that any rigid motion is equivalent to a combination of no more than three line reflections. The rigid motions are used to develop the ideas of congruence. The basic definition is as follows:

> Two geometric figures are congruent if and only if there is a rigid motion which maps one of the figures onto the other.

This approach to congruence is quite natural since students have previously observed that the images of geometric figures have the same size and shape as their pre-images under any of the rigid motions. Furthermore, the necessary correspondence between the points of a figure and its image leads directly to identifying corresponding parts of congruent figures.

In this module the student is also introduced to the three congruence theorems for triangles (Side-Side-Side, Side-Angle-Side, and Angle-Side-Angle). While formal proofs of these

theorems are omitted, investigations and activities to make the theorems plausible are presented.

The module concludes with a review of parallel lines and planes, and perpendicular lines and planes.

TEACHING THE MODULE

Materials

Graph paper, scissors, tracing paper, protractors, rectangular pieces of clear plastic or commercial clear mirrors, rulers, compasses, cellophane tape, geometric models of cubes and prisms.

Vocabulary

Angle-Side-Angle Theorem (ASA)
arrow
congruent
congruent triangles
image
inverse
isosceles triangle
lateral face
line of symmetry
parallel
parallelepiped
perpendicular
prism
rectangular parallelepiped
reflection
reflection image
right angle
rigid motion
rotation
rotation image
Side-Angle-Side Theorem (SAS)
Side-Side-Side Theorem (SSS)
slide
symmetry
theorem
translation
translation image
transversal

Students who have studied from the immediately preceding book in this series have been introduced to the idea of translations, rotations, and reflections (see Unit C, Module 3). However, little background is necessary for successful work in this module. If your students have not had previous experience with the rigid motions, you will want to proceed deliberately and allow for considerable experimentation to enable students to develop their intuitive ideas about these motions. Formal definitions, terminology, and notation have been kept to a minimum in order that students may focus their attention on the motions.

Rigid motions are the tools used to develop notions of congruence. The main exphasis in this module is upon congruent triangles. Students are expected to be able to recognize a pair of congruent triangles given any of these three conditions:

1. Three sides of one triangle congruent to three sides of another triangle (Side-Side-Side Theorem).
2. Two sides and the included angle of one triangle congruent to two sides and the included angle of another triangle (Side-Angle-Side Theorem).
3. Two angles and the included side of one triangle congruent to two angles and the included side of another triangle (Angle-Side-Angle Theorem).

The rigid motions are also related to perpendicular and parallel lines. If two lines are parallel, then each is a translation image of the other. If two lines are perpendicular, then one line is reflected onto itself by a reflection in the other line.

Lesson Schedule

You will probably need to allow 10 to 12 days for complete coverage of the module. However, your schedule should be flexible enough to allow more time if you need to move more slowly with students who have had little or no experience with the rigid motions.

Evaluation

There is not a large body of facts and skills that students should master in the study of the content of this module. Rather, there should be an assimilation of ideas and concepts.Therefore, your day-to-day contact with your students will be an essential part of your evaluation of their progress.

Use the module review questions and the self-evaluation test (Test Yourself) as a guide in deciding what points should be emphasized and as an aid in developing any evaluative devices.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 3 (optics), pp. 24, 25, 61, 88; *Aftermath 4*, pp. 41, 72, Creative Publications

Developmental Math Cards, L9, Addison-Wesley

ESS: Mirror Cards; Tangrams, Webster, McGraw-Hill

Experiments in Mathematics, Stage 1, "Optical illusions," p. 52; Stage 2, p. 58, Houghton Mifflin

Geoboard Activity Cards (triangles, polygons, congruence, transformations), Geocards 61–72, 110–115, 137–140, Cuisenaire Co.

Geoboards and Motion Geometry, Scott, Foresman

Mira Activities for Junior High, Activities 21–26, 37–44, 63–68, 101–138, Creative Publications

Nuffield Project: *Angles, Courses, Bearings; Shape and Size 4*, pp. 23–25, 33–44; *Symmetry; Problems*—Red Set, Nos. 19–21B, Wiley

For map-coloring problems, see Resources for Active Learning, Unit H, Module 4

Manipulative Devices

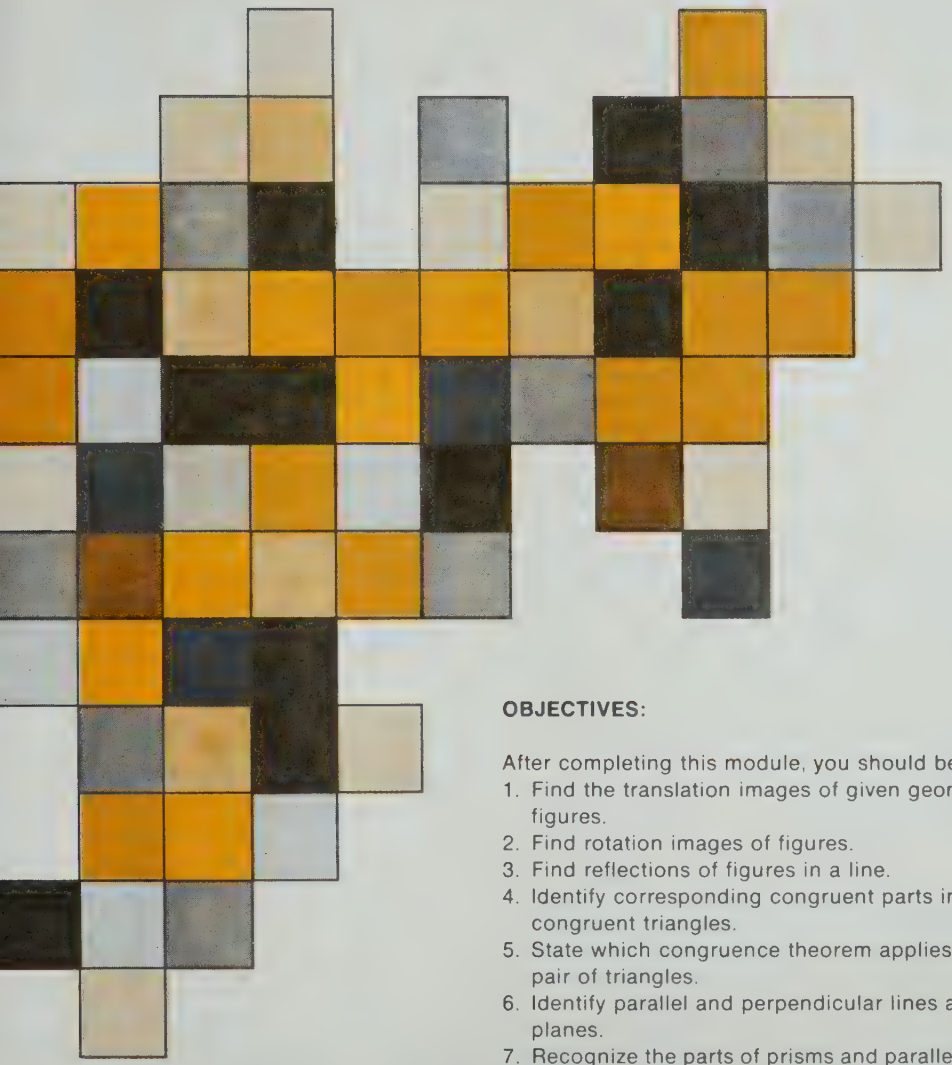
Circle Master Compass (Creative Publications)
Co-ordinate Space Model (Sargent-Welch)
Geoboards (Addison-Wesley)
Geometer (Academic Industries)
Geometric Figures and Solids (Creative Publications; Geyer; LaPine; Sigma)
Mira (Creative Publications; Cuisenaire Co.)
Prisms (Edmund Scientific)
Protractors (Geyer; Selective Educational Equipment; Sigma)
Sage Kit (LaPine; Sigma)

Games and Puzzles

Euclid (Midwest Publications)
Perception Games (Edmund Scientific; Selective Educational Equipment)

UNIT G: *Geometry*

MODULE 1: *Rigid Motions and Congruence*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the translation images of given geometric figures.
2. Find rotation images of figures.
3. Find reflections of figures in a line.
4. Identify corresponding congruent parts in pairs of congruent triangles.
5. State which congruence theorem applies for a pair of triangles.
6. Identify parallel and perpendicular lines and planes.
7. Recognize the parts of prisms and parallelepipeds.

Objective

Given a geometric region in a plane and a specified translation, the student can show the translation image.

PREPARATION

Materials: Scissors, graph paper (*Duplicator Masters*, page 83).

Students who have studied earlier books in this series may have had previous experience with translations; however, this introduction assumes little previous student experience.

INVESTIGATION

The Investigation provides concrete experience with the idea of a translation. The triangular cut-out can be slid, without turning, the same distance and direction as the arrow from A to A'. This can be demonstrated effectively on the overhead projector.

DISCUSSION

Exercise 1 introduces the terms *motion*, *translation* (or *slide*), and *translation image*. The phrase "maps A onto A'" means the same thing as "the image of point A is point A'." This special language refers to the function (or mapping) that pairs each point of the original figure with the corresponding points of the translation image.

The arrow notation in $A \rightarrow A'$ should not be confused with the translation arrows. The arrow in this notation is simply an alternate way of denoting functions. A translation could be thought of as a function t such that $t(A) = A'$.

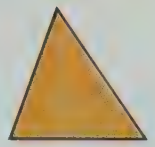
When discussing Exercise 2, have some students show their translations on a grid on the overhead projector or on the chalkboard. Other students can use graph paper and work at their desks.

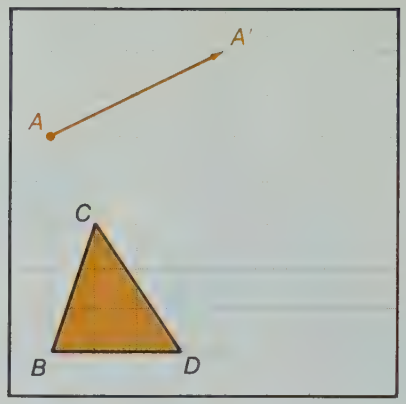
In Exercise 3, observe that the translation image of a given line is another line parallel to the original line unless the translation happens to be along the line itself. In this case, the original line and its image coincide.

Translations

Investigating the Ideas

Make a drawing of $\triangle BCD$ on graph paper. Also, cut out a triangular region that is the same size and shape as $\triangle BCD$.





Can you use the triangular cut-out to show the new location of $\triangle BCD$ if all the points of the plane are slid the same distance and in the same direction as the arrow from A to A'?

Discussing the Ideas

1. A *motion* which slides all points of the plane the same distance and in the same direction is called a **translation** or simply a **slide**. In the figure above, point A' is called the **translation image** of point A.

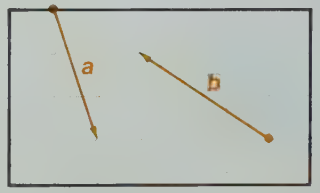
   We say that the translation "maps A onto A'" and denote this by writing: $A \rightarrow A'$

   A Label the translation image of $\triangle BCD$. What points do B, C, and D map onto?

   B What relations do you see between $\triangle BCD$ and its translation image?

2. Arrows such as ***a*** and ***b*** show the distance and direction of translations. Draw a geometric figure on graph paper and show how to find its image for translations by

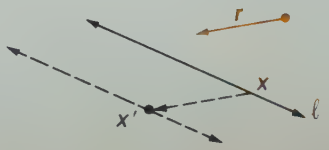
   A arrow ***a***. B arrow ***b***.



3. A Draw a line ℓ and an arrow for a translation. Show the translation image of the line.

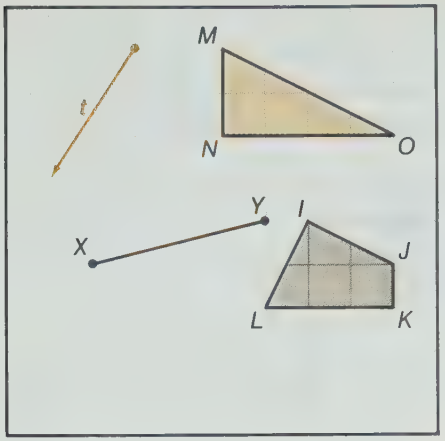
   B How are the line and its translation image related?

   C Can you describe a translation in which the image of line ℓ coincides with the original line?



Using the Ideas

1. Draw the figures shown on graph paper. Then show the translation image of each figure after the translation given by arrow ***t***.



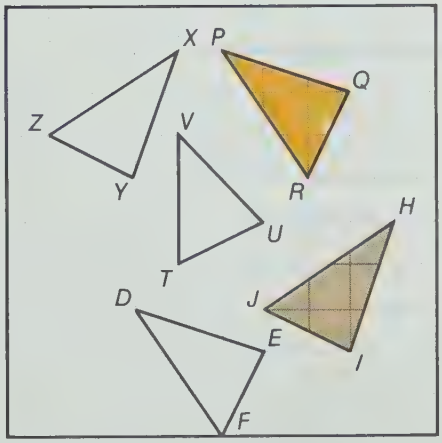
2. Using the results of Exercise 1, complete each sentence.

- **A** The translation image of $\triangle MNO$ is a ? that has the same size and shape as $\triangle MNO$.
- **B** The translation image of $\overline{XY}$ is a ? just as long as $\overline{XY}$.
- **C** The translation image of quadrilateral *IJKL* is a ? with the same size and shape as *IJKL*.

3. **A** In the figure at the right, which triangle is a translation image of $\triangle PQR$?

B Complete this mapping for the translation:

$P \rightarrow$ ▨, $Q \rightarrow$ ▨, $R \rightarrow$ ▨



4. **A** Using the same grid as for Exercise 3, which triangle is a translation image of $\triangle HIJ$?

B Complete this mapping for the translation:

$H \rightarrow$ ▨, $I \rightarrow$ ▨, $J \rightarrow$ ▨

★ **5.** The arrow ***a*** shows a translation so that $X \rightarrow Y$. Is there a translation so that $Y \rightarrow X$? (That is, X is the image of Y.) How is this translation related to the translation given by arrow ***a***?

UTILIZATION

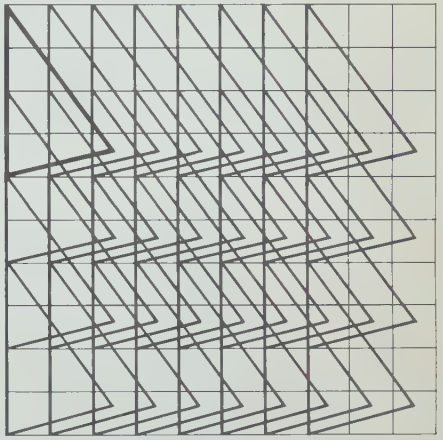
Graph paper should be available for students to use for the exercises.

Exercise 5 introduces the idea of an *inverse translation*; that is, a translation which is opposite in direction but the same distance as the original translation.

EXTENSION

Additional practice with translations is offered on *Workbook* page 67.

Enrichment: Some unusual designs can be constructed based on repeated translations of a given figure. The figure below suggests one such design based on translations of a triangle. Encourage students to make their own designs based on other shapes. Shading or coloring the designs will make them more attractive.



Assignments

Minimum 1–4. Average 1–4. Maximum 1–5.

Objective

Given a geometric region in a plane and two translations, the student can find the translation image when one translation is followed by the other.

PREPARATION

Materials: Scissors, graph paper (*Duplicator Masters*, page 83).

This lesson extends the ideas of the previous lesson. Two translations are performed in succession. If one translation maps X onto Y ($X \rightarrow Y$) and a second translation maps Y onto Z ($Y \rightarrow Z$), then the combination of the two translations has the same effect as a single translation that maps X onto Z ($X \rightarrow Z$).

INVESTIGATION

Many students will be able to show the translation images in the Investigation without using a cut-out of the triangular region. Be certain that students understand that it is the *image of* $\triangle PQR$ translated by arrow a (*not* the original $\triangle PQR$) that is to be translated by arrow b.

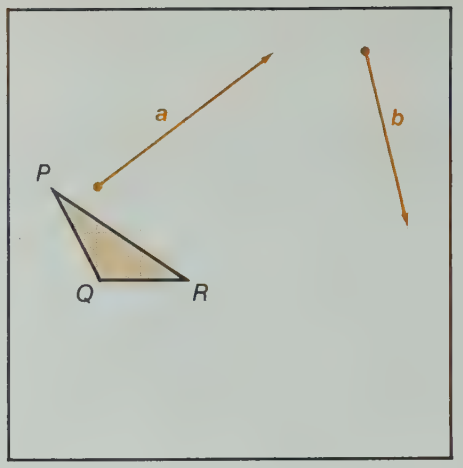
DISCUSSION

The description of the translation by arrow a as "right 4, up 3" suggests that this translation is itself a combination of a horizontal translation followed by a vertical translation. In Exercise 2, some students may notice that the "right 4, up 3" translation followed by a "right 1, down 4" translation has the same effect as the single translation of "right 5, down 1."

Combinations of Translations

Investigating the Ideas

Draw $\triangle PQR$ on graph paper and cut out a triangular region that is the same size and shape as $\triangle PQR$.



Can you use the triangular region to show the following translation images?

1. The translation image of $\triangle PQR$, using arrow ***a***.

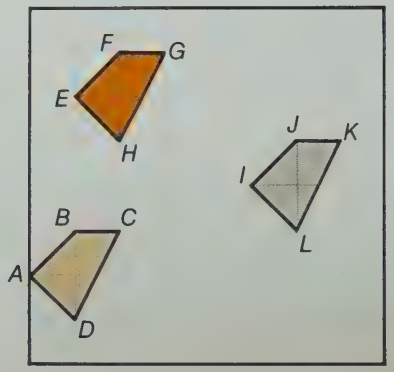
2. The translation image of the image found in step 1, using arrow ***b***.

Discussing the Ideas

1. The translation for arrow ***a*** could be described as "right 4, up 3." How could the translation for arrow ***b*** be described?

2. The combination of translations in the Investigation is a translation by "***a* followed by *b***."
   - **A** Is there a single translation that would produce the same final image as "***a*** followed by ***b***?"
   - **B** How would you describe the translation?

3. **A** Describe the translation that maps *ABCD* onto *EFGH*; that is, $A \rightarrow E$, $B \rightarrow F$, $C \rightarrow G$, and $D \rightarrow H$.
   - **B** Describe the translation that maps *EFGH* onto *IJKL*.
   - **C** The combination of the translations for parts **A** and **B** will map *ABCD* onto *IJKL*. Describe the single translation that will give the same final image as the combination of the translations for parts **A** and **B**.

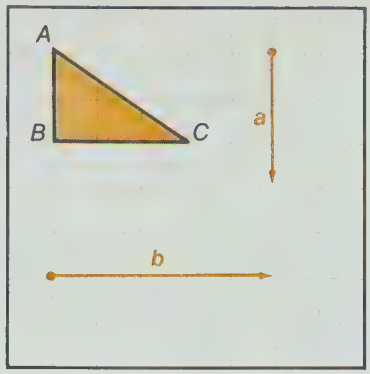


Using the Ideas

1. **A** Draw $\triangle ABC$ on graph paper. Show the image of $\triangle ABC$ by translation ***a***.

B Show the image of $\triangle ABC$ for the combination "***a*** followed by ***b***."

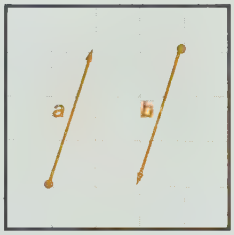
C Show an arrow that gives the same translation as "***a*** followed by ***b***."



2. Draw a square on graph paper. Show the final translation image of the square for the combination of translations "***a*** followed by ***b***" where ***a*** is "right 5, down 2" and ***b*** is "left 4, up 3."

3. **A** Draw a figure of your choice on graph paper.

B Show the translation image of the figure after the combination of translations "left 3, up 5" followed by "right 3, down 5."



4. Two translations are **inverses** of each other if the combination of the two translations leaves all points in their original position. Are the two translations in Exercise 3 inverses of each other?

5. The translations "right 2, up 1" and "left 2, down 1" are inverse translations. Give the inverse translation of each of the following:

A left 1, up 1

B right 4, down 5

C right 2, up 7

D left 4, down 4

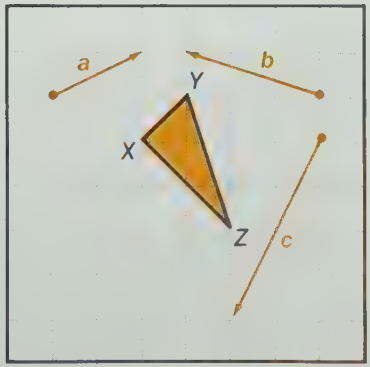
E up 3

F left 2

★**6.** **A** Draw a triangle like $\triangle XYZ$ on graph paper.

B Show the final translation image of $\triangle XYZ$ after the combination of "***a*** followed by ***b*** followed by ***c***."

C Try translations ***a***, ***b***, and ***c*** again but in a different order, such as "***c*** followed by ***a*** followed by ***b***." Is the final translation image in a position that is different from that for part **B**?



UTILIZATION

Graph paper should be avilable for students to use in doing the exercises.

The exercises contain the hint of a mathematical structure: the set of all translations of a plane form a mathematical system called a *group*. Thus, the set of all translations has the following properties:

A. The combination of two translations is a translation. (Exercise 1)

B. Translation is an associative operation. (Exercise 6)

C. There is a zero translation. (No motion)

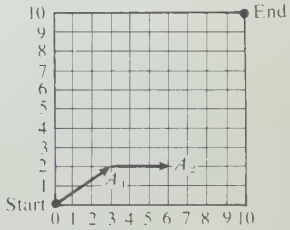
D. Each translation has an inverse translation. (Exercises 3–5)

Such ideas might be discussed with very capable students.

EXTENSION

See page 48 of the *Duplicator Masters* for further exercises involving combinations of translations.

Enrichment: The game of "Translations" can provide reinforcement for translation concepts. Label the six faces of one cube "right 0," "right 1," . . . "right 5." Label the faces of the other cube "up 0," "up 1," . . . "up 5." The "game board" consists of a grid like the one shown below.



The game is played by starting at (0, 0) and showing the translations obtained by tossing the two cubes. The object is to get to a particular point on the grid, say (10, 10). To add variety, some points might be labeled "Go back to start," "Miss a turn," or "Throw again." Allow students to develop their own rules for the game.

Assignments

Minimum 1–3, Average 1–5, Maximum 1–6.

Objective

Given a region in a plane, a centre of rotation, and a specified rotation, the student can show or name the rotation image.

PREPARATION

Materials: Tracing paper.

Students who have studied from previous books in this series will already have encountered the topic of rotations. However, little previous knowledge is assumed for this lesson. Simple angle measure is used in this lesson. You may wish to review the use of the protractor in angle measurements.

INVESTIGATION

The Investigation provides the student with a concrete example of a rotation. Only $\triangle JKL$ will align with the rotation of $\triangle ABC$. You may find it helpful to demonstrate additional rotations on the overhead projector. The students' understanding of rotations will be strengthened if they see the actual motion take place.

DISCUSSION

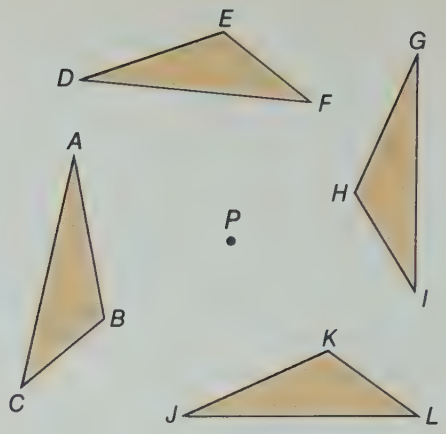
The three elements of a rotation are emphasized in Exercise 3—the *centre*, the *direction*, and the *amount of rotation*.

For Exercises 2 and 4, tracing paper can be used to show the rotations. The original figure is traced and the centre of rotation marked on the tracing paper. The tracing is rotated the specified amount about the centre and the original figure is traced again. Finally, both tracings are appropriately labelled.

Rotations

Investigating the Ideas

Place a piece of tracing paper over these figures. Trace $\triangle ABC$ and point P. Place your pencil point at P and turn the tracing paper.



Which one of the other three triangles will match the tracing of $\triangle ABC$ when the paper is turned around point P?

Discussing the Ideas

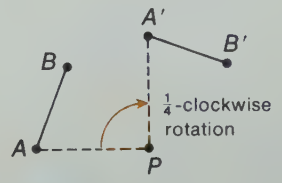
1. A Which point of any of the other triangles matched point A when you turned the paper around point P?
   B Which other points matched each other?

2. The motion that matched the two triangles is called **a rotation with centre at *P***, or **a rotation about point *P***. You can think of each point in the plane, except P, rotating through the same size angle with point P as the vertex of the angle. Draw a figure of your choice and then show a rotation of this figure about some point.

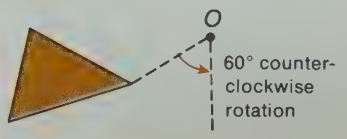
3. A rotation is determined by
   1. its *centre*,
   2. a *direction* (clockwise or counterclockwise), and
   3. *amount of rotation* (usually a fractional part of a complete rotation or a degree measure).

   The figure at the right shows a **$\frac{1}{4}$-clockwise rotation with centre *P***. The rotation image of point A is A', and we write $A \rightarrow A'$.

   What is the rotation image of point B?

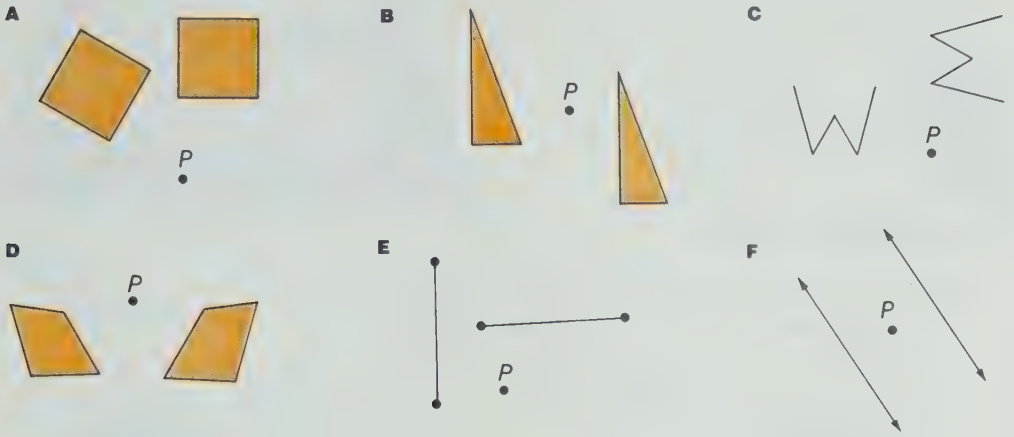


4. Draw a triangle and a point O. Explain how to find the rotation image of the triangle for "a 60° counterclockwise rotation with point O as the centre."

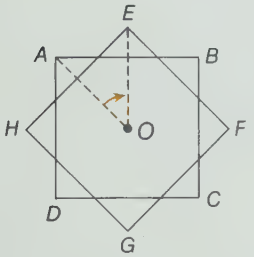


Using the Ideas

1. Using tracing paper, trace one of the figures and point P. Then tell whether the other figure is a rotation image of the first figure with centre at P.

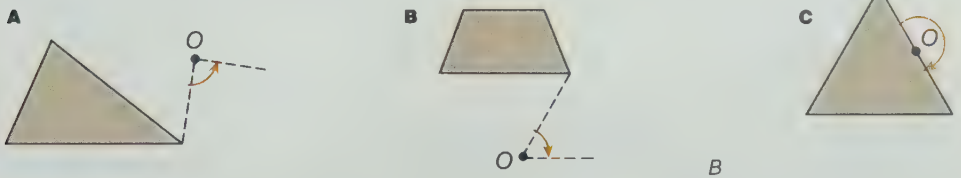


2. A rotation with centre O maps point A of square $ABCD$ onto point E.

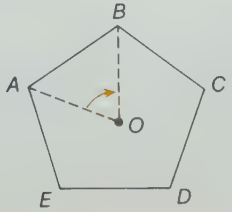


- **A** What is the rotation image of point B? C? D?
- **B** Give the amount and direction for this rotation.

3. Use tracing paper to copy each figure. Then find the rotation image using the amount and direction of rotation shown. Point O denotes the centre of rotation for each part.



★ **4.** For the rotation with centre O shown, $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow D$, $D \rightarrow E$, and $E \rightarrow A$. Consider another rotation with centre at point O so that $A \rightarrow C$. What would be the rotation image for B? C? D? E?



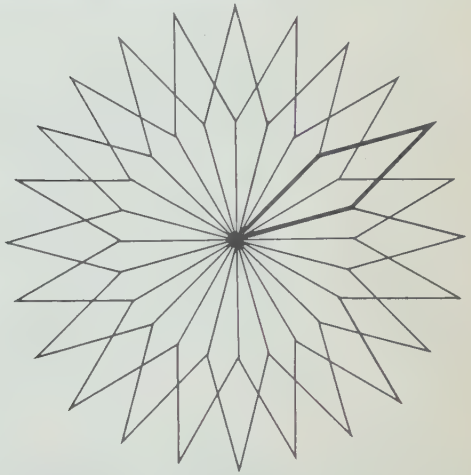
UTILIZATION

Tracing paper or other thin paper should be available for students to use for the exercises.

EXTENSION

Page 68 of the *Workbook* and page 49 of the *Duplicator Masters* provide appropriate additional practice exercises.

Enrichment: Have students cut out a geometric region from an index card in order to use the card as a template to show repeated rotations of the region around one of its vertices as the centre of rotation. The figure below illustrates an example of rotations around a vertex of a parallelogram.



Assignments

Minimum 1–2. Average 1–3. Maximum 1–4.

Objective

Given a region and a line in the same plane, the student can show the reflection image of the region in the line.

PREPARATION

Materials: Rectangular pieces of clear plastic or rectangular, nonfragile mirrors.

It is very desirable to have clear plastic sheets available for this lesson. Such material can be purchased in arts and crafts stores. Used X-ray film, which may be obtained free from some doctors or hospitals, also will provide a satisfactory reflecting surface, especially if taped in a cardboard frame to provide additional sturdiness.

INVESTIGATION

The Investigation introduces reflections by means of an interesting problem. Some students will be surprised that the reflection image of the word CHOICE appears the same as the original word. This effect occurs because each of the capital letters in the word choice has a horizontal line of symmetry. In the word CANDY, however, only the letters C and D appear the same, because they are the only letters in the word that have a horizontal line of symmetry.

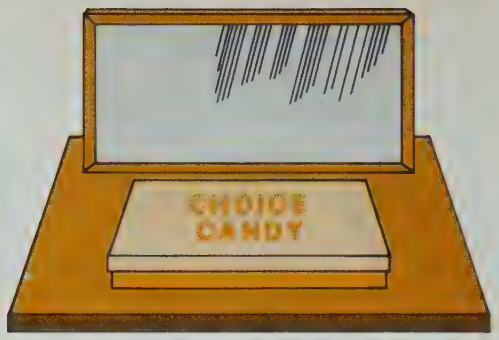
DISCUSSION

Allow students sufficient time to experiment with the plastic mirrors while discussing Exercises 1, 2, and 3. Exercise 4 contains a formal description of a reflection of the point of the plane in a line. This definition should simply serve to summarize students' previous observations; you should not expect students to memorize the definition.

Reflections

Investigating the Ideas

Suppose you saw this box of candy lying in front of a mirror.

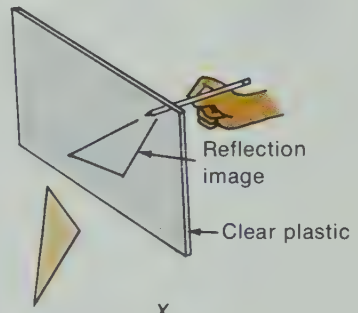


Can you draw the image that you would see in the mirror?

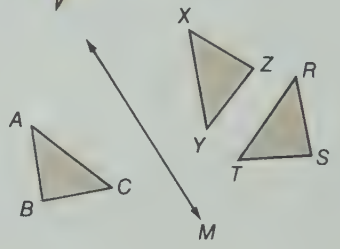
Discussing the Ideas

1. Which word on the box of candy looks the same when you see its reflection in the mirror?

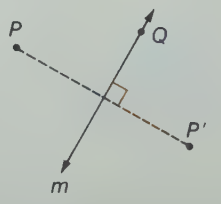
2. You can use a rectangular piece of clear plastic as a mirror to help you draw reflections of figures. Hold the piece of plastic upright on your paper. Look through the plastic to see the **reflection image**. Draw some figures and then use this method to find their reflection images.



3. Suppose line m represents a mirror.
   - **A** Which figure is the reflection image of $\triangle ABC$ in line m?
   - **B** Which point is the reflection image of A? B? C?

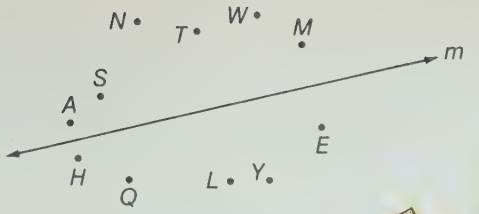


4. A reflection of the points of the plane in a line m can be described as follows:
   - [1] If P is not on m, then $P \rightarrow P'$ so that m is the perpendicular bisector of PP'.
   - [2] If Q is on m, then Q is its own image, $Q \rightarrow Q'$.
   - **A** Draw some points on one side of a line. Use a clear plastic "mirror" to find their reflection images.
   - **B** Connect some points to their images. Is the line the perpendicular bisector of these segments?

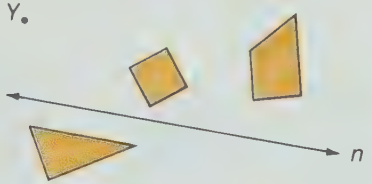


Using the Ideas

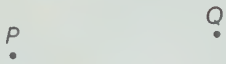
1. Which pairs of points are reflection images of each other in line m?



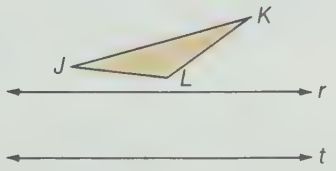
2. Draw or trace figures like the three shown here. Use a clear plastic "mirror" to draw the reflection image of each figure in line n.



3. Mark two points P and Q on your paper. Suppose P and Q are reflection images of each other. Use a plastic mirror to find and draw the reflecting line.

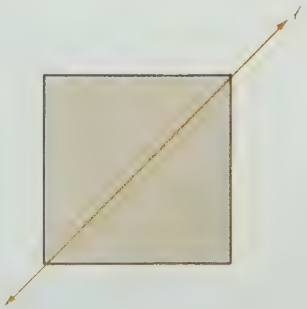


4. Draw two parallel lines r and t and $\triangle JKL$ as shown.



- **A** Find the reflection image of $\triangle JKL$ in line r. Call it $\triangle MNO$.
- **B** Find the reflection image of $\triangle MNO$ in line t. Call it $\triangle PQR$.
- **C** Is there a single reflection in a line so that $\triangle PQR$ is the reflection image of $\triangle JKL$?
- **D** Is $\triangle PQR$ a translation image of $\triangle JKL$?

5. Line ℓ is a **line of symmetry** of the square because the reflection images of all the points of the square in line ℓ will be on the square.



- **A** How many different lines of symmetry does a square have?
- **B** How many lines of symmetry does a rectangle which is not a square have?

6. How many lines of symmetry does each figure have?

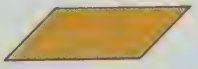
A

Equilateral triangle

B

The letter H

C



Parallelogram

D

Circle

UTILIZATION

In Exercises 2, 3, and 4, students can make their own drawings for the exercises or make tracings of the drawings in the text. Actually, it would be preferable to have students draw figures larger than those shown in the text.

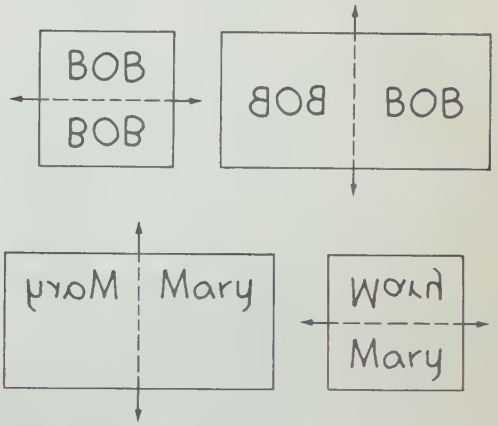
The term *line of symmetry* is introduced in Exercise 5. You might prefer to discuss this term before assigning the exercises.

EXTENSION

Additional practice exercises are provided on *Workbook* page 69 and *Duplicator Masters* page 50.

Remedial: Have students draw other geometric shapes and show their reflections in given lines using their plastic mirrors.

Enrichment: Students can try writing or printing their names upside down and backward. Then they can use their mirrors to check whether they are correct. Some examples are shown below.



Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

Given a geometric figure in a plane and a combination of two motions (reflection, rotation, or translation), the student can show the image for the combination of motions.

PREPARATION

Materials: Graph paper (*Duplicator Masters*, page 83), tracing paper, plastic mirrors, protractors.

This lesson is designed to extend the ideas related to the various motions studied thus far in the module. Use a grid on the overhead projector or the chalkboard to review translations, rotations, and reflections.

INVESTIGATION

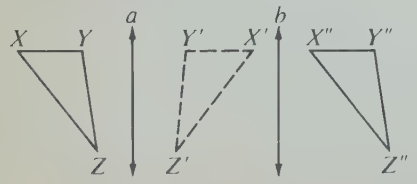
Have students draw $\triangle ABC$ and lines m and n on graph paper. Emphasize that in part B of the Investigation the reflection image of $\triangle ABC$ in line m (part A) is to be reflected in line n.

DISCUSSION

It will be obvious to most students that there is not a single reflection or translation that will map $\triangle ABC$ onto $\triangle A''B''C''$. However, a 180° rotation ($\frac{1}{2}$ turn) about point O, the intersection of lines m and n, is equivalent to the combination of reflections.

Exercise 2 may help students to realize that a combination of reflections in a pair of intersecting lines is equivalent to a rotation about the point of intersection of the lines.

Exercise 3 should help students understand that a combination of reflections in two parallel lines is equivalent to a translation. For example, if $\triangle XYZ$ is reflected in line a followed by a reflection in line b where $a \parallel b$, then $X''Y''Z''$ is a translation image of $\triangle XYZ$.

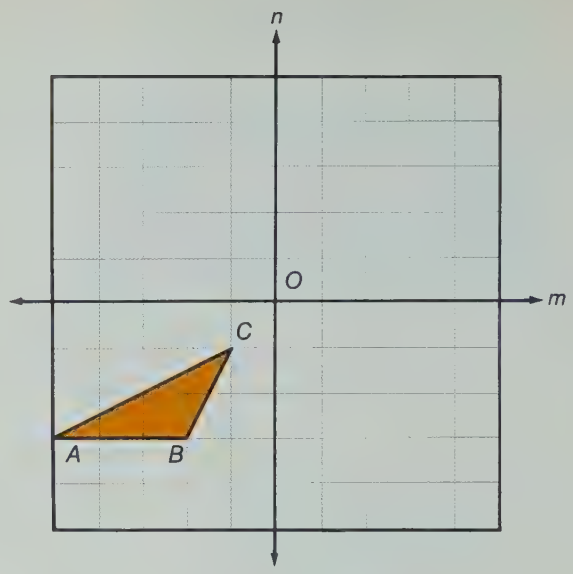


Combinations of Motions

Investigating the Ideas

Draw $\triangle ABC$ and perpendicular lines m and n on a sheet of graph paper.

Can you find the following reflection images?



A Find the reflection image of $\triangle ABC$ in line m. Call it $\triangle A'B'C'$.

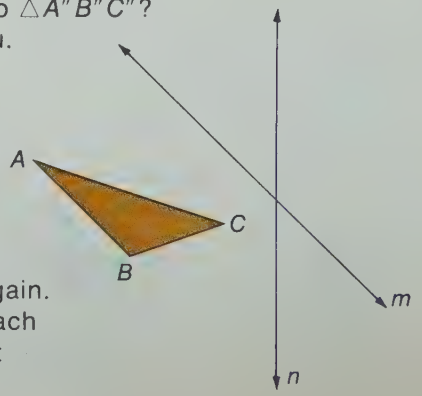
B Find the reflection image of $\triangle A'B'C'$ in line n. Call it $\triangle A''B''C''$.

Discussing the Ideas

1. $\triangle A''B''C''$ is said to be the reflection image of $\triangle ABC$ after a reflection in line m followed by a reflection in line n.
   - **A** Is there a single reflection that will map $\triangle ABC$ onto $\triangle A''B''C''$? Use a clear plastic mirror to help you.
   - **B** Is there a single translation that will map $\triangle ABC$ onto $\triangle A''B''C''$?
   - **C** Is there a rotation that maps $\triangle ABC$ onto $\triangle A''B''C''$? Use a piece of tracing paper to help you.

2. Try the Investigation again with lines m and n intersecting but not perpendicular. What conclusions can you draw about this combination of reflections?

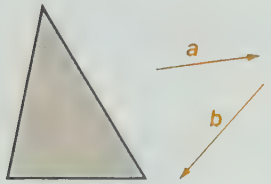
3. Try the same combination of reflections again. This time, have lines m and n parallel to each other. How is this result different from that found in Exercise 2?



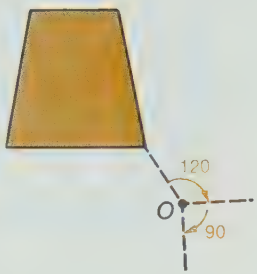
Using the Ideas

For each exercise, draw or trace the figures given. Then perform the combination of motions given and show the final image. Use a plastic mirror, graph paper, or tracing paper to help you find the final image.

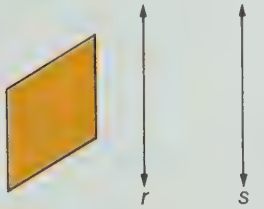
1. Translation: ***a*** followed by ***b***.



2. Clockwise rotations: 120° followed by 90° with centre *O*.

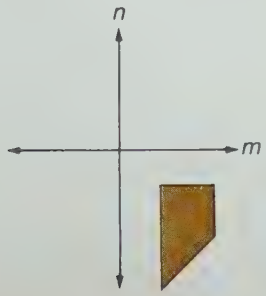


3. Reflection in line *r* followed by a reflection in line *s*.

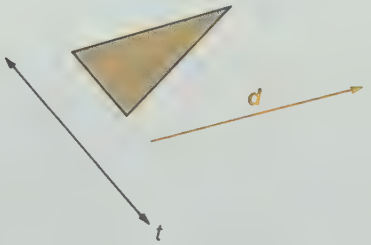


4. 90° clockwise rotation followed by translation ***c***.

5. Reflection in line *n* followed by a reflection in line *m*.



6. Reflection in line *t* followed by translation ***d***.



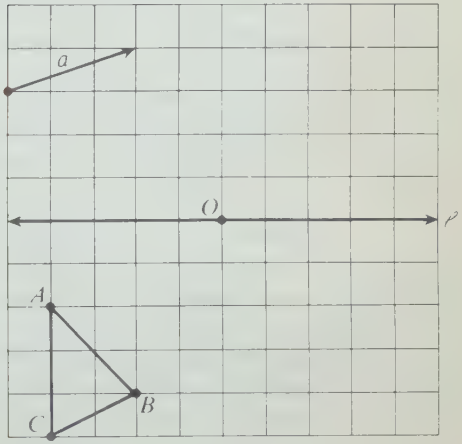
UTILIZATION

For the exercises, students can draw enlarged versions of the figures shown in the text. Encourage students to be reasonably accurate in showing the various motions, but strive for understanding of the motions involved rather than exactness in geometric construction. Allow students to display and discuss their solutions on the overhead projector or at the chalkboard.

EXTENSION

Enrichment: Prepare some exercises for students that involve a combination of three motions. For example:

Find the image of $\triangle ABC$ after a translation by arrow a, followed by a reflection in line ℓ, followed by a $\frac{1}{4}$-counterclockwise rotation about point O.



Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

Given two geometric figures which are images of each other for some rigid motion, the student can make statements of congruence about corresponding parts of the figures.

PREPARATION

Materials: Tracing paper, plastic mirror, graph paper (*Duplicator Masters*, page 83).

Review the three motions (reflections, translations, and rotations) studied in this module. Emphasize that the image of any geometric figure for any of these motions appears to have the same size and shape as the original figure.

DISCUSSION

The term *rigid motion* is introduced in Exercise 1 to describe any of the three motions studied thus far. Rigid motions are sometimes called *distance preserving motions* because, if P and Q are any two points in the plane and P' and Q' are their respective images for some rigid motion, then $\ell(\overline{PQ}) = \ell(\overline{P'Q'})$.

Exercise 2 introduces the idea of *congruence* in terms of rigid motions. Emphasize that the notation used for congruence denotes the correspondence between congruent parts of two geometric figures.

Exercise 3 suggests how a $\frac{1}{2}$-clockwise rotation about point O can be used to show that vertical angles, $\angle AOB$ and $\angle A'OB'$, are congruent. To show that $\angle BOA' \cong \angle B'OA$, we can observe that the half turn about O gives the mappings $B \to B'$, $O \to O$, $A' \to A$. Hence $\angle BOA' \cong \angle B'OA$.

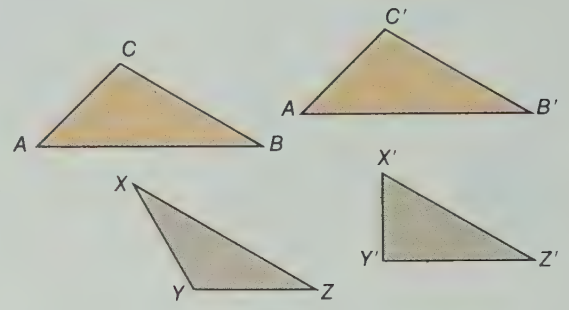
Rigid Motions and Congruence

Discussing the Ideas

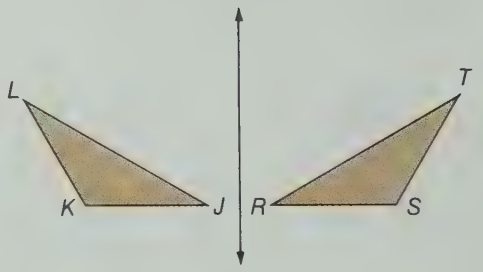
1. Each of the motions you have studied (translations, reflections, and rotations) is called a **rigid motion** because the image of a given figure after one of the motions is the same size and shape as the original figure.

A Which rigid motion is illustrated if $\triangle A'B'C'$ is the image of $\triangle ABC$?

B Could $\triangle X'Y'Z'$ be the image of $\triangle XYZ$ for one of the three rigid motions? Explain your answer.



2. The image of any geometric figure for any rigid motion or combination of rigid motions is a figure which is **congruent** to the original figure. Thus, if $\triangle RST$ is the image of $\triangle JKL$ for some rigid motion, then we say that $\triangle JKL$ is congruent to $\triangle RST$. We write this as $\triangle JKL \cong \triangle RST$. The parts of each figure that match each other are called **corresponding parts** and are **congruent to each other**.



For example, in the figure,

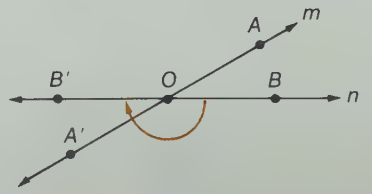
$\overline{KJ} \cong \overline{SR}$ because $K \leftrightarrow S$ and $J \leftrightarrow R$

$\angle LJK \cong \angle TRS$ because $L \leftrightarrow T$, $J \leftrightarrow R$, and $K \leftrightarrow S$.

A What other pairs of corresponding parts can you find?

B What statements of congruence can you write for them?

3. Draw two intersecting lines m and n and mark points A and B as shown.



A Find the images of points A and B after a $\frac{1}{2}$-clockwise rotation about point O. Call the images A' and B'.

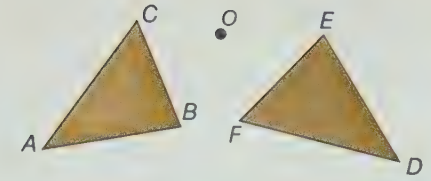
B Is $\overline{OB} \cong \overline{OB'}$? Is $\overline{OA} \cong \overline{OA'}$? Why?

C Since $\angle A'OB'$ is a rotation image of $\angle AOB$, is $\angle AOB \cong \angle A'OB'$?

Using the Ideas

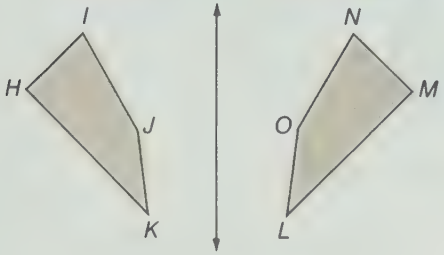
1. $\triangle ABC$ is a rotation image of $\triangle DEF$ with point O as the centre of rotation. Complete each statement:

- **A** $\overline{AB} \cong$ ▒
- **B** $\overline{BC} \cong$ ▒
- **C** $\overline{AC} \cong$ ▒
- **D** $\angle A \cong$ ▒
- **E** $\angle B \cong$ ▒
- **F** $\angle C \cong$ ▒



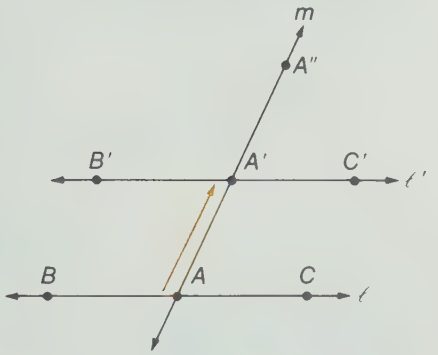
2. Suppose the two quadrilaterals are reflection images of each other in line t.

- **A** Write statements of congruence for the corresponding pairs of segments.
- **B** Write statements that show which pairs of angles are congruent.



3. Lines ℓ and m intersect at point A. Consider a translation of the points in the plane by the arrow from A to A'. The translation maps $B \rightarrow B'$, $C \rightarrow C'$, $A \rightarrow A'$, and $A' \rightarrow A''$.

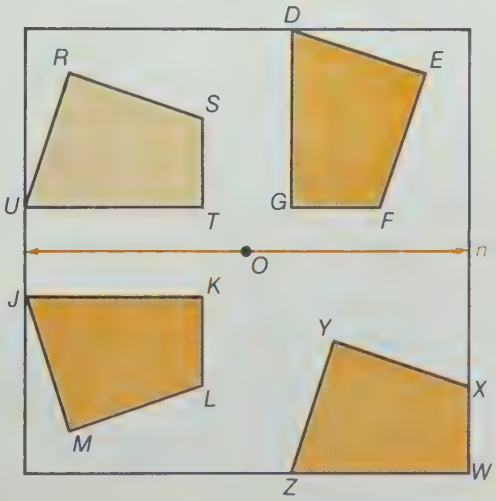
- **A** What is the image of line ℓ?
- **B** $\angle A''A'C'$ is the translation image of $\angle$?.
- **C** Is $\angle A''A'C' \cong \angle A'AC$?
- **D** List other pairs of congruent angles in this figure.



Use the figures on the grid for Exercises 4 and 5.

4. Find two quadrilaterals which are

- **A** rotation images of each other.
- **B** reflection images of each other.
- **C** translation images of each other.

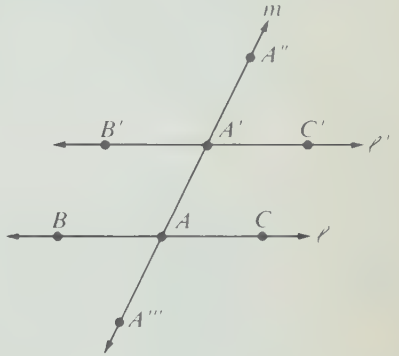


5. Write a statement of congruence for each pair of quadrilaterals named in Exercise 4**A**, **B**, and **C**.

UTILIZATION

The exercises provide practice in relating the rigid motions to pairs of congruent figures and identifying corresponding congruent parts.

Plan to discuss the exercises in detail after students have completed them. The translation in Exercise 3 not only shows that $\angle A'AC \cong \angle A''A'C'$ but also shows that $\angle BAA' \cong \angle B'A'A''$. If A is the image of A''' (see figure below), then $\angle A'''AB \cong \angle AA'B$, $\angle A'''AC \cong \angle AA'C'$. By referring back to Discussion Exercise 3 on page G–12, it can be shown that pairs of vertical angles such as $\angle AA'B'$ and $\angle A''A'C'$ are congruent.



EXTENSION

Remedial: Prepare a duplicated sheet of at least 6 pairs of congruent geometric figures. Have students decide which pairs of segments and angles are congruent, and then write statements of congruence for these pairs of corresponding parts.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–5.

Objective

The student can identify the corresponding parts of two congruent triangles.

PREPARATION

Materials: Tracing paper.

Review the fact that to be congruent, figures must have the same size and shape. Then begin the Investigation.

INVESTIGATION

The Investigation provides a visual puzzle to motivate students. Tracing paper can be placed over the drawing and students can check for congruence by tracing and comparing the various triangles. There are two pairs of congruent triangles: $\triangle CBH \cong \triangle GFA$ and $\triangle JBF \cong \triangle NMF$. It is not necessary for all students to find both pairs.

DISCUSSION

Two triangles are congruent if there is a correspondence between the vertices such that corresponding sides and corresponding angles are congruent. Exercise 1 presents this definition in symbolic form. Point out that an equal number of slash marks through the two sides of the triangles indicate congruent segments. Slash marks are also used on arcs connecting the sides of angles to indicate pairs of congruent angles in drawings.

Emphasize that the notation for congruent triangles indicates the correspondence between the corresponding congruent parts. Thus, if $\triangle PQR = \triangle TSU$ then $\angle P \cong \angle T$, $\angle Q \cong \angle S$, $\angle R \cong \angle U$, $\overline{PQ} \cong \overline{TS}$, $\overline{PR} \cong \overline{TU}$, and $\overline{QR} \cong \overline{SU}$.

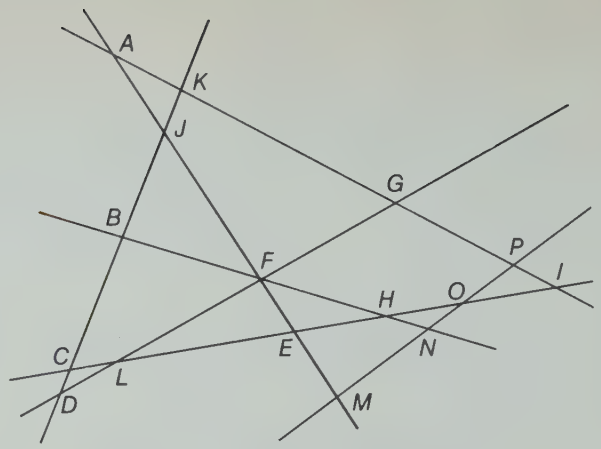
Exercise 3 simply applies the definition of congruence. Since there is a correspondence between the two triangles such that all the corresponding parts are congruent, the two triangles are congruent.

Congruent Triangles

Investigating the Ideas

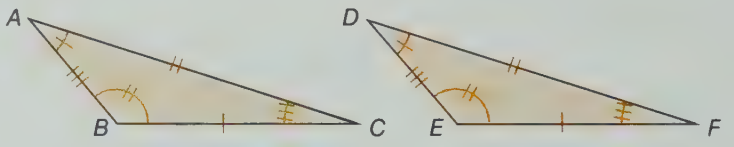
Study the drawing at the right.

Can you locate a pair of congruent triangles in the drawing? Check by tracing the triangles.



Discussing the Ideas

1. The figure below shows how corresponding segments and angles of congruent triangles can be marked.



$$\triangle ABC \cong \triangle DEF \Leftrightarrow \begin{matrix} \angle A \cong \angle D \\ \angle B \cong \angle E \\ \angle C \cong \angle F \end{matrix} \text{ and } \begin{matrix} \overline{AB} \cong \overline{DE} \\ \overline{BC} \cong \overline{EF} \\ \overline{AC} \cong \overline{DF} \end{matrix}$$

What statements can you make about the corresponding parts of the congruent triangles in the Investigation?

2. The statement $\triangle JKL \cong \triangle XYZ$ means that the vertices of the two congruent triangles are matched this way:

$J \leftrightarrow X$, $K \leftrightarrow Y$, and $L \leftrightarrow Z$

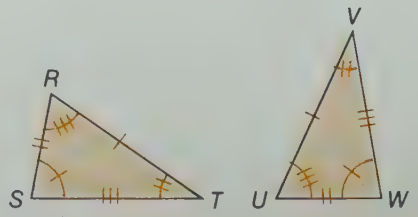
   A The correspondence above means that $\overline{JK} \cong \overline{XY}$. What other pairs of segments must be congruent?

   B The correspondence above also means that $\angle J \cong \angle X$. What other pairs of angles are congruent?

3. A Name the pairs of sides that are congruent.

   B Name the pairs of angles that are congruent.

   C What statement of congruence can be written for the two triangles?



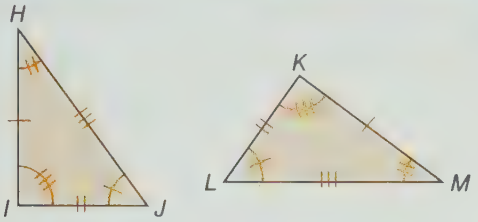
Using the Ideas

1. In the figure, $\triangle DEF \cong \triangle D'E'F'$.

- **A** $\overline{DE} \cong$ ▨
- **B** $\overline{FE} \cong$ ▨
- **C** $\overline{DF} \cong$ ▨
- **D** $\angle D \cong$ ▨
- **E** $\angle E \cong$ ▨
- **F** $\angle F \cong$ ▨

2. Complete each part.

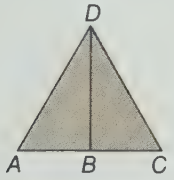
- **A** $\overline{HI} \cong$ ▨
- **B** $\overline{IJ} \cong$ ▨
- **C** $\overline{HJ} \cong$ ▨
- **D** $\angle H \cong$ ▨
- **E** $\angle I \cong$ ▨
- **F** $\angle J \cong$ ▨
- **G** $\triangle HIJ \cong$ ▨



3. $\triangle RST \cong \triangle XWY$. Write statements of congruence for the pairs of congruent segments and pairs of congruent angles.

4. In each figure, a pair of congruent triangles is shown. Decide which vertices correspond, then write a statement of congruence about the triangles.

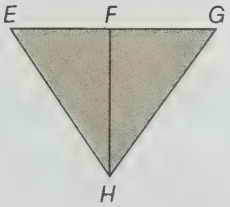
A



EXAMPLE:

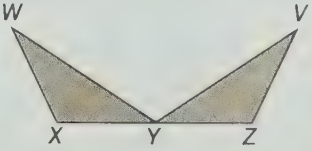
$\triangle ABD \cong \triangle CBD$

B

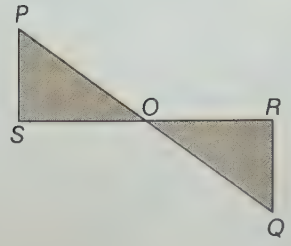


C

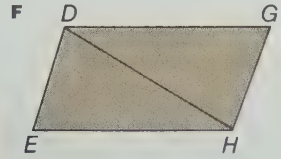
D



E

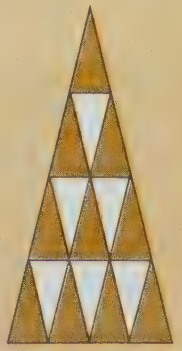


F



think

How many different triangles are in this figure?



UTILIZATION

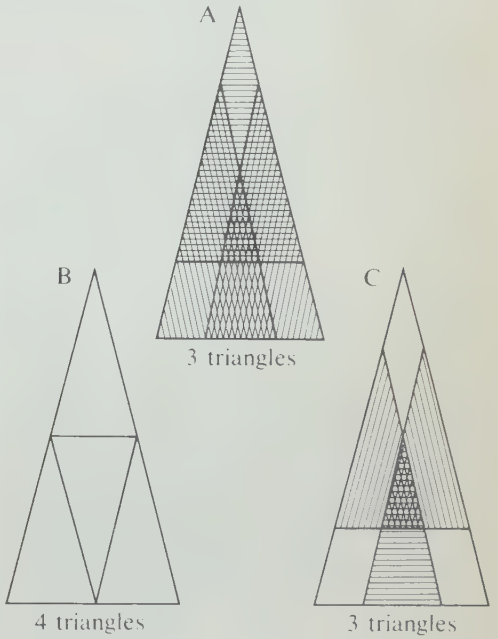
The exercises provide practice in determining pairs of corresponding parts of congruent triangles. In discussing Exercise 4 you may have students name corresponding segments and corresponding angles for each pair of congruent triangles.

EXTENSION

For appropriate further practice exercises, see *Workbook* page 70 and *Duplicator Masters* page 41.

Think Solution

The largest triangle and the 16 smallest triangles are obvious. Then as illustrated in Figures A, B, and C, there are 3, 4, and 3 triangles, respectively. Thus, the total is $1 + 16 + 3 + 4 + 3 = 27$.



Assignments

Minimum 1, 2, 4A–B.
Average 1, 2, 3, 4A–D.
Maximum 1–4.

Objective

The student can apply the Side-Side-Side congruence theorem to appropriate pairs of triangles.

PREPARATION

Materials: Centimetre ruler, compass.

Since the Investigation requires a construction, you may want to review construction of a triangle, given three sides. Have some students demonstrate their construction method on the chalkboard.

INVESTIGATION

Suggest to your students that they make the construction of the triangle as carefully as they can because they will later compare their constructed triangle to those of their classmates. When all have finished, proceed immediately to the Discussion Exercises.

DISCUSSION

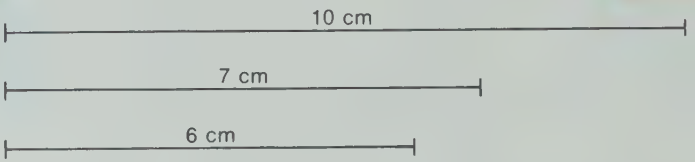
Have the students compare their triangles by superposition. If one construction is placed on the other and held to the light, it is possible to see whether the triangles coincide. When the triangles do not seem to be congruent, ask the students to remeasure the lengths of the sides and review their construction to see where inaccuracies have occurred.

The results of the Investigation should make the acceptance of the Side-Side-Side Theorem (SSS) quite natural. Point out that we need not know anything about the angles of a pair of triangles to apply the Theorem. But once we can establish that two triangles are congruent by the SSS Theorem, then it follows from the definition of congruent triangles that the corresponding angles must be congruent.

The Side-Side-Side Theorem

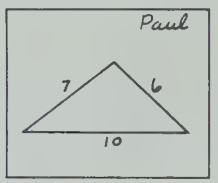
Investigating the Ideas

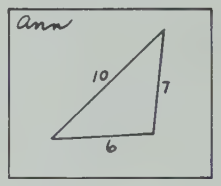
Can you make a drawing of a triangle whose sides are as long as the segments below?



Discussing the Ideas

1. Compare the triangle that you drew with a triangle drawn by one of your classmates. You may have to rotate, translate, or reflect one of the triangles. Do the two triangles seem to be congruent?



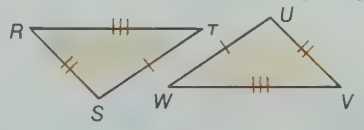


2. From the result of the Investigation, you may be able to see that the following statement or **theorem** of geometry is true.

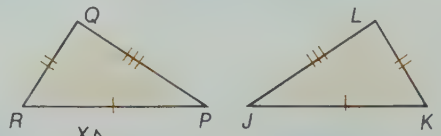
> **The Side-Side-Side Theorem (SSS)**
>
> If the sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

Each pair of triangles below are congruent because of the SSS Theorem. Write a statement of congruence for each pair of corresponding angles.

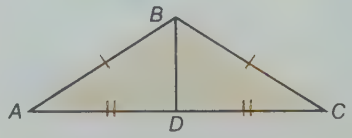
A



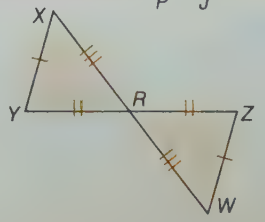
B



C



D



NOTE: $\overline{BD} \cong \overline{BD}$

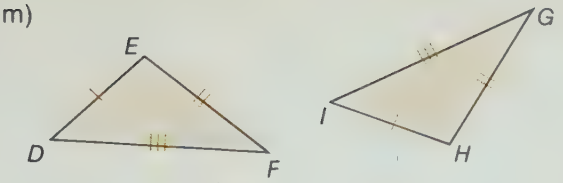
Using the Ideas

1. **A** $\triangle ? \cong \triangle ?$. (SSS Theorem)

B $\angle D \cong \angle ?$

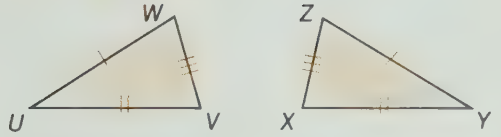
C $\angle E \cong \angle ?$

D $\angle F \cong \angle ?$



2. **A** $\triangle ? \cong \triangle ?$.

B Which pairs of angles are congruent?



3. The two triangles in each figure are congruent because of the SSS Theorem. You will also notice that the two triangles in each figure have a *common side*. Name the common side and then write statements of congruence for corresponding sides of the triangles.

EXAMPLE:

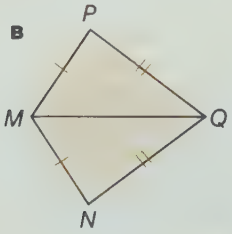
SOLUTION: *Common side* $\overline{AC}$

$\overline{AC} \cong \overline{AC}$, $\overline{AD} \cong \overline{CB}$, $\overline{AB} \cong \overline{CD}$

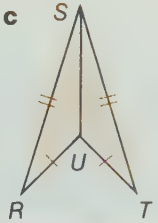
$\triangle ABC \cong \triangle CDA$

A

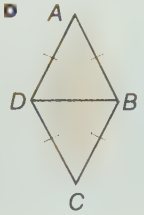
B



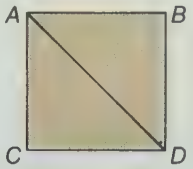
C



D



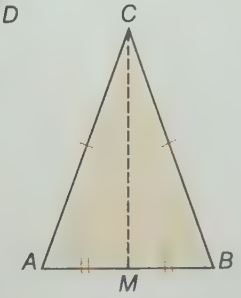
4. *ABCD* is a square. Show that $\triangle ABD \cong \triangle ACD$ by the SSS Theorem.



5. An **isosceles triangle** has a pair of congruent sides. Suppose $\triangle ABC$ is isosceles and $\overline{AC} \cong \overline{BC}$. Let *M* be the midpoint of $\overline{AB}$ so that $\overline{AM} \cong \overline{MB}$.

A Is $\triangle AMC \cong \triangle BMC$? Why?

B Is $\angle A \cong \angle B$?



UTILIZATION

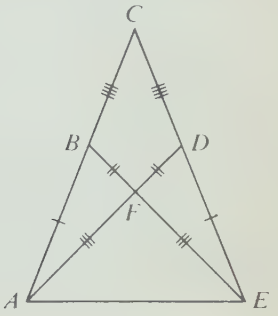
Exercises 1, 2, and 3 provide practice applying the SSS Theorem.

Observe that Exercise 5 contains, essentially, the proof that the base angles of an isoceles triangle are congruent angles.

EXTENSION

Workbook page 71 provides exercises that would be suitable for use in extending practice with the ideas presented in this lesson.

Enrichment: Ask students to find and name three pairs of congruent triangles in the figure.



Solution

$\triangle ABF \cong \triangle EDF$ $\triangle ABE \cong \triangle EDA$

$\triangle ADC \cong \triangle EBC$

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can apply the Side-Angle-Side congruence theorem to appropriate pairs of triangles.

PREPARATION

Materials: Tracing paper, rulers.

Review the SSS congruence theorem. You might tell the students that in this lesson they will explore another way to determine whether or not two triangles are congruent.

INVESTIGATION

Have the students trace the two angles and draw $\overline{AC}$ and $\overline{DF}$ to form two triangles. Then, they should cut out the two triangles and use superposition to decide whether the two triangles are congruent.

We can prove $\triangle ABC$ is congruent to $\triangle DEF$ in the following way: If one of the figures is a rigid motion image of the other, we have $A \leftrightarrow D$, $C \leftrightarrow F$, and $B \leftrightarrow E$. Then, by the definition of rigid motions, $\overline{AC} \cong \overline{DF}$. Since $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, it follows that $\triangle ABC \cong \triangle DEF$ by the SSS theorem.

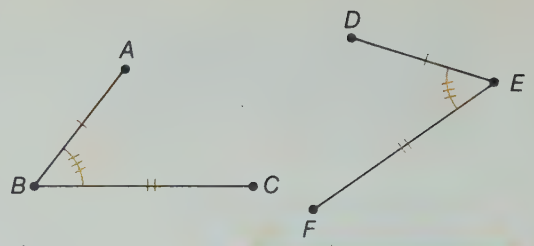
DISCUSSION

Students will generally accept the Side-Angle-Side Theorem quite readily. However, if they are to use this theorem efficiently, you will need to emphasize some techniques for identifying the "included angle." Some of the triangles shown in Exercise 3 can be used to reinforce this idea. Provide additional examples as needed.

The Side-Angle-Side Theorem

Investigating the Ideas

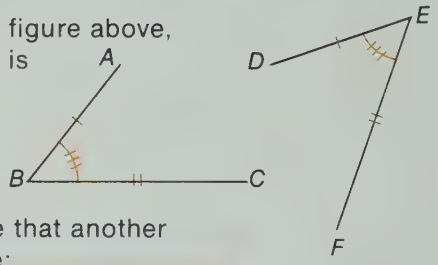
Suppose a rigid motion maps one figure onto the other so that the pairs of corresponding parts are congruent as marked.
Do you think the two triangles with vertices A, B, C and D, E, F will be congruent?



Can you trace the figures and complete the triangles, to check your guess?

Discussing the Ideas

1. From the parts marked congruent in the figure above, the correspondence for the rigid motion is $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$.
   - **A** Is $\overline{AC} \cong \overline{DF}$?
   - **B** Is $\triangle ABC \cong \triangle DEF$? Why?

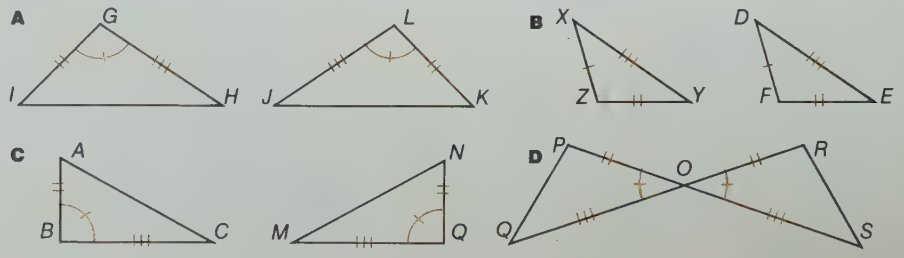


2. From the discussion above, you may see that another congruence theorem for triangles is true:

> The Side-Angle-Side Theorem (SAS)
>
> If two sides and the included angle of one triangle are respectively congruent to the corresponding parts of another triangle, then the triangles are congruent.

In the Investigation picture, $\angle B$ is the "included angle" for sides $\overline{AB}$ and $\overline{BC}$. What sides form the included angle E?

3. Each pair of triangles below are congruent. Which congruence theorem, SSS or SAS, applies to each pair?

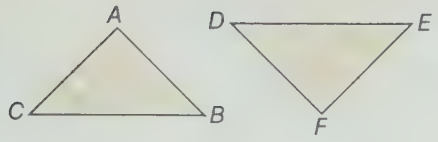


Using the Ideas

1. **A** $\triangle ? \cong \triangle ?$.

B Which congruence theorem is illustrated?

C $\overline{AB} \cong ?$; $\angle A \cong ?$; $\angle B \cong ?$

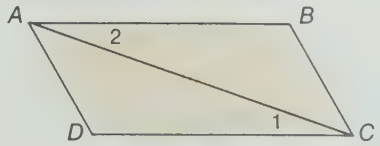


2. Complete each statement.

A $\overline{AB} \cong ?$ **B** $\overline{AC} \cong ?$

C $\angle 1 \cong ?$

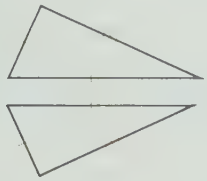
D $\triangle ACD \cong \triangle ?$ because of the ? Theorem.



3. In each pair of triangles, pairs of congruent sides or angles are marked. Tell whether each pair of triangles illustrates the SSS Theorem or the SAS Theorem.

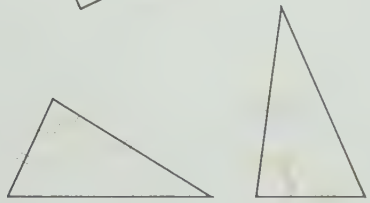
A

B



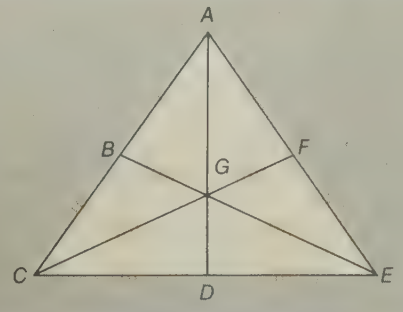
C

D



think

How many pairs of congruent triangles can you find and name for this figure?



UTILIZATION

Exercises 1 and 2 apply the SAS theorem directly. Both the SAS and the SSS theorems are applied in Exercise 3.

EXTENSION

Additional suitable practice exercises are provided on page 72 of the *Workbook*.

Remedial: Have students draw an angle; then draw two segments.

Using straightedge and compass, construct a triangle with two sides and the included angle congruent to the segments and angle. Repeat the construction. The two triangles should be congruent by the SAS congruence theorem.

Enrichment: Ask students to show that the bisector of the apex angle of an isosceles triangle bisects the base of the triangle. (See the figure below.)

Solution: Given $\overline{AD}$ bisects $\angle CAB$. $\overline{AB} \cong \overline{AC}$. Therefore, $\angle CAD \cong \angle BAD$, $\overline{AD} \cong \overline{AD}$. $\triangle BAD \cong \triangle CAD$ by SAS Theorem. $\overline{BD} \cong \overline{CD}$ because they are corresponding parts of congruent triangles. Therefore D must be the midpoint of $\overline{BC}$, so $\overline{AD}$ bisects $\overline{BC}$.

Think Solution

There are 7 pairs of congruent triangles:

$\triangle CGD \cong \triangle EGD$ $\quad$ $\triangle CFE \cong \triangle EBC$

$\triangle CGB \cong \triangle EGA$ $\quad$ $\triangle CFA \cong \triangle EBA$

$\triangle CGA \cong \triangle EGA$ $\quad$ $\triangle ABG \cong \triangle AFG$

$\triangle CDA \cong \triangle EDA$

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–3.

Objective

The student can apply the Angle-Side-Angle congruence theorem to appropriate pairs of triangles.

PREPARATION

Materials: Tracing paper, rulers.

Review the SSS and SAS congruence theorems, pointing out that these two theorems allow us to conclude that two triangles are congruent when given a minimum amount of information about the triangles.

INVESTIGATION

Have the students trace the angles and segment and complete the drawing of $\triangle ABC$. They can compare their drawing with those of their classmates. Note that in some cases students may find it necessary to turn one triangle over in order to get the two triangles to coincide.

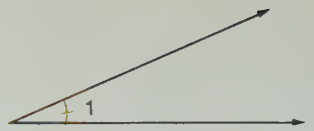
DISCUSSION

In discussing Exercise 1, the term "included side" should be emphasized and illustrated with additional examples, if needed. Exercise 2 contains the formal statement of the Angle-Side-Angle (ASA) congruence theorem suggested by the Investigation. Exercise 3 provides practice in selecting the congruence theorem that applies to the given pair of congruent triangles. Supply additional similar examples on the chalkboard if further reinforcement is needed.

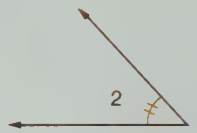
The Angle-Side-Angle Theorem

Investigating the Ideas

Make a drawing or tracing of a triangle, $\triangle ABC$, in which $\angle A \cong \angle 1$, $\angle B \cong \angle 2$, and $\overline{AB} \cong \overline{XY}$.

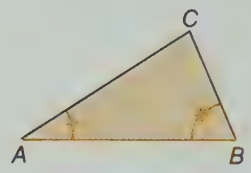


Can you compare your triangle with a classmate's triangle to see whether they are congruent?



Discussing the Ideas

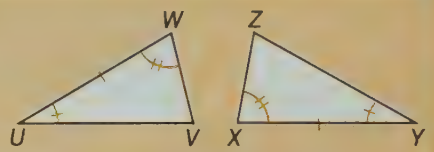
1. In $\triangle ABC$ of the Investigation, $\overline{AB}$ is said to be the "included side" with respect to angles A and B. Does the Investigation suggest another theorem about congruent triangles? Can you state it?



2. The Investigation suggests this congruence theorem:

The ngle- ide- ngle Theorem ( )

If two angles and the included side of one triangle are congruent to the corresponding parts of another triangle, then the triangles are congruent.



A What is the "included side" for $\triangle UVW$? for $\triangle XYZ$?

B Which side is congruent to $\overline{WV}$? to $\overline{UV}$?

C Which angle is congruent to $\angle V$?

3. Each pair of triangles are congruent. Which congruence theorem (SSS, SAS, or ASA) applies to each pair?

A

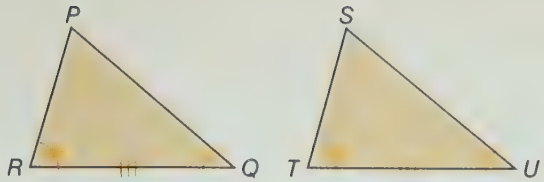
B

C

Using the Ideas

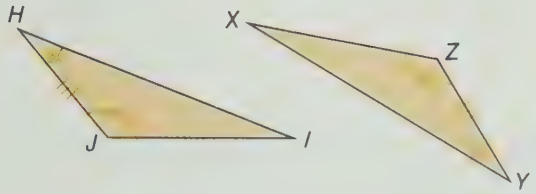
1. Complete each statement.

- **A** $\triangle PQR \cong \triangle$? because of the ? Theorem.
- **B** $\angle P \cong \angle$?
- **C** $\overline{RP} \cong$? and $\overline{PQ} \cong$?



2. Complete each statement.

- **A** $\triangle XYZ \cong \triangle$?
- **B** $\overline{JI} \cong$?, $\overline{HI} \cong$?, and $\angle X \cong \angle$?

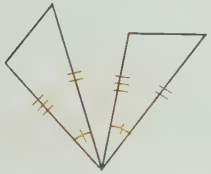


3. Each part illustrates either the SSS, SAS, or ASA congruence theorem. Give the correct theorem.

A

B

C



D

★**4.** Tell which single rigid motion could be used to map one of the triangles in each part of Exercise 3 onto the other triangle.

think

Make a strip of squares, eight squares long by one square wide.

1. Fold the strip into a model of a cube. (Some squares will be hidden.)
2. Can you make a cube using a strip only seven squares long?

UTILIZATION

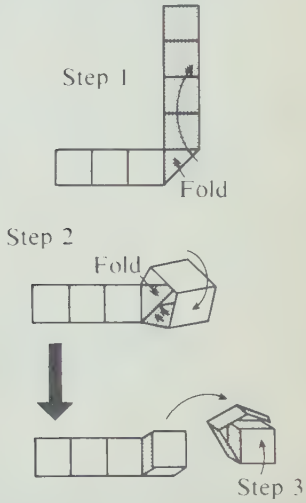
Exercises 1 and 2 provide application of the ASA theorem. Exercise 3 provides practice with all three congruence theorems as they apply to the pairs of congruent triangles.

EXTENSION

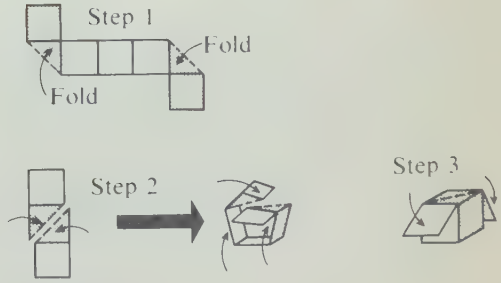
Refer to *Workbook* page 73 and *Duplicator Masters* page 52 for additional suitable practice exercises.

Think Solution

1. Mark off the strip into $2\frac{1}{2}$ centimetre squares and follow the illustrations.



2. Mark off the strip into seven squares.



Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can identify and make drawings of parallel lines, perpendicular lines, and right angles.

PREPARATION

Materials: Clear plastic mirrors, rulers, tracing paper.

The standard definitions of parallel and perpendicular lines are not stated in this lesson because the intent here is that the student be able to see how the concepts of parallelism and perpendicularity can be related to the rigid motions. However, in case you should want to provide the standard definitions, they are as follows:

Line r is parallel to line t $(r \parallel t)$ if r and t are in the same plane and do not intersect.

Line r is perpendicular to line t $(r \perp t)$ if r and t intersect so that pairs of adjacent angles are congruent.

INVESTIGATION

Have students trace the three drawings. If clear plastic mirrors (see Preparation on page G–8) are not available, students can fold their drawing on the reflecting line and show the reflection images.

DISCUSSION

Observe that in Exercise 4 perpendicular lines are defined in terms of reflections. Each of the four non-straight angles formed are called right angles, and by reflection are seen to be congruent.

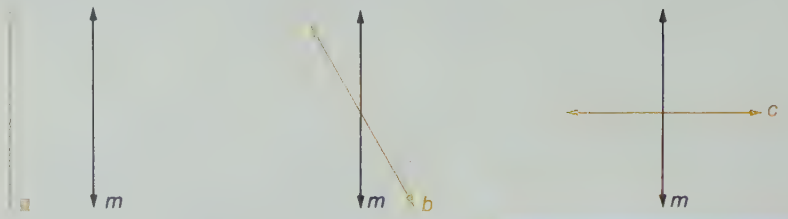
In discussing Exercise 5, bring out the fact that the translation image of a given line by an arrow not along the line is a line parallel to the given line. This fact will already be familiar to students from their work with translations.

Exercise 6 reviews vertical angles and corresponding angles formed by a transversal of two parallel lines.

Parallel and Perpendicular Lines

Investigating the Ideas

Trace or copy these figures on your paper. Then use a clear plastic mirror to complete this Investigation.



Can you find the reflection image in line m for each of the lines a, b, and c?

Discussing the Ideas

1. If lines a and m do not intersect, they are **parallel lines** and we write $a \parallel m$. Suppose the reflection image of a is a'. Are lines a and a' parallel?

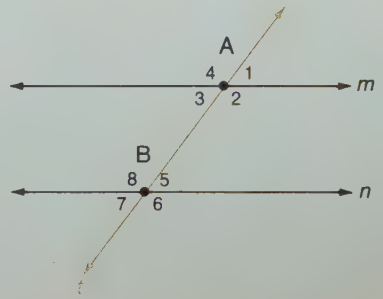
2. What kind of lines are b and m, *parallel* or *intersecting*? What can you say about line b and its reflection image?

3. How are line c and its reflection image related?

4. Since c is reflected onto itself by m, we say that c and m are **perpendicular** lines and write this as $c \perp m$. What kind of angle do perpendicular lines form?

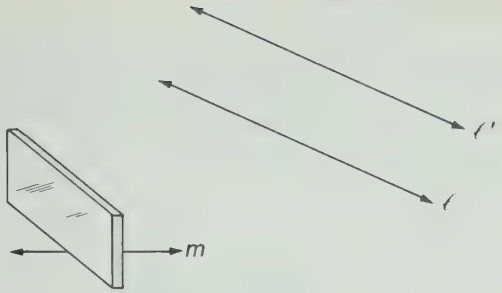
5. Suppose a line ℓ is translated by some arrow **a**. What is the relation between line ℓ and its translation image?

6. Lines m and n are parallel. Line t is a **transversal** of m and n.
   - **A** Is $\angle 1 \cong \angle 3$? (HINT: Think about a $\frac{1}{2}$-clockwise rotation with centre A.)
   - **B** Is $\angle 1 \cong \angle 5$? (HINT: Think about a translation from A to B.)



Using the Ideas

1. Draw a line ℓ on your paper. Use a translation to show a line ℓ' which is parallel to ℓ.



2. Draw a line m on your paper. Use a clear plastic mirror to help locate a line n which is perpendicular to m.

3. Perpendicular lines form **right angles** (SYMBOL:). Are two right angles congruent to each other?

4. Make a drawing of lines ℓ, m, and t such that $\ell \parallel m$ and $\ell \perp t$. What relationship is there between lines m and t?

5. Make a sketch of lines r, s, and t for these conditions:

   $r \perp s$ at point A
   $r \perp t$ at point B and $A \neq B$

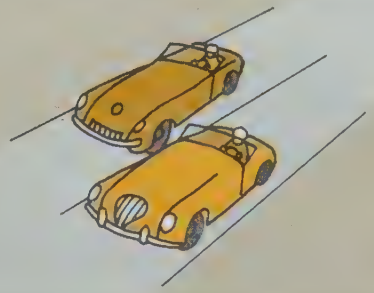
   How are lines s and t related?

6. Make a sketch of lines a, b, and c such that $a \parallel b$ and $b \parallel c$. How are lines a and c related?

Think

Two race cars sped two laps around a track. The first car averaged 200 km/h on the first lap and 240 km/h on the second lap. The second car averaged 220 km/h for both laps.

Which car won the race, or was it a tie?



UTILIZATION

The emphasis in the exercises is on construction of parallel and perpendicular lines. Accuracy of construction is not as important as a clear understanding and representation of the ideas.

Exercise 5 suggests a way to use reflections to construct a pair of perpendicular lines.

EXTENSION

Further appropriate practice exercises are provided on page 74 of the *Workbook*.

Think Solution

Let d = the distance of one lap.

First car: $\frac{d}{200}$ = time for first lap

$\frac{d}{240}$ = time for second lap

Total time: $\frac{d}{200} + \frac{d}{240} = \frac{22d}{2400} = \frac{11d}{1200}$

Second car, total time:

$$\frac{d}{220} + \frac{d}{220} = \frac{2d}{220} = \frac{1d}{110}$$

Since d is the same distance for both cars, we can just compare the two fractions $\frac{11}{1200}$ and $\frac{1}{110}$.

$\frac{11}{1200} = \frac{121}{13\,200}$ and $\frac{1}{110} = \frac{120}{13\,200}$.

Since $\frac{11}{1200} > \frac{1}{110}$, this means that the time for the first car was more than the time for the second car; the second car was the winner of the race.

Assignments

Minimum 1–4. Average 1–5. Maximum 1–6.

Objective

The student can identify parallel and perpendicular planes in parallelepipeds and prisms.

PREPARATION

Materials: Graph paper (*Duplicator Masters*, page 83), scissors, cellophane tape (models of cubes, parallelepipeds and prisms are optional).

This lesson extends to planes, the notions of parallelism and perpendicularity. You may want to conduct a brief review of the ideas of the previous lesson before beginning this one.

INVESTIGATION

Have students study the drawing in the Investigation and write their choices of opposite faces for the cube on a piece of paper. Students who lack good spatial intuition may find this problem difficult. Students should construct a model of the cube from the given pattern to check their choices.

DISCUSSION

Use the models that the students have made to illustrate parallel planes and perpendicular planes. Emphasize that the faces of the cubes are only portions of planes. Planes are not limited in extent as are the faces of the cubes.

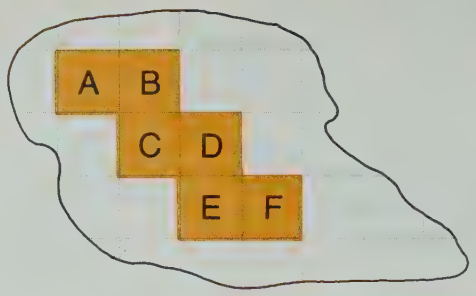
Encourage students to observe in discussing Exercise 1 that parallel planes are two planes that do not intersect. Students may express this idea by saying that they are "always the same distance apart."

Exercise 3 illustrates the geometric theorem that, if a line is perpendicular to two intersecting lines at their point of intersection, then it is perpendicular to the plane of the intersecting lines. Thus, in part A, the pencil can be held so that it is perpendicular to exactly one of the lines; however, if the pencil is held so that it is perpendicular to two of the lines (part B), it will also be perpendicular to the third line.

Parallel and Perpendicular Planes

Investigating the Ideas

Suppose you cut out this shape from graph paper and fold it along the dotted segments to form a cube.



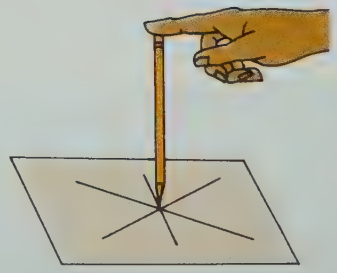
Can you decide which pairs of lettered faces will be opposite each other on the cube?

Discussing the Ideas

1. The pairs of faces of the cube that are opposite each other lie in **parallel planes**. How would you define parallel planes?

2. Two of the faces that intersect, such as A and B, are in **perpendicular planes**. Which other pairs of faces of the cube are in perpendicular planes?

3. Draw three lines that intersect at one point on your paper.
   - A Can you hold your pencil so that it is perpendicular to exactly one of the lines?
   - B Can you hold your pencil so that it is perpendicular to two of the lines but not the third line?



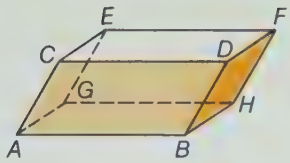
4. What objects in your room suggest parallel planes?

5. What objects in your room suggest perpendicular planes?

Using the Ideas

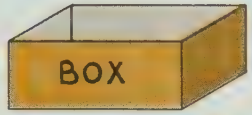
1. The figure shows a **parallelepiped**. The opposite faces are congruent parallelograms.

- **A** How many faces does it have?
- **B** Which face is opposite *DFHB*?
- **C** Are the opposite faces parallel?



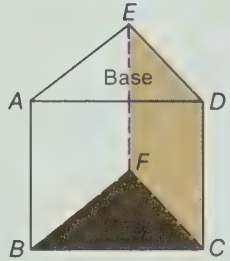
2. A "box" is a *rectangular parallelepiped*.

- **A** Are the intersecting faces perpendicular to each other?
- **B** What shape is each face?



3. **Prisms** have bases which are congruent polygons in parallel planes and **lateral faces** which are parallelograms.

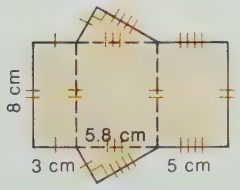
- **A** Name the bases in this figure.
- **B** Quadrilateral *ABCD* is a lateral face. Name the other lateral faces.



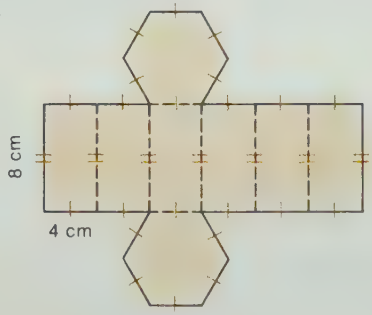
Triangular Prism

4. Prisms are named according to their kind of base. What kind of prism would these patterns make?

A



B



★**5.** Make the models shown in Exercise 4.

UTILIZATION

The words *parallelepiped*, *rectangular parallelepiped*, *prisms*, and *lateral faces* are introduced in the exercises. For many students, it would be most helpful to treat Exercises 1, 2, 3, and 4 as discussion questions. Exercise 5 could be used as a class activity.

EXTENSION

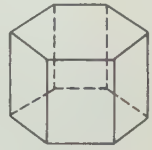
The exercises on page 75 of the *Workbook* and page 53 of the *Duplicator Masters* would be suitable for further practice assignments.

Enrichment: The models constructed for Exercise 5 could be used to illustrate Euler's Formula for polyhedrons (see Unit C, page C–119):

If F is the number of faces, V is the number of vertices, and E is the number of edges, then $V + F - E = 2$.

For the triangular prism, $F = 5$, $V = 6$, and $E = 9$, $V + F - E = 5 + 6 - 9 = 2$.

For the hexagonal prism, $F = 8$, $V = 12$, and $E = 18$, $V + F - E = 8 + 12 - 18 = 2$.



Assignments

Minimum 1–4, oral. Average 1–4, oral; 5. Maximum 1–4 oral, 5.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

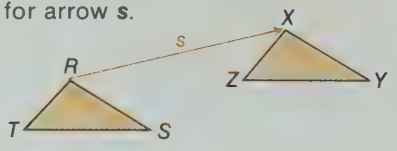
UTILIZATION

Use the review exercises in ways that will be most beneficial for your students. If the exercises are assigned for individual work, most students will benefit from a discussion of any problems that gave them difficulty.

The exercises on page 76 of the *Workbook* provide further review of the major ideas introduced in this module.

REVIEWING THE IDEAS

1. $\triangle XYZ$ is the translation image of $\triangle RST$ for arrow **s**.



A What is the image of point T?

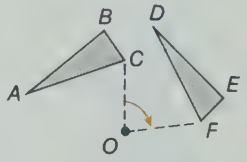
B What is the image of $\overline{RS}$?

2. **A** Draw a figure on graph paper.

B Show a "right 2, down 3" translation of the figure.

C Translate the image of part **B** "left 4, up 1."

3. The figure shows a rotation with centre O.

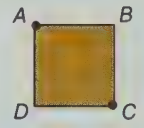


A Which direction is the rotation?

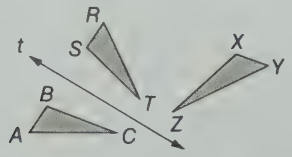
B What is the image of point B?

C What is the image of $\angle BAC$?

4. Draw a square $ABCD$. Show the rotation image of $ABCD$ after a $\frac{3}{4}$-counterclockwise rotation about point C.



5. Which triangle is the reflection image of $\triangle ABC$ in line t?



6. How many lines of symmetry does a rectangle have?

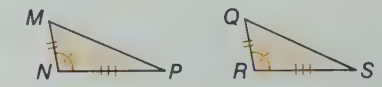
7. Draw an equilateral triangle. Show all its lines of symmetry.

8. True or false?

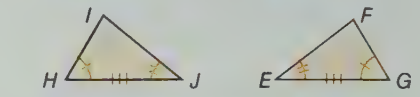
The image of any geometric figure for any rigid motion is a figure congruent to the original figure.

9. Write a statement of congruence for the triangles in each part and tell whether the SSS, SAS, or the ASA congruence theorem applies.

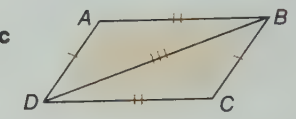
A



B

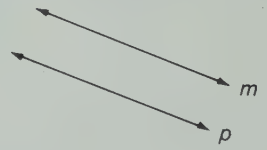


C

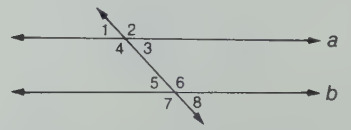


10. Line p is a translation image of line m.

What is the relation of m to p?



11. Lines a and b are parallel.

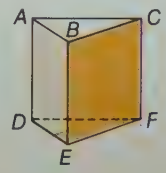


A Name all the angles that are congruent to $\angle 3$.

B Name all the angles that are congruent to $\angle 2$.

12. **A** Name the bases of the prism.

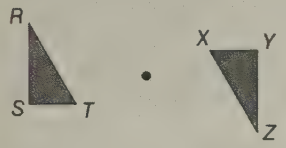
B What shape are the lateral faces of the prism?



TEST YOURSELF

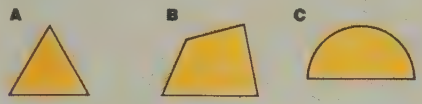
1. Which rigid motion can be thought of as a motion that moves all points of the plane the same distance and in the same direction?

2. In the figure, $\triangle RST \cong \triangle XYZ$. What single rigid motion would map $\triangle RST$ onto $\triangle XYZ$?



3. Point X is the reflection image of Y in line m. How is $\overline{XY}$ related to line m?

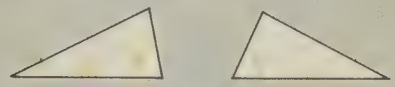
4. Which figure does not have a line of symmetry?

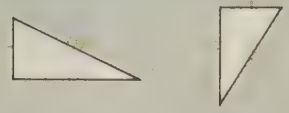


5. How many lines of symmetry does a square have?

6. Suppose $\triangle GHI \cong \triangle RST$.
   - A $\overline{HI} \cong$?
   - B $\angle G \cong$?

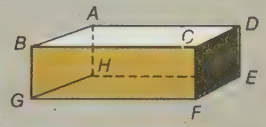
In Exercises 7, 8, and 9, some pairs of congruent angles and segments are marked. Tell which congruence theorem (SSS, SAS, or ASA) can be used to show the pair of triangles are congruent.

7. 

8. 

9.

10. This figure is a rectangular parallelepiped.



   - A Faces $ABCD$ and ? are parallel to each other.
   - B Faces $BGFC$ and $CDEF$ are ? to each other.

RESEARCH PROJECTS

A Investigate the topic of mazes. Try constructing some mazes and see if your classmates can find their way into or out of your maze. (See *The Second Scientific American Book of Puzzles and Diversions* by Martin Gardner; New York: Simon and Schuster, 1960, pp. 112–118.)

B Mathematicians think that every map in a plane can be colored with no more than four colors regardless of the number of countries on the map. However, no one has been able to prove this. Obtain some outline maps and color the states, provinces, or counties on the map with four or fewer colors. Try making your own maps and coloring them.

TEST YOURSELF

This self-evaluation test should provide students with a good measure of their understanding of the essential concepts introduced in this module. Students who demonstrate the ability to work effectively with the ideas covered here may be expected to perform well on an achievement test for the module.

Note that the correlation between the self-test items and the module objectives is indicated on the annotated student text page.

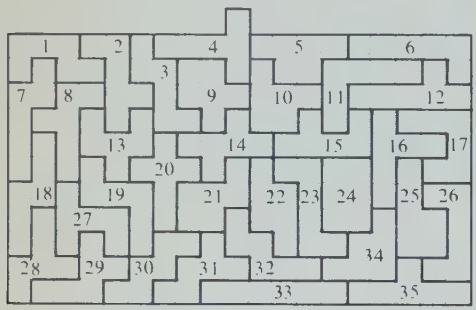
RESEARCH PROJECTS

For students interested in Project A, recommend the booklet *Puzzles and Graphs* by John N. Fujii; Washington, D.C.: National Council of Teachers of Mathematics, 1966, pp. 39–63. This book contains some excellent material on mazes as well as some challenging "Bonus Puzzles."

Students interested in Research Project B may find Martin Gardner's story "The Island of Five Colors" delightful reading. It can be found in *Fantasia Mathematica* edited by Clifton Fadiman; New York: Simon and Schuster, 1961. (Distributed in Canada by Musson Book Company, Don Mills, Ontario.)

MATHEMATICAL RECREATION

1. There are 35 distinct hexominos. These hexominos are shown in the arrangement below covering an 11 by 19 rectangle with one square unit located in the exterior of the rectangle at the center of the long side. It is not possible to construct a rectangle using all 35 hexominos.



2. There are 11 pieces which can be folded to form cubes, The numbers of these pieces are 2, 3, 4, 8, 14, 18, 20, 23, 28, 30, 31.

MATHEMATICAL RECREATIONS

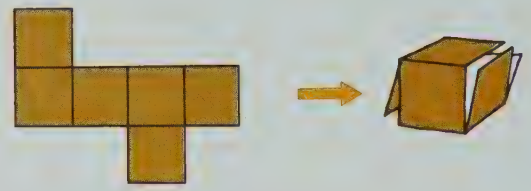
HEXOMINOS

A **hexomino** is a pattern of six connected squares.

You can cut out hexominos from graph paper.

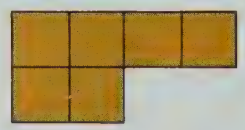
The hexomino in **A** can be folded along the segments to form a cube.

A



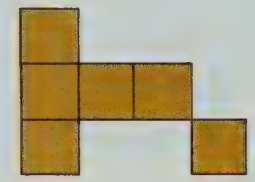
Hexomino **B** cannot be folded to form a cube.

B

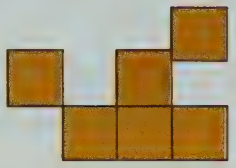


Regions like **C** and **D** are not hexominos.

C



D



1. How many different hexominos can you show on graph paper?

2. How many hexominos can be folded to form cubes?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. translation **2.** rotation **3.** $m \perp$ bisector of $\overline{XY}$
4. B **5.** 4 **6.** A $\overline{ST}$; B $\angle R$ **7.** ASA **8.** SSS
9. SAS **10.** A $HGFE$; B perpendicular

UNIT G: GEOMETRY

Module 2: Measurement

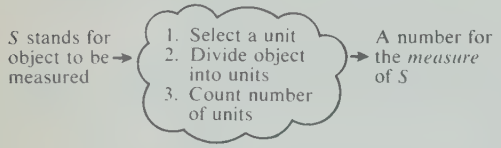
General Objectives

To review measurement concepts of length, area, and volume.
To strengthen understanding of metric units of measurement.
To review angle measurement and the sum of the measures of the angles of polygons.
To emphasize applications of the Pythagorean Theorem.
To develop various mensuration formulas, namely, areas of rectangles, parallelograms, and triangles, and volumes of prisms and pyramids.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
80	**RED** Given a line segment, the student can find the length of the segment in terms of metric units.	G-32, 33	**WB** 77 **DM** 54		
81	**RED** Given a polygon and the lengths of its sides, the student can find the perimeter of the polygon.	G-30, 31 G-32, 33	**WB** 77 **DM** 54	**SP** f-2 **DS** b-1	
82	**RED** Given a rectangular parallelogram region, the student can find the area of the figure.	G-34, 35	**WB** 78 **DM** 55	**ASC** AP-7, 8 **SP** f-3 **PD** e-2	**SWM 3** 118, 213, 256-257
83	**RED** Given a triangular region, the student can find its area.	G-36, 37	**WB** 79 **DM** 56	**ASC** AP-9 **SP** f-3 **PD** e-2	**SWM 3** 119, 210-211
84	**RED** The student can compute sums and differences of angle measures using degrees, minutes, and seconds.	G-38, 39	**WB** 80	**SP** e-1	
85	**RED** Given a right triangle and the measures of its legs, the student can find the area of the square on the hypotenuse of the triangle.	G-42, 43	**WB** 81		
86	**RED** Given a rectangular solid, the student can find its volume.	G-46, 47	**DM** 59	**ASC** AP-12	**SP** f-4
87	**YELLOW** Given the measures of some of the angles of a polygon, the student can find the sum of the measures of the angles of the polygon and the missing angle measure.	G-40, 41	**DM** 57	**SP** e-4 **PD** g-3, 4	**SWM 3** 224-225
88	**YELLOW** Given a right triangle and the lengths of two of its sides, the student can apply the Pythagorean Theorem to find the length of the third side.	G-44, 45	**DM** 58	**DS** f-3, 4	
89	**YELLOW** Given the dimensions of a prism or a pyramid, the student can find the volume.	G-50, 51	**WB** 83 **DM** 60	**ASC** AP-12, 13	**SWM 3** 304-307
90	***GREEN*** *Given the dimensions of a rectangular container, the student can express its capacity and mass using metric units.*	G-48, 49	**WB** 82		
	Reviewing the Ideas	G-52	**WB** 54		

MATHEMATICS

The measurement process can be thought of as a function. The domain of the function is a set of points; the range is a non-negative real number. The diagram below shows the basic steps in the measurement process.


S stands for object to be measured
1. Select a unit
2. Divide object into units
3. Count number of units
A number for the measure of S

Before an object can be measured, a unit must be chosen. The following illustration shows the kind of unit used in the four types of measurement discussed in this module.

Length	Area	Volume	Angle Measure
Segment	Square	Cube	Angle

Throughout history many different units have been used for measurement. In order to avoid confusion and to communicate effectively, people have come to accept standard units of measure. The majority of the people in the world use the metric system because it is extremely well suited to decimal notation.

TEACHING THE MODULE

Materials

Tracing paper; metre sticks; metric tape; centimetre rulers; centimetre graph paper; scissors; string; thumbtacks; tagboard; cardboard boxes; models of prisms and pyramids; metric scales; scientific balances; plastic food bags; cellophane tape; rice, beans, sand, or other loose material; decks of playing cards.

Vocabulary

angle
apex angle
area
base
base angles
complementary angles
cube
degree
exterior angle
gram
height
hypotenuse
legs
length
litre
mass
metre
minute
parallelogram
perimeter
prism
pyramid
Pythagorean Theorem
rectangle
second
supplementary angles
surface area
trapezoid
triangle
unit
vertical angles
volume

In teaching this module, there are three main points of emphasis:

1. The concept of measurement
2. Practice in using measurement devices
3. Development and application of the various formulas for perimeter, area, and volume.

Although students are probably familiar with the concept of measurement, it should be stressed again. Because measurement is a very practical skill, some time should be devoted to this aspect of measure. Therefore, it is essential that students actively participate in carrying out the Investigations and other measurement activities. Finally, students are expected to know and be able to apply the simple area and volume formulas presented in this module.

The metric system of measurement is emphasized in this module. It is expected that most students will have had some exposure to this system, but little background is assumed.

Lesson Schedule

You will need to allow at least 12 days for coverage of all lessons in this module, if the lessons are covered at the rate of one per day. It is quite possible, if you devote some time to field work in measurement, special projects, or extra time to particular topics, that 4 or 5 additional days might be used.

Evaluation

Paper and pencil tests on the content of this module must necessarily emphasize the mensuration formulas developed in the module.

The students' ability to use measurement devices, their ability to use estimation in measurement, and their understanding of the measurement process can be evaluated most effectively through your direct contact and observation of the students in their daily work.

RESOURCES FOR ACTIVE LEARNING

Activities

Cloudburst, Vol. 5 (applications of length, area, capacity, etc.), 5100–5500 series; (Pythagorean Theorem), 6800 series, Midwest Publications

Creative Geometry Experiments, "Soma Cube," No. 16, Midwest Publications

Cubes, Cuisenaire Co.

Developmental Math Cards, K17, L1, Addison-Wesley

ESS: Senior Balancing, "Extensions," Problem Cards S1–S7, Webster, McGraw-Hill

Experiments in Mathematics, Stage 2, pp. 40–41, Houghton Mifflin

Geoboard Activity Cards, "Angles," Geocards 28–37, Cuisenaire Co.

Let's Play Games in Metrics (games and activities coded "S"), National Textbook Co.

Measure and Find Out, Book 2, "The Metric System," pp. 4–36, Scott, Foresman

The Metric System, Addison-Wesley

Manipulative Devices

Angle indicator (marker) (LaPine; Selective Educational Equipment)

Calipers (Creative Publications)

Cubes—centimetre (1 gram mass) and colored (Addison-Wesley)

Geoboards (Addison-Wesley)

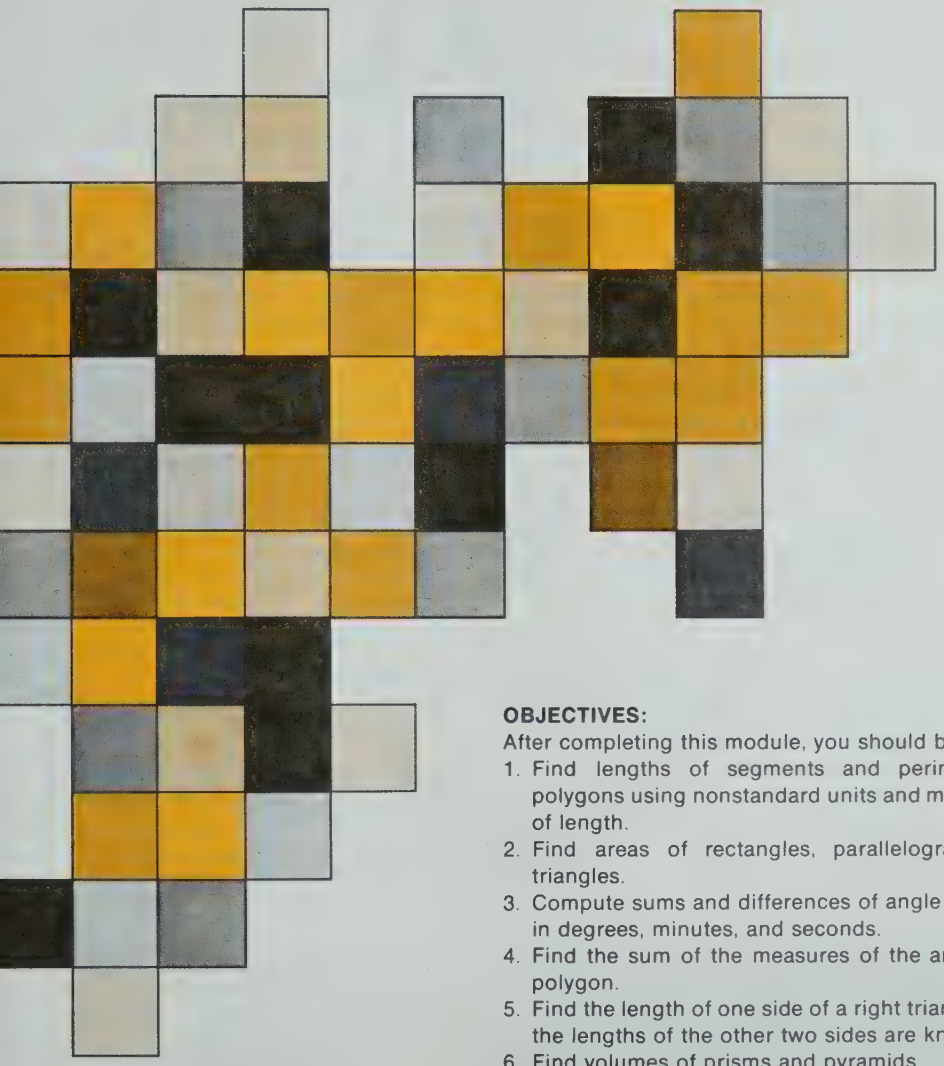
Geometric figures and solids (Creative Publications; Geyer; LaPine; Sigma)

Isometric geoboards (Selective Educational Equipment)

Metric Aids (Addison-Wesley; Balla; Creative Publications; Dick Blick; Educational Teaching Aids)

Games and Puzzles

Soma cubes and other block puzzles (Creative Publications; Edmund Scientific)



OBJECTIVES:

After completing this module, you should be able to:

1. Find lengths of segments and perimeters of polygons using nonstandard units and metric units of length.
2. Find areas of rectangles, parallelograms, and triangles.
3. Compute sums and differences of angle measures in degrees, minutes, and seconds.
4. Find the sum of the measures of the angles of a polygon.
5. Find the length of one side of a right triangle when the lengths of the other two sides are known.
6. Find volumes of prisms and pyramids.
7. Find the capacity of rectangular prisms using metric units.

Objective

The student can use nonstandard units of length to measure segments and find perimeters of polygons.

PREPARATION

Material: Tracing paper.

This lesson introduces the idea of linear measure with nonstandard units. It would be better not to use standard rulers in this lesson since the arbitrary choice of units of length is a point that should be emphasized here.

INVESTIGATION

The Investigation consists of two parts. First, the students are asked to determine how far it is from X to Y using $\overline{AB}$ as a unit. Ask students to pretend that they do not have rulers to measure either segment and let them devise some ways to solve the problem. Tracing paper should be supplied for those who want to trace $\overline{XY}$, then mark off segments congruent to $\overline{AB}$ on $\overline{XY}$. Other students may simply mark a segment congruent to $\overline{XY}$ on the edge of a piece of paper, then mark off segments congruent to $\overline{AB}$ along $\overline{XY}$.

The second part of the Investigation involves using a step or pace as a unit to estimate the width or length of the classroom. You can expect a variety of estimates and measurements.

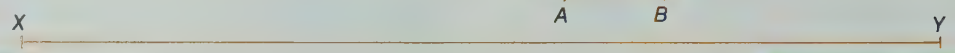
DISCUSSION

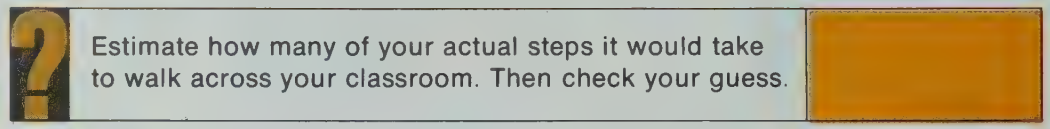
The emphasis in the Discussion should be on the measurement process: the choice of a unit segment, dividing the segment to be measured into segments congruent to the unit segment, and then counting the units. Of course, when there is not a whole number of units, the unit is subdivided into smaller units such as halves, tenths, and so on.

Measurement of Segments

Investigating the Ideas

Suppose $\overline{AB}$ represents one of your steps (or paces) when you walk. About how many steps would you take to walk from X to Y?





Estimate how many of your actual steps it would take to walk across your classroom. Then check your guess.

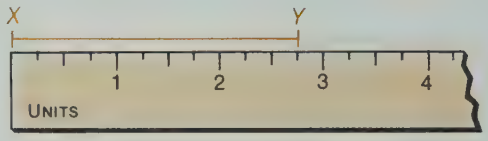
Discussing the Ideas

1. To find the **length** of a segment, we first choose a *unit segment*. Then we count the number of times the unit segment is contained in the segment to be measured. If $\overline{AB}$ is the unit segment in the Investigation, what is the length of $\overline{XY}$?

2. If one of your steps is the unit, what is the length of your classroom?

3. Why might two students get different numbers for the length of the classroom using their steps as units?

4. The length of $\overline{XY}$, **to the nearest unit**, is 3 units. We write: $\ell(\overline{XY}) = 3$ units. How can you find the length of $\overline{XY}$ **to the nearest half unit**?



5. A Use a segment as long as $\overline{MN}$ as the unit. Measure the distance across your desk top.

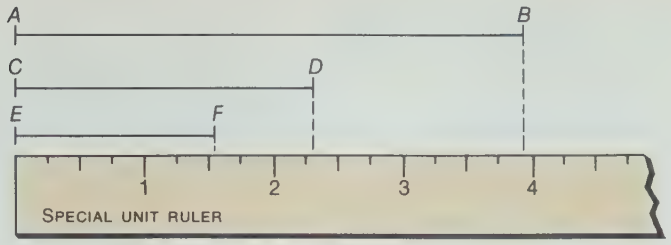
   B Divide $\overline{MN}$ into ten smaller units. Measure the distance across your desk top to the nearest tenth of a unit.

Using the Ideas

1. **A** Give the length of each segment to the nearest unit.

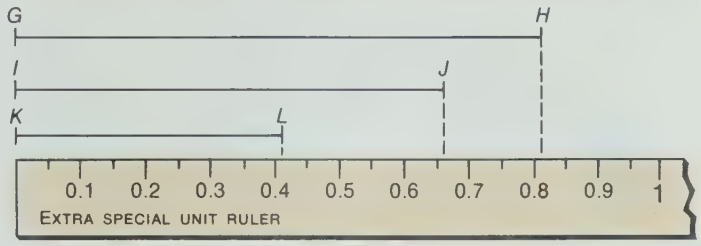
B Give the length of each segment to the nearest half unit.

C Find the sum of the lengths for part **B**.

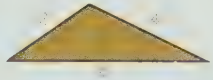


2. **A** What is the length of each segment to the nearest tenth of a unit?

B Find the sum of the lengths for part **A**.



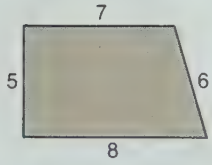
3. The **perimeter** of a region bounded by segments is the sum of the lengths of the segments bounding the region.



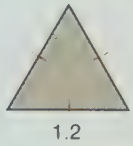
Perimeter = 3 + 6 + 4 = 13 units

Find the perimeter of each region.

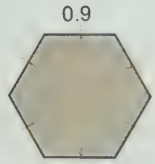
A



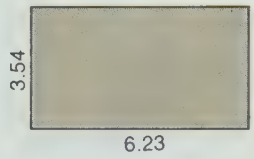
B



C



D



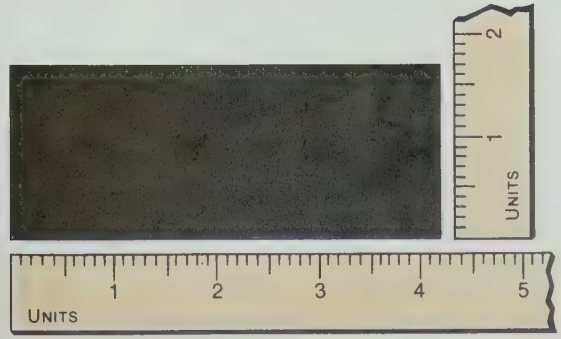
4. Give the length and width of the rectangle

A to the nearest unit.

B to the nearest half unit.

C to the nearest tenth unit.

5. What is the perimeter of the rectangle using the measurements of Exercise 4, part **A**? **B**? **C**?



UTILIZATION

The exercises involve a review of ideas concerning measurement of segments and perimeter of regions using nonstandard units. Since this is largely review, you will want to proceed through this introductory material quite rapidly.

EXTENSION

Enrichment: Suggest that the students make rulers with units of their own choice. The rulers should be constructed from tagboard or heavy paper. Students might base their units on hands' width, the length of their feet, or simply choose an arbitrary segment as a unit. Have students subdivide the units into tenths and measure various objects with the rulers.

This project should help students appreciate the need for standard units of measurement.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can give lengths of segments using metric units and can express a metric length in larger or smaller metric units.

PREPARATION

Materials: Centimetre rulers, metre sticks or metric tape measures (*Duplicator Masters*, page 88).

Discuss with the students the disadvantages of nonstandard units of length. Some students might be interested in the history of the metric system and could be asked to do some research on this topic and report their findings to the class.

INVESTIGATION

Have the students read through the Investigation section. Before carrying out the suggested measurement activity, call attention to the symbols used for *metre* (m), *decimetre* (dm), and *centimetre* (cm). Emphasize that these are *symbols*, not abbreviations, and thus require no period.

Students can use the ruler pictured in the text to make the measurement for part A. For B and C, metre sticks or metric tape measures will be needed. Give students an opportunity to record and report their measures.

DISCUSSION

Students should strive for mastery of the prefixes associated with the metric system. The prefixes deca- and hecto- are used infrequently and need less emphasis.

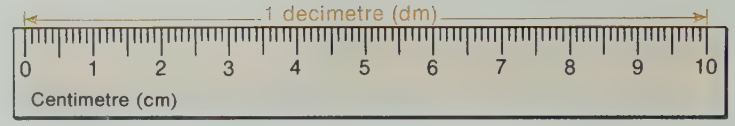
When discussing Exercise 3, students can use the finger widths in millimetres found in part A of the Investigation as examples to be expressed in larger units. Thus, if the width of a finger is reported as 15 mm, this could be expressed as 1.5 cm, 0.15 dm, or 0.015 m.

If necessary, provide additional practice in changing given measures to larger or smaller units before assigning the exercises on page G–33.

Metric Units of Length

Investigating the Ideas

The **metre** (m) is the standard unit for measuring length. Smaller or larger units are simply related to the metre by powers of ten.



Thus,

1 decimetre is 0.1 metre.	10 dm = 1 m
1 centimetre is 0.01 metre.	100 cm = 1 m
1 millimetre is 0.001 metre.	1000 mm = 1 m

Can you carry out one or more of the measurement activities below and record your results?

A Find the width of each finger on one of your hands in millimetres.

B Find your height to the nearest decimetre.

C Find the length or width of your classroom to the nearest metre.

Discussing the Ideas

1. The prefixes in the metric system tell the part or multiple of the basic unit. Give the missing number for each ▮.
   - A 1 centimetre (cm) = ▮ metre (m)
   - B 1 hectometre (hm) = ▮ m
   - C 1 decametre (dam) = ▮ m
   - D 1 kilometre (km) = ▮ m

Prefix	Meaning
milli-	one thousandth
centi-	one hundredth
deci-	one tenth
deca-	ten
hecto-	one hundred
kilo-	one thousand

2. The most frequently used units of length are the metre, kilometre, centimetre, and millimetre. Which unit or units would most likely be used to measure each of the following:
   - A The length of a hockey field.
   - B Airline distance between cities.
   - C The thickness of a coin.
   - D A person's waist.

3. What would be the width of your middle finger in centimetres if it were 19 millimetres wide?

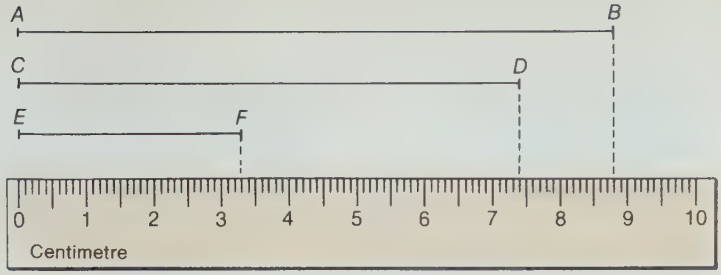
Using the Ideas

1. Give the length of each segment to the nearest

A centimetre.

B half centimetre.

C millimetre.



2. Express each measure in metres.

A 500 cm	**D** 2 km	**G** 47 dam	**J** 25 cm
B 30 dm	**E** 1.6 km	**H** 15 hm	**K** 0.001 km
C 1200 mm	**F** 227 cm	**I** 358 dm	**L** 0.1 dam

3. Express the measurement given in each statement in metres.

A Some Australian earthworms are 152 cm long.

B A giraffe may be 5480 mm in height.

C A large blue whale was 0.0276 km long.

D A large African elephant may be 366 cm tall.

E A large polar bear may be 34 dm in length.

4. Each measurement is given in metres. Express it in millimetres.

A The world's smallest antelope is only about 0.25 m tall.

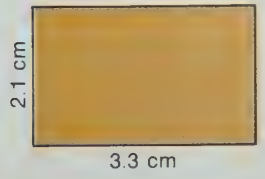
B The white-tooth shrew is about 0.038 m long.

C The eye of a giant squid is about 0.3 m in diameter.

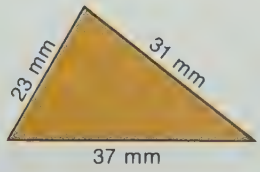
D A small hummingbird has a wingspan of about 0.005 m.

5. Find the perimeter of each region.

A



B



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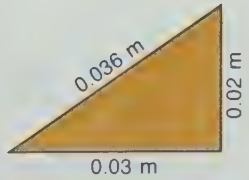


More practice, page S-22, Set 41

UTILIZATION

Assign the exercises as independent work. Exercises 3 and 4 concern measurements of some very large and some very small creatures in the animal kingdom.

EXTENSION

For additional practice exercises appropriate to this lesson, see Supplementary Exercise Set 41 on page S-22, *Workbook* page 77, and *Duplicator Masters* page 54.

Remedial and Enrichment: Prepare a worksheet containing a list of objects around the classroom or school to be measured using metric units. For those students who have had little experience with metric measurement this can be a valuable experience. A sample worksheet is shown below.

Object	Unit	Measure
Pencil thickness	mm	
Table height	cm	
Chalkboard	m	
Penny thickness	mm	
Door height	cm	
Book thickness	mm	
Gymnasium length	m	

Assignments

Minimum 1, 2, 5. Average 1–5.
Maximum 1–5.

Objective

The student can find the area of rectangular or parallelogram regions by using appropriate formulas.

PREPARATION

Materials: Centimetre graph paper (*Duplicator Masters*, page 83).

No special preparation is required before students undertake this Investigation.

INVESTIGATION

This activity provides an effective way to introduce the concept of area. The number of unit squares (the squares of the graph paper) within the outline of the hand is the *area* of the region inside the outline. To determine the total number of squares, the students should combine the partial squares that were covered by the hand to obtain an approximation of a whole number of squares. Students will be interested in comparing results to determine who has the largest hand and who has the smallest hand.

DISCUSSION

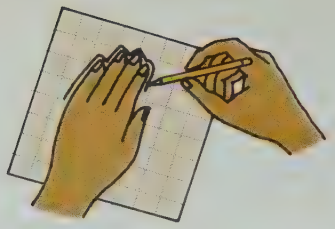
You might use students' hand outlines to motivate the introduction of the area formula for rectangles. For example, ask, "What is the largest rectangle you can show that is completely contained in your hand's outline?" "How many squares are in it?" "How can you find the area of this rectangle quickly?"

It is expected that almost all students will be able to recall and use the area formula for rectangles: $A = lw$.

Exercise 4 illustrates the concept of "conservation of area." The total area of the odd-shaped region is the same as the rectangle formed by reassembling the two pieces. This idea is vital to the development of the formula for the area of parallelograms.

Investigating the Ideas

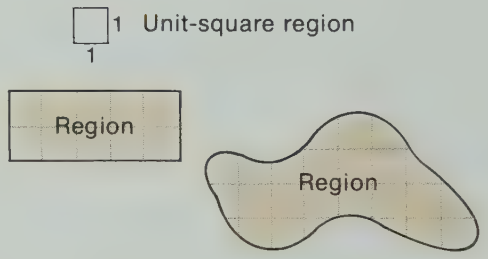
Place your hand on a sheet of graph paper and trace around it.



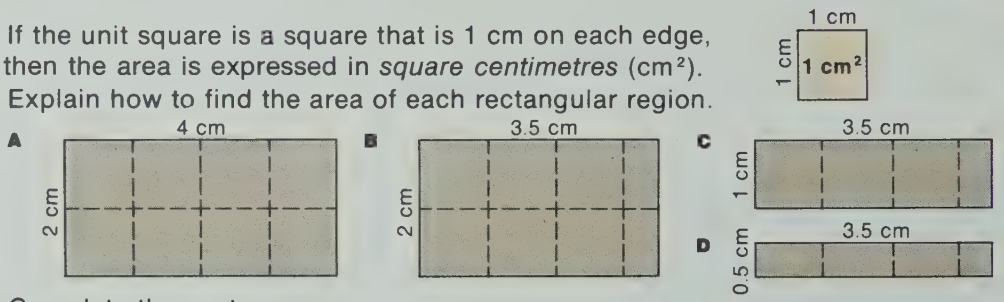
About how many squares does your hand cover?

Discussing the Ideas

1. To find the **area** of a region, we think of dividing it into *unit-square regions* and then counting the number of units. Even if the borders are curved, we still "think" of counting unit-square regions to find the area. What is the area of your hand using the square region of the graph paper as the unit?



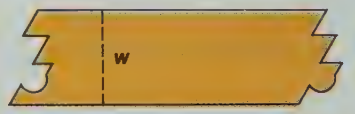
2. If the unit square is a square that is 1 cm on each edge, then the area is expressed in *square centimetres* (cm²). Explain how to find the area of each rectangular region.



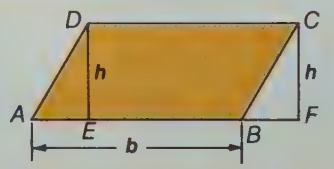
3. Complete the sentence.

The formula for the area A of a rectangle with length ℓ and width w is ?.

4. "Cutting apart" and "gluing together" can sometimes help you to find area. Use this figure to explain this idea.



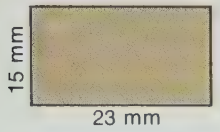
5. Parallelogram *ABCD* has a **base** of length b and **height** of length h.
   - A What is the length and width of rectangle *DEFC*?
   - B Using the letters b and h, give a formula for the area of the parallelogram.
   - C What is the area if $b = 3.7$ cm and $h = 1.1$ cm?



Using the Ideas

1. Find the area of each rectangular region.

A



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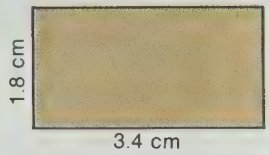

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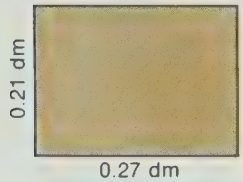
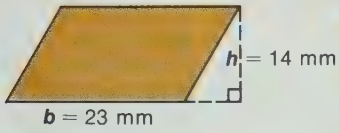


2. A rectangular room is 4.7 metres long and 4.4 metres wide. What is the area of the room?

3. What would it cost to carpet the floor of the room in Exercise 2 if carpeting costs $11.98 per square metre?

4. What is the area of each parallelogram?

A



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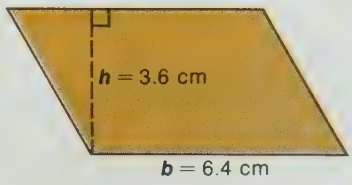

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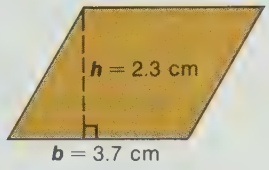
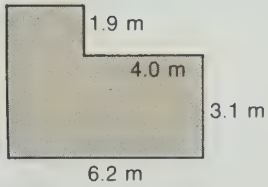


5. Find the area of each region.

A



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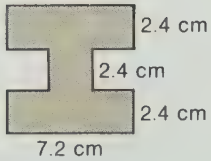
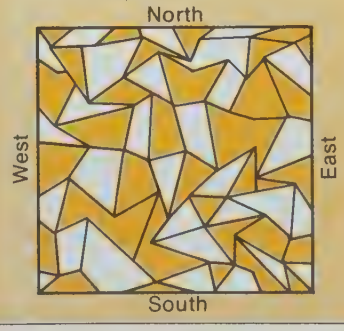


6. A house has a rectangular base that is 22.4 metres long and 18.9 metres wide. The house is located on a rectangular lot 53.6 metres long by 32.6 metres wide.

- **A** What is the area of the base of the house?
- **B** What is the area of the lot?
- **C** What is the area of the part of the lot not occupied by the house?

think

What is the smallest number of "fences" you can cross in going from the west side to the east side of the square if you cannot cross at any of the points of intersection?



More practice, page S-22, Set 42

UTILIZATION

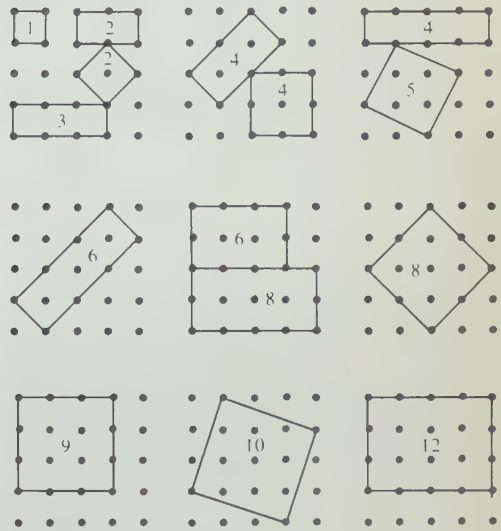
Encourage students to use the proper symbols to denote area. The symbols m^2, cm^2, mm^2 represent square metres, square centimetres, and square millimetres, respectively.

EXTENSION

To provide appropriate further practice, make selective assignments from Supplementary Exercise Set 42 on page S-22, *Workbook* page 78, *Duplicator Masters* page 55, and *Arithmetic Skill Cards* AP-7 and AP-8.

Enrichment: Ask students to find how many rectangles or squares with different areas they can show on a 5 by 5 nail geoboard where the unit of area is one of the small geoboard squares.

Solution: Areas for each whole number from 1 to 16 with the exception of 7, 11, 13, 14, and 15 are possible. Samples are shown below.



(Of course, the 16 unit square area consists of all the geoboard units.)

Assignments

Minimum 1–4. Average 1–5. Maximum 1–6.

Objective

Given the base (b) and height (h) of a triangle, the student can find the area of the triangle by applying the formula $A = \frac{1}{2}bh$.

PREPARATION

Materials: Paper, scissors, and rulers.

Review the area formula for rectangles before students begin the Investigation.

INVESTIGATION

If you prefer, students can draw their triangles on graph paper and use the squares on the graph paper as unit squares. If other paper is used, students will need to measure length and width of the rectangle found and use the area formula.

The triangular region should be folded so that the first fold brings the vertex of the largest angle to the base, with the fold line parallel to the base. This will avoid the problem of the vertex falling outside the base, even though the rectangle can be formed by additional folds when this is the case. Clearly the area of $\triangle ABC$ must be twice the area of rectangle $GDEF$ because of the double thickness of the paper formed by the folds.

DISCUSSION

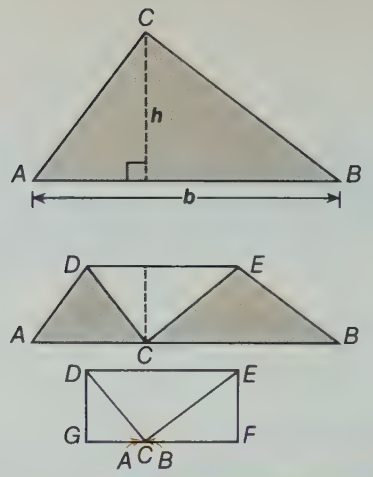
Exercises 1 and 2 develop the area formula for a triangle based on the results of the Investigation. Exercise 3 can be used as an alternate approach in the development of the formula.

When discussing Exercise 4, emphasize that either leg of a right triangle can be considered as the base and the other leg the height. Therefore, the area of a right triangle is half the product of the lengths of the legs.

Areas of Triangles

Investigating the Ideas

Draw a large triangular region on a sheet of paper. Let $\angle C$ be the largest angle of the triangle. Let ***h*** be the height of the triangle to the base ***b***.
Cut out the region.
Fold vertex C to base ***b*** so that the fold line $\overline{DE}$ is parallel to $\overline{AB}$.
Fold vertices A and B to point C.
Measure rectangle $DEFG$ and compute its area.



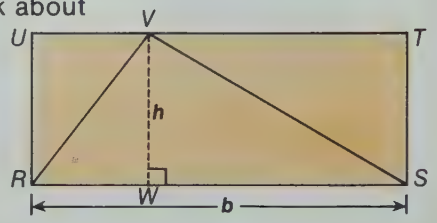
Can you determine the area of the original triangle by knowing the area of rectangle $DEFG$?

Discussing the Ideas

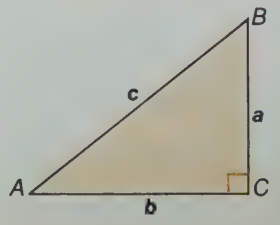
1. **A** Explain why the width of rectangle $DEFG$ is $\frac{1}{2} \cdot h$, which is the height of $\triangle ABC$.
   B Why is the length of the rectangle $\frac{1}{2} \cdot b$, where ***b*** is the base of $\triangle ABC$?

2. Since the area of rectangle $DEFG$ is $\left(\frac{1}{2} \cdot b\right) \cdot \left(\frac{1}{2} \cdot h\right) = \frac{1}{4}bh$, why is the formula for the area ***A*** of the triangle $A = \frac{1}{2}bh$?

3. The figure below suggests another way to think about the area of a triangle.
   A What is the area of rectangle $RSTU$?
   B Find two pairs of congruent triangles in the figure.
   C Explain the formula: Area $(\triangle RSV) = \frac{1}{2} \cdot b \cdot h$



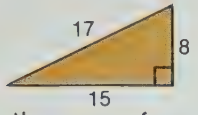
4. In right $\triangle ABC$, $\angle C$ is a right angle, ***a*** and ***b*** are legs, and ***c*** is the *hypotenuse*.
   A If ***b*** is chosen as the base, what is the height?
   B If ***a*** is the base, what is the height?
   C What rule can you state for finding the area of a right triangle?



Using the Ideas

1. Find the area of each triangular region.

A



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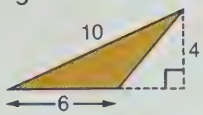

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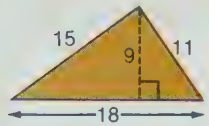
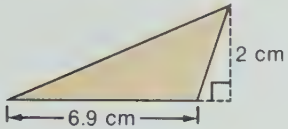


2. Give the area of each triangular region.

A



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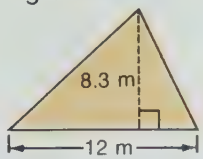

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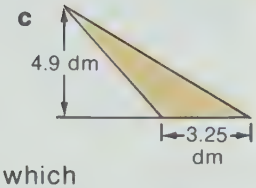
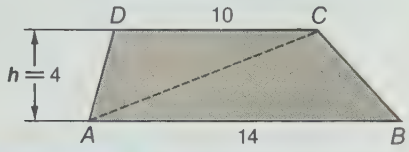
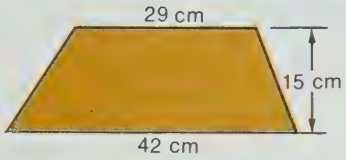


3. A **trapezoid** is a quadrilateral with a pair of parallel sides which are called the **bases** of the trapezoid.
   - A Find the area of $\triangle ABC$.
   - B Find the area of $\triangle ADC$.
   - C Find the area of the trapezoid $ABCD$.

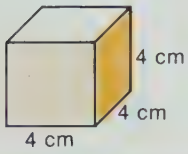


4. Use the method suggested in Exercise 3 to find the area of the trapezoid shown at the right.



5. The **surface area** of a space figure is the sum of the areas of each face of the figure. Find the surface area of each figure.

A



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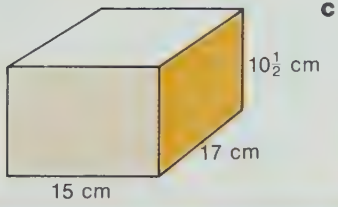
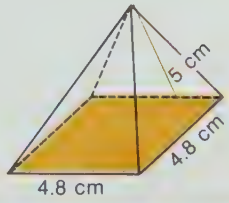


C



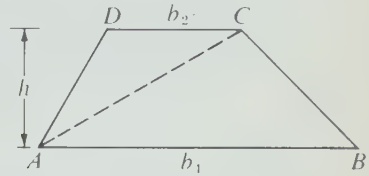
think

Think of any number except zero. Add 5 to it and square the sum. Subtract 16 and remember the resulting number. Call this number *A*. Now subtract 3 from the original number and square the difference. Subtract the square from *A*. Divide the resulting number by the original number. What number do you get as the answer? Try this procedure again. What is your answer this time? Do you think the result will always be the same?

More practice, page S-23, Set 43

UTILIZATION

The area of a trapezoid is found by using areas of triangles in Exercises 3 and 4; the area formula for trapezoids, $A = \frac{1}{2} h \cdot (b_1 + b_2)$, is not given. However, you may want to show the development to some students.



Area $(\triangle ABC) = \frac{1}{2} b_1 h$; Area $(\triangle ADC) = \frac{1}{2} b_2 h$
Area (Trapezoid $ABCD$) =
$\frac{1}{2} b_1 h + \frac{1}{2} b_2 h = \frac{1}{2} h(b_1 + b_2)$

Exercise 5 defines *surface area* of space figures. This extension of area to space figures should present little difficulty for most students.

EXTENSION

For students who would profit from further practice with the concepts of this lesson, provide assignments from Supplementary Exercise Set 43 on page S-23, *Workbook* page 79, *Duplicator Masters* page 56, or *Arithmetic Skill Card* AP-9.

Think Solution

The results of the operations given can easily be shown algebraically:

Let n = the original number (not zero)

$(n + 5)^2 - 16 = A$

$(n - 3)^2$ is to be subtracted from A and the result divided by n. Thus,

$$\frac{[(n+5)^2 - 16] - (n-3)^2}{n}$$

$$= \frac{(n^2 + 10n + 25 - 16) - (n^2 - 6n + 9)}{n}$$

$$= \frac{16n}{n} = 16$$

Assignments

Minimum 1, 2. Average 1, 2, 5. Maximum 1–5.

Objective

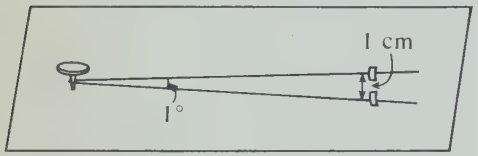
The student can find sums and differences of angle measures which are expressed in degrees, minutes, and seconds.

PREPARATION

Materials: String, thumbtacks, tape.

Prepare a model that illustrates an angle whose measure is 1 degree (1°) by following these steps:

A. Cut a piece of string or thread so that it is 114.6 cm long.
B. Find the middle of the string. Use a thumbtack to fasten the middle of the string to a piece of cardboard.
C. Pull the ends of string tight and place them 1 cm apart. Tape the ends of the string to the cardboard. The size of the angle is very close to 1 degree.



DISCUSSION

In this lesson, the student is given an opportunity to review the basic meaning of angle measure using whole numbers and then to extend it to include the idea of measuring in minutes and seconds. Note that the *degree* is defined in terms of a right angle, not vice versa. The model suggested in the preparation section can be effectively used to help students visualize an angle of 1°. To emphasize the smallness of 1 minute, you might point out that a model of a 1-minute angle would have to have sides about 34.4 metres long in order for its opening to be 1 cm. A model of a 1-second angle would have to have sides a little more than 2 kilometres long in order to have a separation of 1 cm.

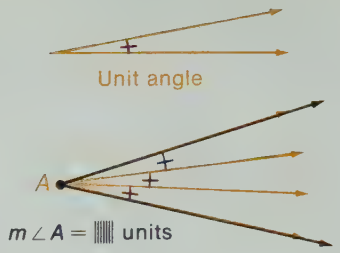
Exercises 4 and 5 provide examples of addition and subtraction of angle measures. You may need to supply additional examples for student practice before assigning the exercises on page G–39.

Angle Measurement

Discussing the Ideas

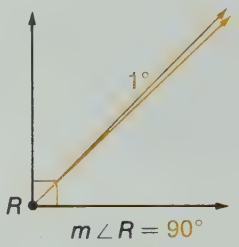
1. To measure an angle, first choose a unit angle. Then, on the given angle, "stack up" adjacent angles congruent to the unit angle. Finally, count the number of units.

   Using the given unit angle, what is the measure of $\angle A$?



2. Although a unit angle of any size may be used, the standard unit is the angle of one **degree** (1°). The size of a degree is such that the measure of a right angle is 90 degrees.

   Estimate the measure of each angle below in degrees.



3. When units smaller than one degree are needed, the unit angle of 1° is divided into 60 congruent angles, each called a **minute** (1′), and each angle of 1′ is divided into 60 congruent subunits called **seconds** (1″).

   60 minutes (60′) = 1°
   60 seconds (60″) = 1′

   What part of a degree is each of the following?

   A 30′ **B** 15′ **C** 10′ **D** 6′ **E** 1′ **F** 30″ **G** 15″

4. The example shows how sums of angle measures are found.

   Explain how 52° 105′ 81″ was rewritten as 53° 46′ 21″.

   EXAMPLE:

$$\begin{array}{r} 37^\circ\ 48'\ 35'' \\ +\ 15^\circ\ 57'\ 46'' \\ \hline 52^\circ\ 105'\ 81'' \end{array} = 52^\circ\ 106'\ 21'' = 53^\circ\ 46'\ 21''$$

5. Explain each step in the subtraction example below.

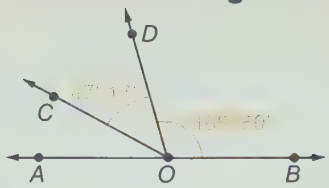
   EXAMPLE:

$$\begin{array}{r} 37^\circ\ 48'\ 35'' \\ -\ 15^\circ\ 57'\ 46'' \end{array} \rightarrow \begin{array}{r} 37^\circ\ 47'\ 95'' \\ -\ 15^\circ\ 57'\ 46'' \end{array} \rightarrow \begin{array}{r} 36^\circ\ 107'\ 95'' \\ -\ 15^\circ\ \ 57'\ 46'' \\ \hline 21^\circ\ \ 50'\ 49'' \end{array}$$

Using the Ideas

1. If $\angle AOB$ is a straight angle, what is the measure of each of the following angles?

A $\angle COB$ **B** $\angle COA$ **C** $\angle AOD$



2. Two angles are **complementary** if the sum of their measures is 90°. What is the measure of an angle which is complementary to an angle whose measure is 70° 19′?

3. Two angles which are **supplementary** must have the sum of their measures equal to 180°. What is the measure of an angle which is supplementary to an angle whose measure is 125° 38′ 15″?

4. Angles 1 and 2 are **vertical angles**. Angles 3 and 4 are also vertical angles. The $m\angle 1$ is 30°.

A What is $m\angle 2$? **B** What is $m\angle 1 + m\angle 3$?

C What is $m\angle 3$? **D** What is $m\angle 4$?

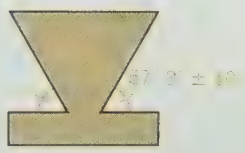
E Is it true that vertical angles have the same measure?

5. A ray of light striking a mirror is reflected so that the ray makes congruent angles with the mirror. Thus, $\angle APC \cong \angle BPD$. If $m\angle APC$ is 23° 40′, what is $m\angle APB$?

6. A machinist must grind the angle on a dovetail piece as shown in the figure.

A What is the greatest measure the angle may have?

B What is the smallest measure the angle may have?



7. Find the sums of the angle measures.

A 72° 49′ 31″ + 38° 6′ 19″

B 64° 57′ 26″ + 82° 14′ 38″

C 152° 50′ 10″ + 27° 9′ 50″

D 35° 17′ 43″ + 91° 22′ 30″

8. Find the differences of the angle measures.

A 84° 53′ 46″ − 25° 39′ 37″

B 90° − 38° 17′ 40″

C 111° 11′ 11″ − 22° 22′ 22″

D 142° 19′ 25″ − 106° 34′ 17″

More practice, page S-23, Set 44

UTILIZATION

Complementary and *supplementary* angles are discussed in terms of angle measure in Exercises 2 and 3. When finding measures of complements or supplements of angles involving degrees, minutes, and seconds, students should rewrite 90° as 89° 59′ 60″ and should rewrite 180° as 179° 59′ 60″.

When discussing Exercise 4, you will want to emphasize that *vertical angles* exist in pairs and that a given pair of vertical angles must have the same measure.

EXTENSION

Suitable additional practice assignments may be made from Supplementary Exercise Set 44 on page S-23 and *Workbook* page 80.

Enrichment: Some students may be interested in doing some research on the geometry of a billiard table. See *Mathematics, A Human Endeavor* by Harold R. Jacobs; San Francisco: W. H. Freeman and Company, 1970, pp. 2–16.

Assignments

Minimum 7, 8. Average 1–3, 7, 8. Maximum 1–8.

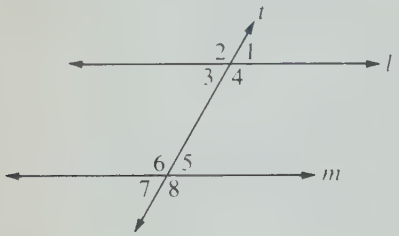
Objective

The student can compute the sum of the angle measures of a convex polygon.

PREPARATION

Materials: Heavy paper or tagboard, rulers, scissors.

Draw parallel lines l and m with transversal t. Have students identify the pairs of congruent angles. In particular, encourage students to observe that $\angle 3 \cong \angle 5$ and $\angle 4 \cong \angle 6$.



INVESTIGATION

If students copy the angles of the triangle carefully, side by side with the same vertex, they will observe that regardless of the shape of the triangle, the two outside, nonadjacent rays will fall along a line. (See student page annotation.) This suggests the theorem that the sum of the angle measures of any triangle is 180°.

DISCUSSION

Exercise 2 suggests a proof of the angle-sum theorem for triangles. Most students should be able to understand the reasoning of this problem but should not be expected to reproduce it.

Exercise 4 has useful applications. If the measures of two angles of a triangle are known, the measure of the third angle can be found by subtracting the sum of the measures of the other two angles from 180°.

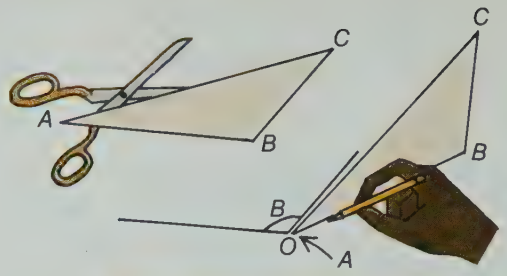
Exercises 5 and 6 illustrate how convex polygons can be divided into triangles by arbitrary segments drawn from one vertex to another. Since a polygon having n sides can be divided into $(n - 2)$ triangles by this method, the sum of the angle measures of any convex polygon must be $(n - 2) \cdot 180°$.

Sums of Angle Measures of Polygons

Investigating the Ideas

Draw a triangle on a piece of cardboard and cut out the triangular region.

Mark a point O on your paper. Then use the triangular region to copy each of its three angles side-by-side so that each of the vertices is at point O.

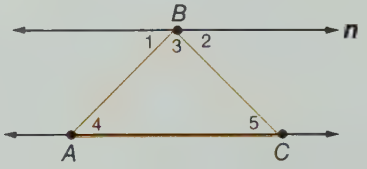


Can you try the Investigation with several triangles of different shapes?

Discussing the Ideas

1. What does the Investigation suggest about the three angles of any triangle?

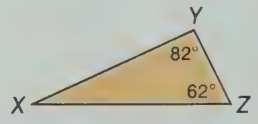
2. Suppose that $n \parallel \overleftrightarrow{AC}$.



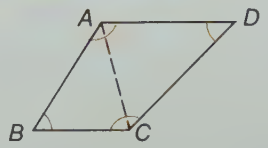
   - **A** Is $\angle 1 \cong \angle 4$?
   - **B** Is $\angle 2 \cong \angle 5$?
   - **C** What is the sum $m\angle 1 + m\angle 2 + m\angle 3$?
   - **D** Explain why $m\angle 3 + m\angle 4 + m\angle 5 = 180°$?

3. Complete: The sum of the measures of the angles of every triangle is ?.

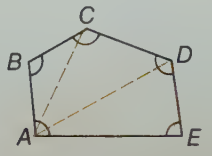
4. Explain how to find $m\angle X$ in $\triangle XYZ$.



5. **A** What is the sum of the measures of the angles of $\triangle ABC$?
   - **B** What is the sum of the measures of the angles of $\triangle ACD$?
   - **C** What is the sum of the measures of the angles in any convex quadrilateral?



6. Explain how to find the sum of the angle measures of a pentagon.



Using the Ideas

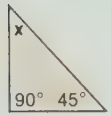
1. Give the measure for **x** in each triangle.

A

B

C

D



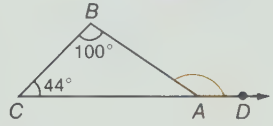
2. The measures of two angles of a triangle are given. Find the measure of the third angle.

A 45°, 45°	**C** 43°, 63°	**E** 27° 23′, 59° 8′	**G** 15° 28′, 69° 28′
B 50°, 50°	**D** 75°, 28°	**F** 14° 50′, 98° 15′	**H** 60° 4′ 5″, 36° 25″

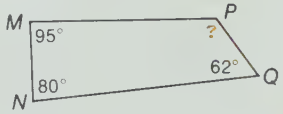
3. What is the measure of each angle of an equilateral triangle?

4. The **apex angle**, $\angle R$, of an isosceles triangle has a measure of 96°. What are the measures of the **base angles**?

5. An **exterior angle** at vertex A of $\triangle ABC$ is $\angle BAD$. What is $m\angle BAD$?



6. What is $m\angle P$ of quadrilateral $MPQN$?



7. **A** What is the sum of the measures of the five angles of a regular pentagon?

B What is the measure of each angle?

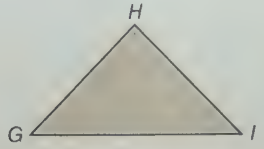
8. **A** What is the sum of the measures of the angles of a regular hexagon?

B What is the measure of each angle?

9. **A** What is the sum of the measures of the angles of a regular decagon?

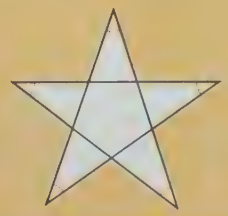
B What is the measure of each angle?

10. In $\triangle GHI$, $m\angle H$ is twice $m\angle G$, and $m\angle I$ is 5° more than $m\angle G$. What are the measures of the angles?



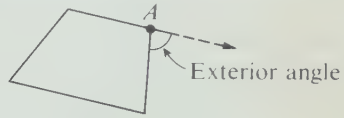
think

What is the sum of the measures of the angles at the tips of this regular 5-pointed star?



UTILIZATION

Exercise 4 introduces the term *apex angle*. This is the angle included by the congruent sides of an isosceles triangle. In discussing Exercise 5, encourage students to observe that the measure of the exterior angle at any vertex of a triangle is the sum of the measures of the other two angles. An exterior angle of a polygon is formed by extending one side of the polygon through a vertex and the other side of the polygon containing that vertex.



Solution, Exercise 10

10. Since the other two angles are compared with $\angle G$, let $m\angle G = n$. Then $m\angle H = 2n$, $m\angle I = n + 5$.

$$n + (2n) + (n + 5) = 180$$
$$4n = 175$$
$$n = 43° 45' \quad (\angle G)$$
$$2n = 87° 30' \quad (\angle H)$$

and $n + 5 = 48° 45' \quad (\angle I)$

EXTENSION

Additional practice exercises are provided on *Duplicator Masters* page 57.

Think Solution

An object like a match or toothpick can be used to find this solution. Start at the side of any angle, rotate the match through the angle, and then slide the match along this terminal side to the next angle. Continue this process until all angles have been covered. The match will rotate through a total of 180°.

Another solution involves a knowledge of the fact that the sum of the measures of the angles of a pentagon (figure formed inside the star) is 540°. Working from this fact to the sets of supplementary angles formed between the pentagon angles and triangle angles, the angle sum can be found to be 180°.

Assignments

Minimum 1–4. Average 1–8. Maximum 1–10.

Objective

The student can find the area of the square on the hypotenuse of a right triangle given the lengths of the legs of the triangle.

PREPARATION

Materials: Graph paper (*Duplicator Masters*, page 83).

On the overhead projector or chalkboard, show some square regions like the one pictured below. Discuss ways of finding the area of these squares.

INVESTIGATION

Have students read the Investigation, and make sure that everyone understands the activity to be carried out. Call attention to Discussion Exercise 1 and suggest that students make a chart like this and complete it for the Investigation.

DISCUSSION

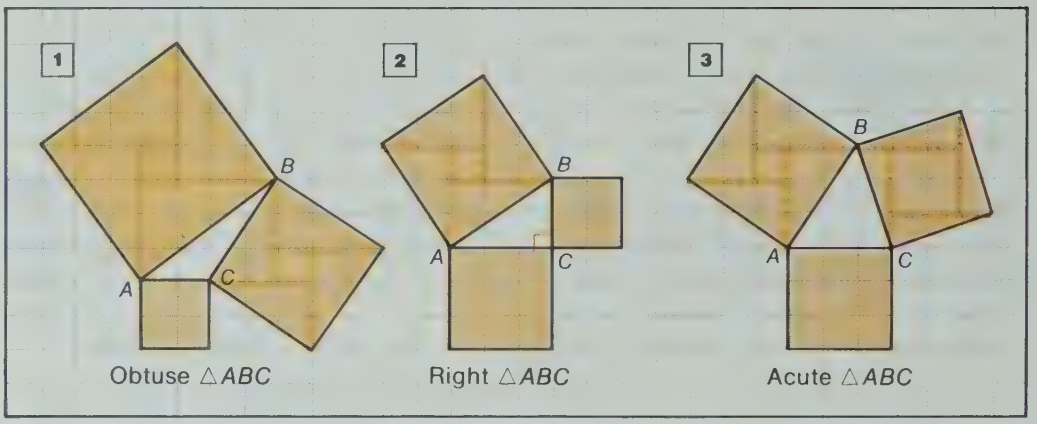
Students should discover from the Investigation that only in the case of a right triangle did the area of the square on $\overline{AB}$ (side opposite $\angle C$) equal the sum of the areas of the squares on the other two sides. If possible, have the students repeat the Investigation with obtuse, right, and acute triangles of their own choice. They should observe that for an obtuse triangle the sum of the areas of the squares on the two shorter sides *is less than* the area of the square on the other two sides. For the acute triangle, the sum of the areas on the two shorter sides *is greater than* the area of the square on the longest side. The Pythagorean Theorem deals with the case of *equality*, which exists when the triangle is a right triangle.

In this lesson and the next the legs of right triangles will be labeled a and b consistently, while the hypotenuse will be labeled c, so the formula for Pythagorean Theorem can always be written as $a^2 + b^2 = c^2$.

The Pythagorean Theorem

Investigating the Ideas

In the figures below, squares have been constructed on the three sides of obtuse, right, and acute triangles.



Can you find and record the area of each square in each of the figures?

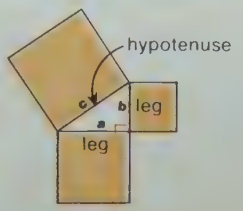
The dotted segments may help you count the unit squares.

Discussing the Ideas

1. Copy and complete the table.

	Area of square on AB	Area of square on AC	Area of square on BC	Sum of the areas of squares on AC and BC
Obtuse △ABC	25	4	13	
Right △ABC				
Acute △ABC				

2. The Investigation suggests a generalization for right triangles called the **Pythagorean Theorem.**

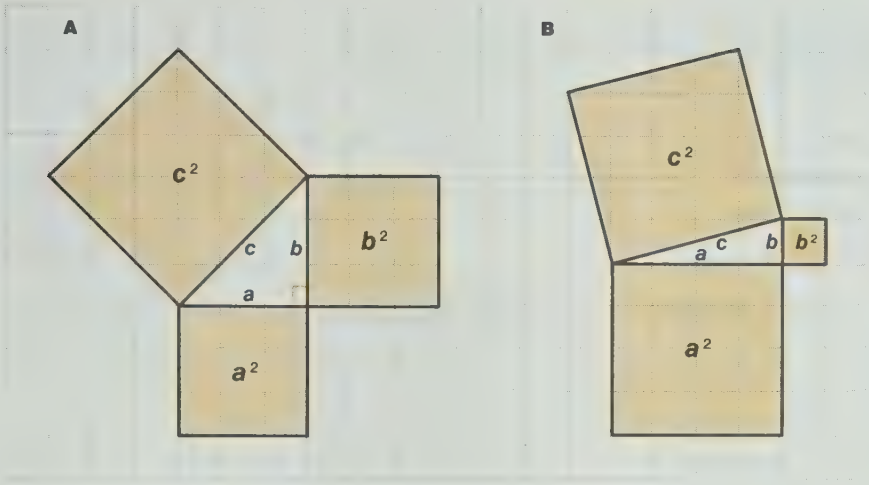


In any right triangle, the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.

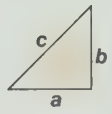
What equation does the statement above suggest if a right triangle has legs of length ***a*** and ***b*** and hypotenuse of length **c**?

Using the Ideas

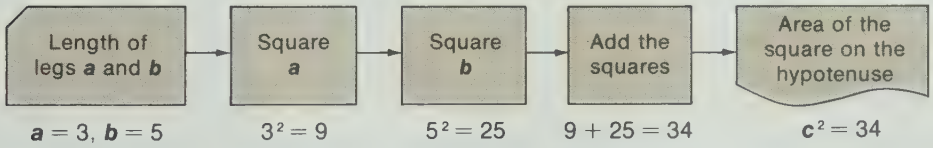
1. Find the area of each square in each figure. Unit squares are indicated by the grid.



For Exercises 2 through 10, find the area of the square on the hypotenuse of each right triangle with legs of length ***a*** and ***b***. Study the flow chart.



$a^2 + b^2 = c^2$



2. $a = 6$ $b = 3$

3. $a = 1$ $b = 1$

4. $a = 4$ $b = 5$

5. $a = 3$ $b = 4$

6. $a = 0.6$ $b = 0.8$

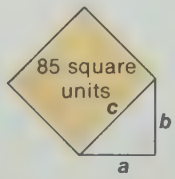
7. $a = 10$ $b = 10$

8. $a = 1\frac{1}{2}$ $b = 2$

9. $a = 5.4$ $b = 6.5$

10. $a = 12.5$ $b = 8.2$

★ **11.** The area of the square on the hypotenuse of a right triangle is 85 square units. What are the possible lengths of legs ***a*** and ***b*** if both ***a*** and ***b*** are whole numbers? (HINT: There is more than one possibility.)



UTILIZATION

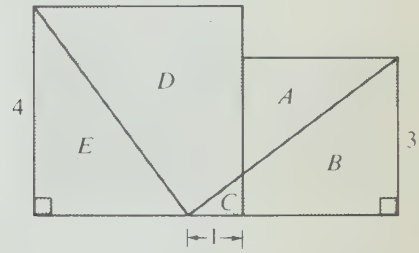
When students have completed the exercises, have selected students place their solutions for Exercises 2 through 10 on the chalkboard.

EXTENSION

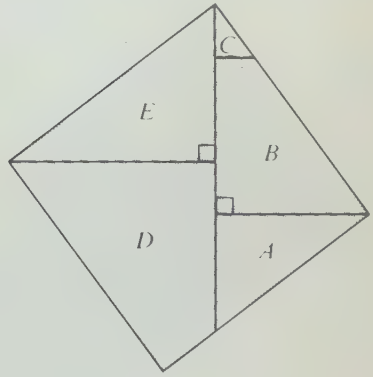
Workbook page 81 offers exercises that may be used for extra practice assignments.

Enrichment: Challenge students to solve this dissection problem, which is based on the Pythagorean Theorem.

Draw a 4-cm square and a 3-cm square side by side. Cut along the lines and boundaries shown in the figure.



1. Use the five regions to reconstruct the two squares.
2. Use all five regions to form one square 5 cm on each side. (See solution below.)



Assignments

Minimum 1–7. Average 1–10. Maximum 2–11

Objective

Given the lengths of two sides of a right triangle, the student can find the length of the third side.

PREPARATION

Review the Pythagorem Theorem and the formula $a^2 + b^2 = c^2$.

INVESTIGATION

The Investigation takes the ideas of the previous lesson one step farther. Since the sum of the areas of the squares on legs a and b is 100, the area of the square on side c is 100 square units; therefore the length of hypotenuse c is 10 units.

DISCUSSION

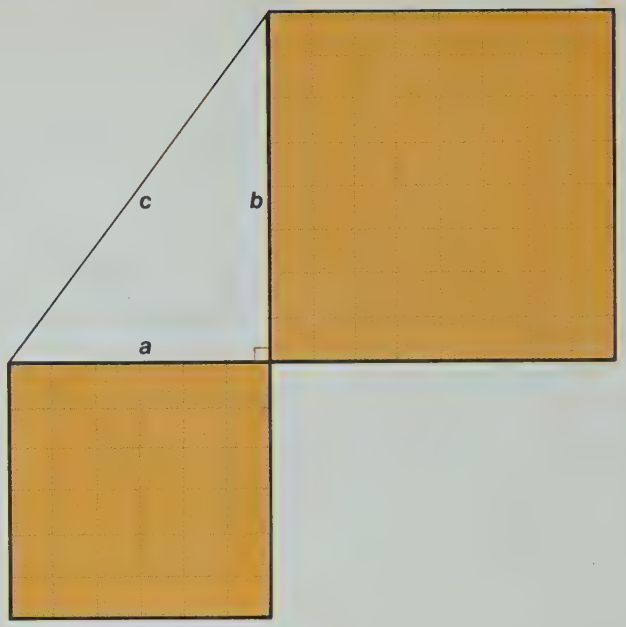
Exercise 2 illustrates how the formula for the Pythagorean Theorem may be applied to find the length of the hypotenuse. Exercise 3 shows how the formula may be applied to determine the length of a leg of a right triangle. Give other examples if needed, but use care that lengths of all sides are whole numbers or simple rational numbers such as $\frac{3}{8}$, $\frac{1}{2}$, and $\frac{5}{8}$ so that $(\frac{3}{8})^2 + (\frac{1}{2})^2 = (\frac{5}{8})^2$. Avoid examples which would introduce irrational numbers.

Using the Pythagorean Theorem

Investigating the Ideas

What is the area of each of the squares on legs ***a*** and ***b***?

What is the area of the square on hypotenuse **c**?



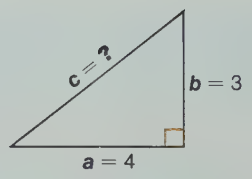
Can you find the length of the hypotenuse of this right triangle?

Discussing the Ideas

1. Explain how you found the length of the hypotenuse of the right triangle in the Investigation.

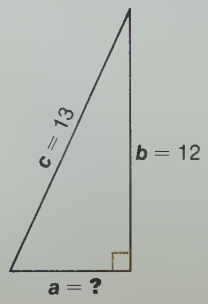
2. Study and complete the solution for finding the length of the hypotenuse of the right triangle.

$$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2$$
$$\mathbf{c}^2 = 4^2 + 3^2$$
$$\mathbf{c}^2 = 16 + 9$$
$$\mathbf{c}^2 = 25$$
$$\mathbf{c} = \text{▮▮▮}$$



3. Study and complete the solution for finding the length of leg ***a*** of this right triangle.

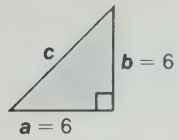
$$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2$$
$$13^2 = \mathbf{a}^2 + 12^2$$
$$169 = \mathbf{a}^2 + 144$$
$$169 - 144 = \mathbf{a}^2$$
$$\text{▮▮▮} = \mathbf{a}^2$$
$$\text{▮▮▮} = \mathbf{a}$$



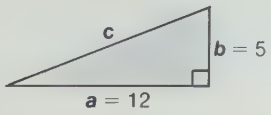
Using the Ideas

1. Find the length of hypotenuse **c** for each right triangle.

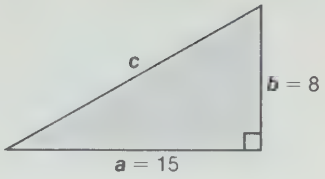
A



B

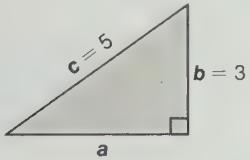


C



2. Find the length of leg ***a*** or ***b*** of each right triangle.

A



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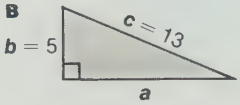
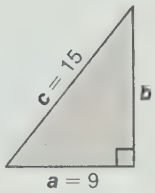
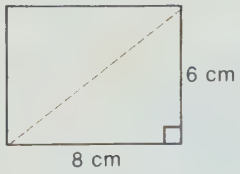


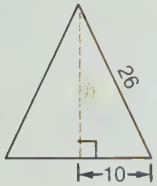
C



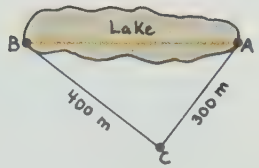
3. How long is the diagonal of a rectangle if the length is 8 cm and the width is 6 cm?



4. What is the height ***h*** of this triangle?

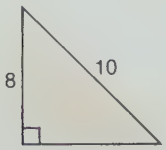


5. Point *C* is chosen so that $\angle ACB$ is a right angle. How far is it across the lake from *A* to *B* if the lengths of $\overline{AC}$ and $\overline{BC}$ are 300 m and 400 m respectively?

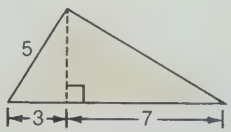


6. Find the area of each region. (HINT: You must first use the Pythagorean Theorem to find the third side of a triangle in each figure.)

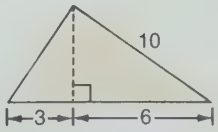
A



B



C



7. Find the length of the third side of each right triangle.

A $a = 12$ $b = 16$

B $a = 1.2$ $c = 1.3$

C $a = 7$ $b = 24$

D $a = 40$ $c = 41$

E $b = 30$ $c = 34$

F $a = 24$ $c = 26$

More practice, page S-24, Set 45

UTILIZATION

Assign the exercises according to the needs of your students. It will be helpful to have a number of them demonstrated at the chalkboard.

EXTENSION

For additional appropriate practice, see Supplementary Exercise Set 45 on page S-24 and *Duplicator Masters* page 58.

Enrichment: Some students might be interested in further explorations related to the Pythagorean Theorem.

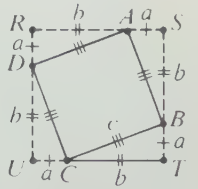
1. The converse of the Pythagorean Theorem is true and is stated as follows:

If a triangle has sides of lengths a, b, and c and $a^2 + b^2 = c^2$, then the triangle is a right triangle with c as the hypotenuse.

Which of the following are possible lengths for the sides for a right triangle? (*Answers*: A, C, D, E)

A. $a = 3$, $b = 4$, $c = 5$
B. $a = 4$, $b = 5$, $c = 6$
C. $a = 6$, $b = 8$, $c = 10$
D. $a = 7$, $b = 24$, $c = 25$
E. $a = 12$, $b = 5$, $c = 13$

B. The figure below can be used to give a proof of the Pythagorean Theorem. Each small triangle in the figure is a right triangle with legs a and b and hypotenuse c.



Area of square $RSTU$ =
$(a + b)^2 = a^2 + 2ab + b^2$

The area of square $RSTU$ is also the sum of the areas of square $ABCD$ and the four right triangles around this square.

Area of square $RSTU$ =
$c^2 + 4(\frac{1}{2}ab) = c^2 + 2ab$

Therefore, $a^2 + 2ab + b^2 = c^2 + 2ab$
or $a^2 + b^2 = c^2$.

Assignments

Minimum 1–5. Average 1–6. Maximum 1–7.

Objective

Given the length, width, and height of a rectangular parallelepiped, the student can find the volume by applying the formula $V = lwh$ or $V = Bh$.

PREPARATION

Materials: Metre sticks; centimetre rulers; shoe boxes or other models of rectangular solids.

The concept of volume will be review for most of the students, and you should plan your lesson accordingly. You might find it helpful, however, to discuss the concept and discuss standard units of volume before beginning the Investigation.

INVESTIGATION

All students should be encouraged to try part 1 of the Investigation. If you have a supply of plastic or wooden centimetre cubes, you might have some students stack up enough of the cubes to represent the volume of their math book.

For parts 2 and 3 of the Investigation, students should make estimates individually and then work in small groups to measure the objects and check their estimates.

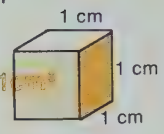
DISCUSSION

The Discussion Exercises will help students to understand why the volume formula for rectangular solids (more simply known as boxes) is $V = lwh$ or $V = Bh$. A thorough understanding of the ideas presented in this lesson is essential to prepare students for the later extension of volume concepts to prisms and pyramids.

Volume

Investigating the Ideas

To find the **volume** of a space region, you can think of dividing it into unit-cube regions and counting the number of cubes it takes to fill the space.



A cubic centimetre (cm^3) is a unit of volume.

A cubic decimetre (dm^3) contains 1000 cubic centimetres.

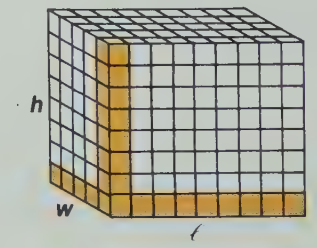
A cubic metre (m^3) contains 1000 cubic decimetres.

Can you estimate the volume of some of the space regions below and find a way to check your estimate?

1. The volume of your math book in cubic centimetres.
2. The volume of a shoe box or a grocery bag in cubic decimetres.
3. The volume of your classroom in cubic metres.

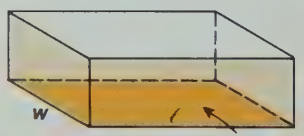
Discussing the Ideas

1. A Count the cubes along the edge ℓ.
   B Count the cubes along the edge w.
   C How many cubes in the bottom layer?
   D How many layers in height h?
   E How many cubes in the solid box?



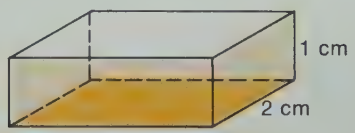
2. Explain why the volume V of the rectangular solid (box) is $V = \ell \cdot w \cdot h$

3. A What is the area B of the base of the box in terms of ℓ and w?
   B Explain why the volume of the box is given by the formula $V = B \cdot h$

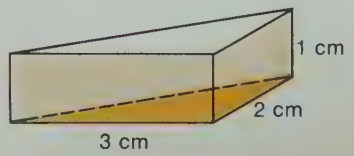


4. What is the volume of each solid?

A

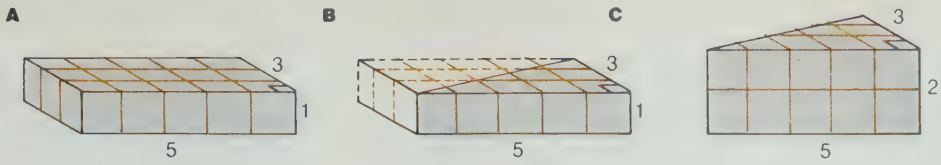


B

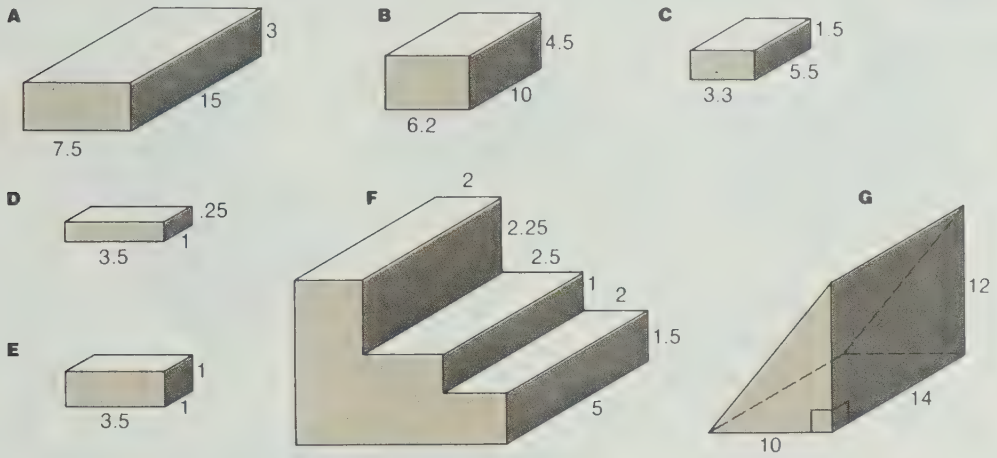


Using the Ideas

1. Find the volume of the shaded portion of each solid.

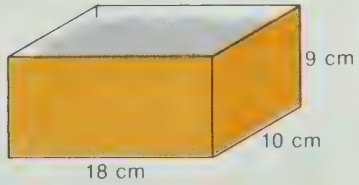


2. Find the volume. All dimensions are in centimetres.



3. Use the dimensions on the container at the right to complete each part.

- **A** Find the volume.
- **B** How many litres of water will the container hold? (1 = 1000 cm³)

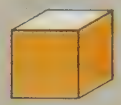


★ **4.** This L-shaped piece has a volume of 3 cm³. What is the volume of a similar L-shaped piece that is twice as long, twice as wide, and twice as high?

★ **5.** What is the volume of the L-shaped piece if every dimension is tripled?

Think

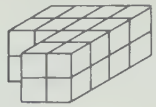
A certain cube has the same number of units of volume as it has units of area. How long is each edge of the cube?



More practice, page S-24, Set 46

UTILIZATION

If students have difficulty visualizing the larger figure described in Exercise 4, use cubes to show how the volume increases by 8 times (2^3) when each dimension is doubled.



If all of the dimensions are tripled (see Exercise 5), the volume increases by 27 times (3^3).

EXTENSION

More practice exercises could be assigned from Supplementary Exercise Set 46 on page S-24, *Duplicator Masters* page 59, and *Arithmetic Skill Card* AP-12.

Enrichment: An ancient problem was known as "Doubling the Cube," that is, constructing a cube whose volume was double that of a given cube. The ancient mathematicians could not solve this problem using the standard geometric tools of compass and unmarked straightedge.

Present the students with a cube that is ten centimeters on each edge (1000 cm³). Challenge them to construct a cube which has twice the volume, or 2000 cm³. The length of one edge of the cube is $\sqrt[3]{2000}$ cm. Students can find, by trial and error, a number whose cube is about 2000. It is about 12.6.

Think Solution

The problem can be solved by trying cubes whose sides have lengths 1, 2, 3, ...

Length of side:	1	2	3	4	5	6
Volume:	1	8	27	64	125	216
Surface area:	6	24	54	96	150	216

Algebraically, we have

Volume = s^3 and area = $6s^2$, where s = length of one edge.

If $s^3 = 6s^2$, divide both members by s^2 to get $s = 6$.

Assignments

Minimum 1, 2A–E. Average 1–3.
Maximum 1–5.

Objective

The student can give the capacity of a rectangular parallelepiped in metric cubic units and also in terms of litres (or millilitres).

PREPARATION

Materials: Empty milk or other watertight cartons, scissors, rulers, metric scales or scientific balance, plastic food bags.

Several days in advance of this lesson, ask students to bring in clean empty milk cartons. If one carton per student is not available, some students can work in pairs in the Investigation.

INVESTIGATION

The purpose of the Investigation is to provide a physical model of a container whose capacity is one litre. Students having little experience with this unit of capacity will benefit greatly from this activity. If cartons are not available, have the students make a model of an open box that will contain 1000 cm^3.

DISCUSSION

The Discussion should emphasize the relationship of the centimetre, gram, and millilitre. 1 cm^3 of water is one millilitre (0.001 litre) and has a mass of 1 gram. 1000 cm^3 (1 cubic decimetre) has a capacity of 1 litre, and 1 litre of water has a mass of 1 kilogram.

If possible, obtain metric scales and demonstrate that 1 litre of water has a mass of 1 kilogram. Place a plastic bag inside one of the liter cartons and fill the box with water. Then show that this mass balances a standard kilogram mass. Discussion Exercise 4 will help to prepare students for the written exercises.

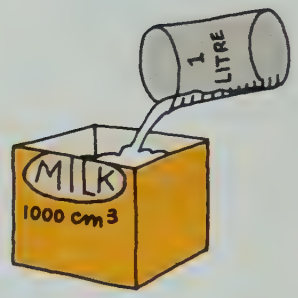
Metric Capacity

Investigating the Ideas

Obtain a large empty milk carton. Find the interior length and width of the carton in centimetres.

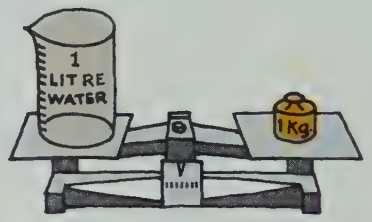
Can you cut off the top of the carton so that the bottom part will have a volume of 1000 cm^3?

Discussing the Ideas



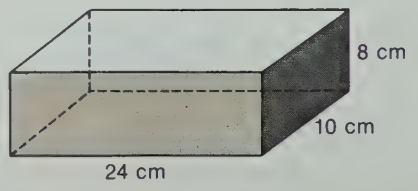
1. If you fill the bottom part of the carton with water, it would have a **capacity** of 1 **litre** (ℓ) of water. What part of a litre of water is 1 cm^3 of water?

2. One cubic centimetre of water is 0.001 litre or 1 **millilitre** (ml).
   - A What part of a litre is 500 ml?
   - B What part of a litre is 10 ml?



3. The **mass** of one litre of water under standard conditions is 1 **kilogram** (kg), or 1000 grams. What is the mass of only one millilitre of water?

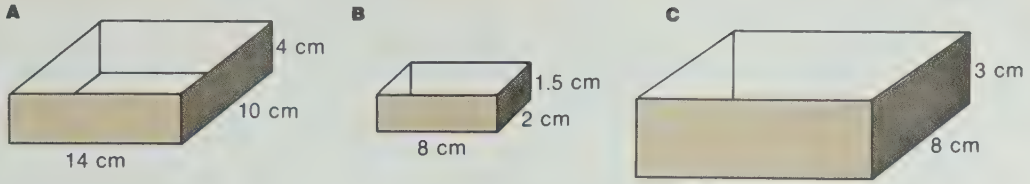
4. A What is the volume of the box in cubic centimetres?
   - B What is the capacity of the box in millilitres?
   - C What is the capacity of the box in litres?
   - D What is the mass of water that the box would contain?



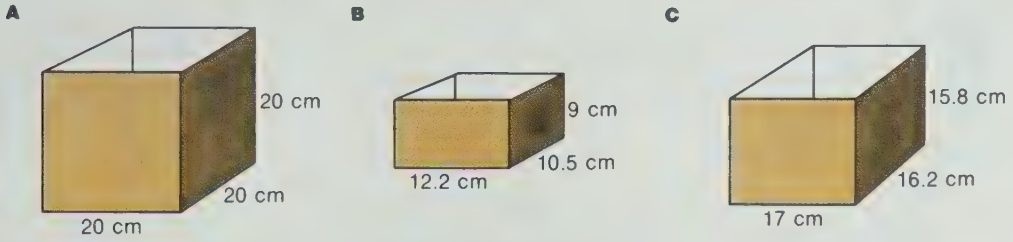
5. Find the capacity of some common containers in litres or millilitres. Then give the mass of water that each container will hold.

Using the Ideas

1. Give the capacity of each container in millilitres.



2. Give the capacity of each container in litres.

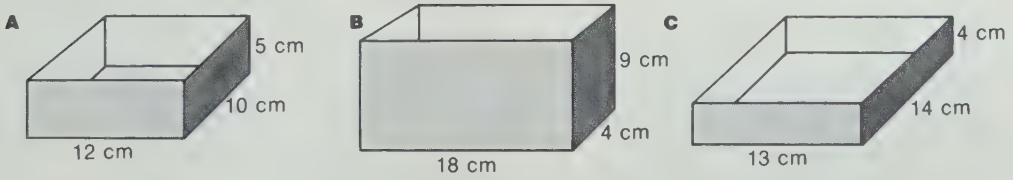


3. Give the mass of water in grams that each container in Exercise 1 will hold.

4. Give the mass of water in kilograms that each container in Exercise 2 will hold.

5. A **kilolitre** (kl) is 1000 litres. What is the capacity of a tank in kilolitres if the tank is a cube one metre on each edge?

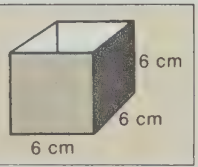
6. Which container is large enough to hold 680 grams of water?



7. How much larger in volume must the two other containers in Exercise 6 be in order to hold 680 grams of water?

think

When the open-top cube is one twelfth full of paint, there is just enough paint to cover the five faces of the open-top cube. How thick will the paint be on the box if you could use **all** of the paint in the cube?



UTILIZATION

Plan to allow time for discussion of the exercises. It is important that measures of capacity can be given in standard cubic units, but when liquid measurement is involved, capacity is often given in litres. Therefore students should be able to convert efficiently from cubic centimetres to millilitres or from litres to cubic centimetres.

EXTENSION

Page 82 of the *Workbook* provides exercises that would be suitable for use as extra practice assignments.

Enrichment: Some students may work together to make a collection of common boxes, cartons, bottles, and other containers. Make a display of the containers with labels stating the capacity in cubic centimetres and also in millilitres or litres.

This would also be an opportune time to encourage some students to undertake Research Project C on page G-53.

Think Solution

An interesting way to solve this problem is to think of slicing the paint in the box into 5 equally-thick slices, each of which will just cover one face of the box. Since the paint in the box is $\frac{1}{12}$ of 6 cm or 5 mm high, each slice will be 1 mm thick.

Assignments

Minimum 1–4. Average 1–5. Maximum 1–7.

Objective

The student can find the volume of a prism or a pyramid, given the appropriate dimensions or areas.

PREPARATION

Materials: Tagboard or cardboard; scissors; tape; rulers; sand, rice, beans, or other loose material; 2 decks of playing cards, or other packets of cards that are all the same size and shape.

Have students begin immediately on the construction of the models for the Investigation.

INVESTIGATION

Some time and material will be saved if the Investigation is carried out by pairs or small groups of students. If the models are carefully made, students will discover that three "pyramidsful" must be used to fill the prism. Thus, the volume of the pyramid is only $\frac{1}{3}$ the volume of the prism.

DISCUSSION

When discussing Exercise 1, use decks of playing cards, sheets of paper, magazines, or books, stacking them to demonstrate the fact that the volume of the stack remains the same even when it is pushed over or twisted into a variety of shapes. From this, students should conclude that the volume formula for a prism must be area of the base times height.

Relate Discussion Exercise 2 to the Investigation. Since it takes 3 pyramids to fill a prism whose base and height are equal to the base and height of the pyramid, the volume of the pyramid is $\frac{1}{3}$ that of the prism, or $V = \frac{1}{3}Bh$.

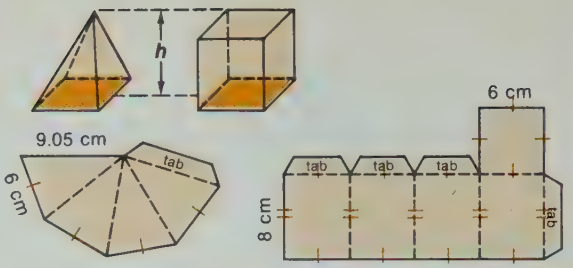
Exercises 1 and 3 illustrate Cavalieri's Principle, which can be stated as follows:

If two solids lie between two parallel planes and if the two cross sections cut by any plane parallel to the including planes have equal areas, then the volumes of the two solids are equal.

Thus, if you think of stacking equal-area cross sections, h layers high, for each pyramid, then the volumes must be the same.

Investigating the Ideas

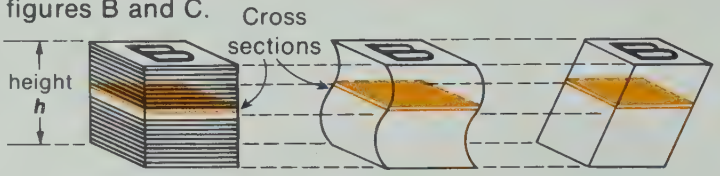
Make copies of the patterns on cardboard, cut them out, and form models for an open-top **prism** and a **pyramid** with its base removed. Both models will have the same dimensions for the base and height.



Can you find how many "pyramidsful" of sand, rice, or other fine material it takes to fill the prism?

Discussing the Ideas

1. The prism constructed for the Investigation is a **right prism** because the lateral edges are perpendicular to the base. Imagine a right prism (figure A) made of a large number of thin sheets of steel and then slid to form figures B and C.



   A Is the height of each figure the same?

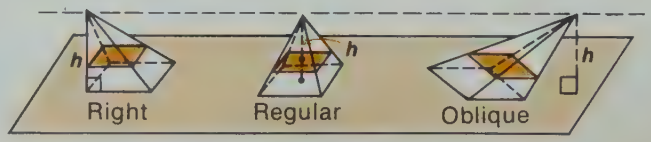
   B Are the cross sections of each figure the same?

   C Is the volume of each figure the same?

   D Explain the formula $V = B \cdot h$

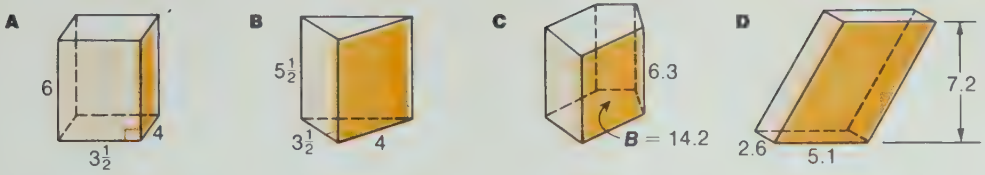
2. Explain why the formula for the volume of a pyramid with base B and height h is $V = \frac{1}{3} \cdot B \cdot h$

3. Do you think that an oblique pyramid has the same volume as a right pyramid with the same base and height? What do these figures suggest? The shaded cross sections are all at the same distance from their bases.

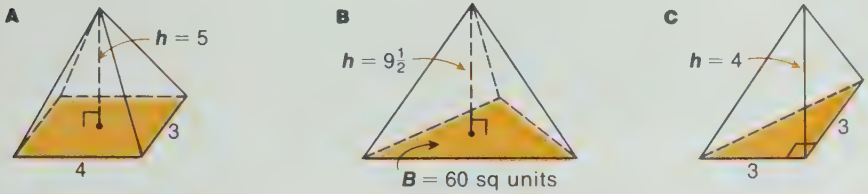


Using the Ideas

1. Find the volumes of these prisms.



2. Use the formula $V = \frac{1}{3} \cdot B \cdot h$, where B is the area of the base and h is the height, to find the volume of each pyramid.

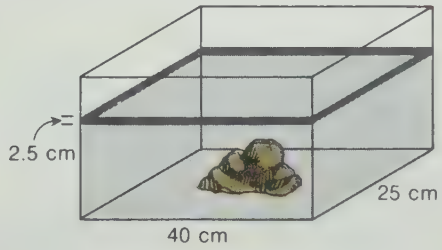


3. The Great Pyramid of Egypt had a square base of 230.4 metres on each side. Originally, the pyramid had a height of 146.5 metres. What was the volume of the pyramid?

4. A concrete sidewalk is to be constructed that is 22 metres in length, 1.5 metres wide, and an average thickness of 15 centimetres.

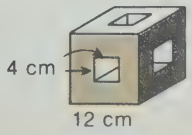
- **A** About how many cubic metres of concrete are needed?
- **B** Concrete has a mass of about 2300 kilograms per cubic metre. About how many kilograms of concrete are needed?

5. When an irregular object is submerged in water, it displaces a volume of water equal to its own volume. If a submerged object causes the water level in the tank shown to rise 2.5 centimetres, what is the volume of the object?



6. A solid cube 12 centimetres on each edge has a square hole cut completely through the centres of the faces of the cube.

- **A** What is the volume of the part of the cube that was removed?
- **B** What is the volume of the remaining part of the cube?



More practice, page S-25, Set 47

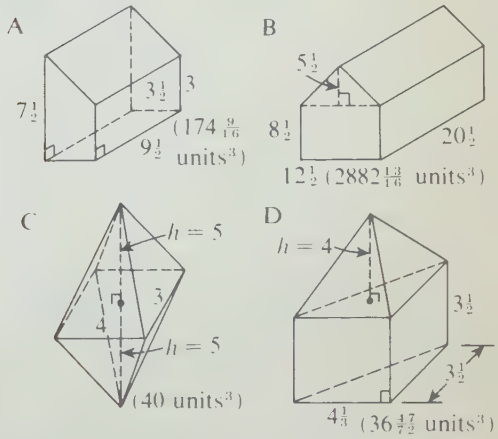
UTILIZATION

For convenience, we allow a shortcut in the language of volumes, just as we did in the language of areas; that is, we say "area of a square" when we really mean "area of a square region." Also, we speak of the "volume of a polyhedron" when we really mean "volume of the region of space bounded by and including the polyhedron." Accept this usage by students, but occasionally ask a question to remind them of the exact meaning of "volume of a polyhedron."

EXTENSION

To provide additional practice, make selective assignments from Supplementary Exercise Set 47 on page S-25, *Workbook* page 83, *Duplicator Masters* page 60, and *Arithmetic Skill Cards* AP-12 and AP-13.

Enrichment: Ask students to find the volume of these polyhedrons. (Answers are given in parentheses.)



This would be a good time to encourage interested students to explore Research Project B on page G-53.

Assignments

Minimum 1–2. Average 1–4. Maximum 1–6.

Objective

The student can demonstrate the ability to work with the major concepts presented in this module.

UTILIZATION

There were several formulas developed in this module which should be mastered by the student. These are summarized below.

Area Formulas

Rectangles: $A = w \cdot l$, where w is the width and l is the length.

Parallelograms: $A = b \cdot h$, where b is the base and h is the height.

Triangles: $A = \frac{1}{2} \cdot b \cdot h$, where b is the base and h is the corresponding height (altitude).

The Pythagorean Theorem states that for every right triangle, the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse. The formula associated with the theorem is $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c the length of the hypotenuse.

Volume Formulas

Volume of a "box": $V = l \cdot w \cdot h$, where l is the length, w is the width, and h is the height.

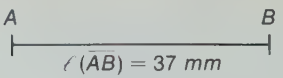
Volume of a prism: $V = B \cdot h$, where B is the base and h is the altitude.

Volume of a pyramid: $V = \frac{1}{3} \cdot B \cdot h$, where h is the altitude.

Additional exercises that may be useful in reviewing the major concepts of this module are provided on page 84 of the *Workbook*.

REVIEWING THE IDEAS

1. What is the length of $\overline{AB}$ in



A centimetres?

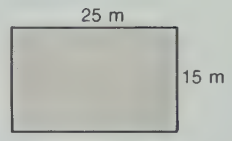
B decimetres?

C metres?

2. Find the perimeter of each region.

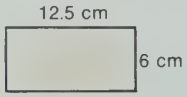
A

B

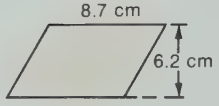


3. Find the area of each region.

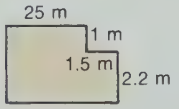
A



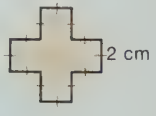
B



C



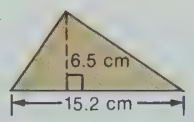
D



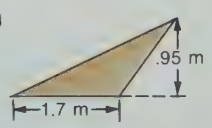
4. If a piece of adding machine tape 6 centimetres wide has an area of one square metre, how long is it?

5. What is the area of each region?

A



B



6. Find the sum and difference.

A

$$\begin{array}{r} 29^\circ 38' \ 7'' \\ +\ 42^\circ 51' 29'' \\ \hline \end{array}$$

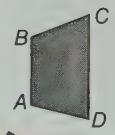
B

$$\begin{array}{r} 125^\circ 18' 20'' \\ -\ 53^\circ 40' 38'' \\ \hline \end{array}$$

7. What is the measure of the complement of an angle measuring $37^\circ 15'$?

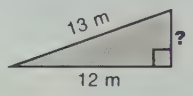
8. How large is the third angle?

9. What is the sum of the measures of angles A, B, C, and D?

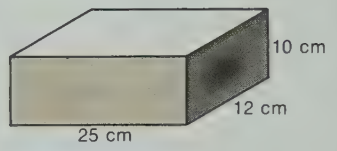


10. What is the area of the large square if □ is a unit square?

11. Find the third side of the triangle.



12. Find the volume.



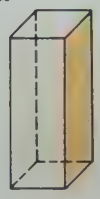
13. **A** What is the mass of 200 millilitres of water?

B What size container will hold 1000 grams of water?

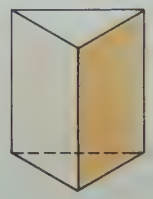
14. What is the capacity in litres of an aquarium 40 cm by 20 cm by 18 cm?

15. The base of each solid below has an area of 40 square units. The height of each solid is 12 units. Find the volume of each solid.

A



B



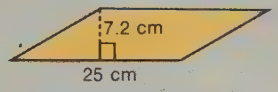
C

TEST YOURSELF

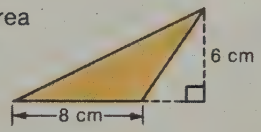
1. What is the perimeter of a rectangular plot of ground 120.8 metres long by 57 metres wide?

2. A rectangular room is 6.1 metres long and 4.3 metres wide. How many square metres of carpet are needed to cover the floor from wall to wall?

3. What is the area of the parallelogram?



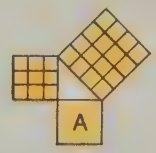
4. What is the area of the shaded region?



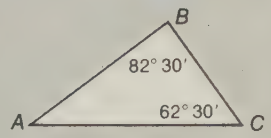
5. Find the sum and difference.

A $\begin{array}{r} 37^\circ 14' 34'' \\ +\,81^\circ 29' 30'' \\ \hline \end{array}$ B $\begin{array}{r} 90^\circ \\ -\,17^\circ 38' 34'' \\ \hline \end{array}$

6. What is the area of square A on the leg of the right triangle?

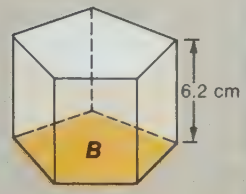


7. What is the measure of $\angle A$?



8. What is the length of each edge of a cube that will contain one litre of water?

9. The area ***B*** of the base of the prism is 66 cm². What is the volume of the prism?



10. What is the volume of a pyramid 18 cm high and with a square base 10 cm on each edge?

RESEARCH PROJECTS

A Make a model that will demonstrate the Pythagorean Theorem. (See *Instructional Aids in Mathematics, 34th Yearbook*; Washington, D.C.: National Council of Teachers of Mathematics, 1973, pp. 378–379.)

B Investigate Plateau's Problem, the problem of determining the least area that will span a given shape. (See *What is Mathematics?* by R. Courant and H. Robbins; New York: Oxford University Press, 1941, pp. 385–397.)

C Find out how the volume of irregular objects can be found by finding the amount of water they displace when they are submerged. Find the volume of several irregular-shaped solids whose volume determined by other methods would be different. (See *Mathematics: Man's Key to Progress* by Richard A. Denholm; Pasadena, California: Franklin Publications, 1968, pp. 84-85; available from Grolier Ltd.)

TEST YOURSELF

Students' performance on this self-evaluation test should serve as a good measure of their ability to work with the major concepts presented in this module. Encourage them to check their work against the answer key provided on page G-54 and to review the lessons related to any topics that proved troublesome. This should be effective preparation for an achievement test covering this module. You might wish to design such a test yourself, or you might prefer to administer the appropriate Module Achievement Test that is available from the publisher as part of the evaluation program designed for use with this series.

RESEARCH PROJECTS

There are probably more proofs of the Pythagorean Theorem (Research Project A) than of any other theorem in mathematics. Students may also be interested in examining the book *The Pythagorean Proposition* by Elisha Scott Loomis, 2nd Edition; Washington, D.C.: National Council of Teachers of Mathematics, 1968.

The investigation proposed in Project B is an extension of the problem presented in Exercise 5 on page G-51. A small rectangular aquarium with a centimetre ruler taped along a vertical edge of the aquarium can be used for experiments related to this project.

Research Project C will be attractive to many students, especially after they have seen a demonstration of it. This would be an excellent project for a mathematics fair or exhibit.

MATHEMATICAL RECREATION

The cycloid has two interesting properties. It is a *tautochrone* which means that, disregarding resistance or friction, a ball moving under the force of gravity of a cycloid with its arch downward will execute simple harmonic motion. In other words, the ball will travel paths of equal amplitude in equal times.

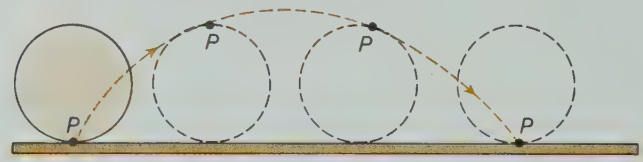
The second property is that the cycloid in the same position as above, is also a *brachistochrone*, which means that a ball will travel along the curve from a higher point to a lower point under the force of gravity in the shortest possible time.

Some students may be interested in making more permanent models of the cycloid by cutting the curve from plywood or hardboard. (For suggestions, see *Mathematical Models* by H. Martyn Cundy and A. P. Rollett; New York: Oxford University Press, 1954, pp. 185–186. Distributed in Canada by Oxford University Press, Don Mills, Ontario.)

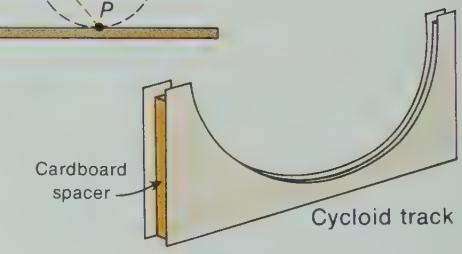
MATHEMATICAL RECREATIONS

The Cycloid

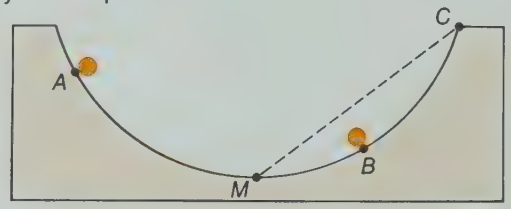
If a circular disk is rolled along a line, the path of a point *P* on the circle is called a **cycloid** (the curve in color). The cycloid has some interesting properties.



Cut out copies of two **cusps** of a cycloid from heavy cardboard. Glue a third piece of cardboard between them to form a cycloid track.



Try this experiment:



Place two marbles on the track on opposite sides of point *M* (the midpoint and lowest part of the cusp). If the marbles are released at the same time, they will reach point *M* at the same time even though the distances along the curve are not the same. Furthermore, a marble released at point *C* will roll along the curve to point *M* more quickly than a marble which rolls along a straight line from *C* to *M*.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. 355.6 m **2.** 26.23 m² **3.** 180 cm² **4.** 24 cm²
5. A 118° 44′ 4″; B 72° 21′ 26″ **6.** 7 square units
7. 35° **8.** 10 cm **9.** 409.2 cm³ **10.** 600 cm³

UNIT G: GEOMETRY

Module 3: Similar Triangles and Trigonometry

General Objectives

To introduce similar polygons, with particular emphasis on similar triangles.
To relate ideas of ratio and proportion to similar polygons.
To focus attention on relationships in similar right triangles.
To introduce the trigonometric ratios in terms of right triangles.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
91	**RED** Given a series of parallel lines that cut off congruent segments on one transversal, the student can find the lengths of the segments cut off on other transversals.	G-56, 57	**WB** 85	**DS** b-3	
92	**RED** Given similar polygons, the student can identify corresponding angles and write proportions for corresponding sides.	G-58, 59		**DS** b-1, 4	**SWM 3** 170-173
93	**RED** Given similar triangles, the student can write and solve proportions for the triangles in order to find the length of a specified side of one of the triangles.	G-60, 61	**WB** 86 **DM** 61	**DS** b-1	**SWM 3** 268-269
94	**YELLOW** Given proportion problems involving similar right triangles, the student can solve them.	G-62, 63 G-64, 65	**WB** 87 **WB** 88		
95	***GREEN*** *Given a right triangle, the student can give the trigonometric ratio (sine, cosine, or tangent) for a specified angle.*	G-66, 67	**WB** 89	**DS** h-1	
96	***GREEN*** *Given problems involving trigonometric ratios, the student can solve them.*	G-68, 69	**DM** 62	**DS** h-1	
	Reviewing the Ideas	G-70	**WB** 90		

MATHEMATICS

Quite a few new mathematical concepts are introduced in this module.

The first section introduces the theorem that, if a series of parallel lines cut off congruent segments on one transversal of the lines, then congruent segments will be cut off on any other transversal of the parallel lines. This basic theorem is applied in the case of dividing two sides of a triangle into congruent segments and thus provides background for work with similar triangles in the later sections.

Similar polygons are defined on page G–58. The Angle-Angle-Angle Similarity Theorem and the Side-Side-Side Similarity Theorem are applied to triangles on pages G–60 and G–61. Formal proofs of these theorems are not given, but the truth of these theorems is made plausible to the students through appropriate activities.

Similar right triangles are introduced in order to prepare the student for the introduction to right triangle trigonometry and the trigonometric ratios.

TEACHING THE MODULE

Materials

Centimetre rulers, compasses, graph paper, index file cards, lined notebook or tablet paper, compasses.

Vocabulary

adjacent sides
congruent polygons
corresponding angles
corresponding sides
cosine (cos)
hypotenuse
opposite side(s)
parallel lines
proportional
similar polygons
similar triangles
sine (sin)
tangent (tan)
transversals
trigonometric ratios

The lessons in this module offer a good opportunity to review and strengthen student understanding of ratio and proportion and to further improve computational skills. You should not expect students to demonstrate great computational skills or problem-solving ability with respect to the material on tri-

gonometry. Rather, regard this material as an extension of the work with similar triangles and point out to your students that they will have other opportunities to study trigonometry in their later work in mathematics.

Lesson Schedule

Although there are only seven lessons in this module, it is quite likely that you will want to spend more than one day on some lessons. Therefore you should plan to allow 8 to 10 days if all the material is covered, with additional time reserved for projects, reinforcement, and evaluation.

Evaluation

Since nearly all of the material in this module will be new to the students, you should not expect great proficiency in problem solving, especially in the work on trigonometry. Unless you spend extra time in developing such skills, it is unlikely that students will be very efficient. Keep in mind that the main purpose of the material in the module is to provide the student with some background in these topics, which will be studied in greater depth in high school mathematics.

RESOURCES FOR ACTIVE LEARNING

Activities

Cloudburst, Vol 5 (linkages), Nos. 5777–87, 8878, Midwest Publications

Developmental Math Cards, K17, L20, Addison-Wesley

Experiments in Mathematics, Stage 2, pp. 10–13; Stage 3, pp. 12–13, 24–25, Houghton Mifflin

Geoboard Activity Cards (Triangles and similarity), Geocards 9–18, 73–76, 99–106, Cuisenaire Co.

Math Experiments with the Geo Strips, HR experiments, Midwest Publications

Mathematics: Man's Key to Progress, Book A, pp. 54–55; Book B, pp. 88, Rand McNally

Mira Activities for Junior High, Activities 45–80, Creative Publications

Nuffield Project: *Problems*—Red Set, Nos. 16, 18; *Shape and Size 4* "Similarity," pp. 26–33, Wiley

Manipulative Devices

Centimetre rulers (Addison-Wesley)

Circle Master Compass (Creative Publications)

Clinometer (Selective Educational Equipment; Sigma)

Geoboards (Addison-Wesley)

GeoStrips (Mind/Matter; Selective Educational Equipment)

Hypsometer (Yoder Instruments)

Logarithm and Trigonometric Function Chart (LaPine; Sargent-Welch)

Mira (Creative Publications; Cuisenaire Co.)

Multi-model Geometric Construction Set (Yoder Instruments)

Pantograph (Yoder Instruments)

Pattern Blocks and stickers (Webster, McGraw-Hill)

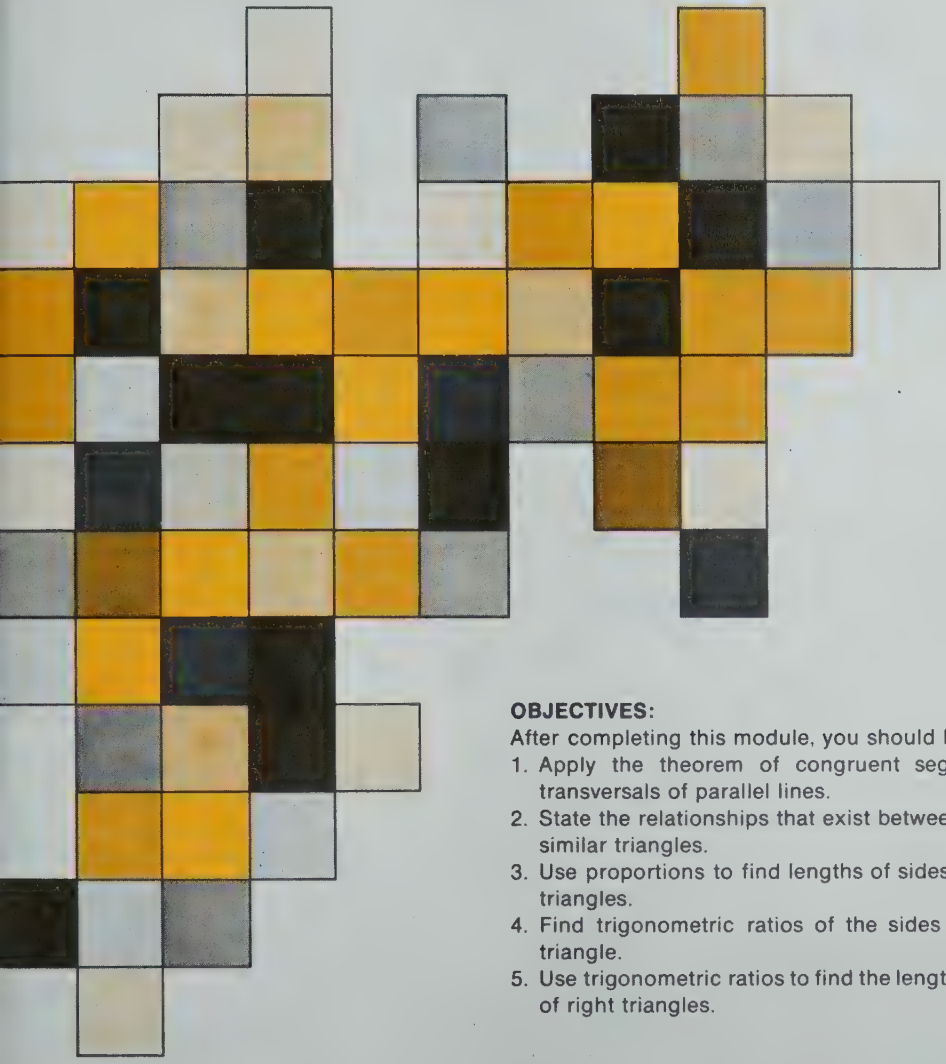
Protractors (Geyer; Selective Educational Equipment)

Shadow Stick (General Learning Corp.)

Transit (Mind/Matter; Yoder Instruments)

Games and Puzzles

Configurations (Wff'N Proof)



OBJECTIVES:

After completing this module, you should be able to:

1. Apply the theorem of congruent segments on transversals of parallel lines.
2. State the relationships that exist between a pair of similar triangles.
3. Use proportions to find lengths of sides of similar triangles.
4. Find trigonometric ratios of the sides of a right triangle.
5. Use trigonometric ratios to find the lengths of sides of right triangles.

Objective

The student can demonstrate an understanding of the theorem that, if a series of parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on all transversals.

PREPARATION

Materials: Index file cards (one per student), lined notebook or tablet paper.

Review the definition of *parallel lines* and a *transversal* of parallel lines. (See Unit G, Module 1, page G-22.)

INVESTIGATION

Students are not likely to be familiar with the method presented in the Investigation for dividing a segment into congruent parts. To extend the activity, suggest that the students try dividing other edges of the card into 3, 4, or 6 congruent parts. If you wish, students might check the accuracy of this construction by measuring the segments to see whether they are about the same length.

DISCUSSION

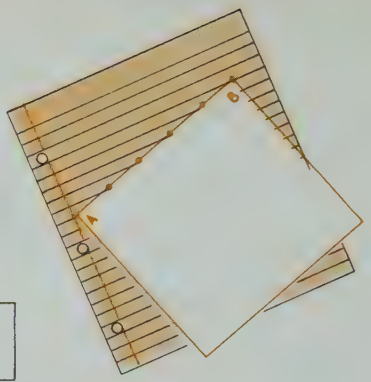
It is important that students understand the statement of the theorem given in Exercise 1 and can apply it in appropriate cases. You might point out that the vertical edge of a piece of notebook paper acts as a transversal of the parallel lines of the paper. Since the segments along this edge (transversal) are congruent (with the possible exceptions of those at the very top or bottom of the page), then the parallel lines will cut off congruent segments on any other transversals of the lines.

Exercise 2 contains the essential steps in the proof of the theorem. You may have to rely on your more able students to carry the burden of the discussion of the proof, but all students can benefit from the reasoning.

Parallel Lines and Transversals

Investigating the Ideas

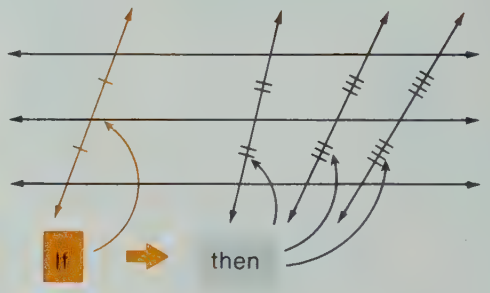
Suppose that you want to divide one edge of a small unlined card into five pieces of equal length but do not have a ruler. The figure shows how you can use a sheet of lined notebook paper to do this.



Can you use this method to divide a segment into seven pieces of equal length?

Discussing the Ideas

1. The Investigation illustrates a geometric theorem which can be stated as follows:

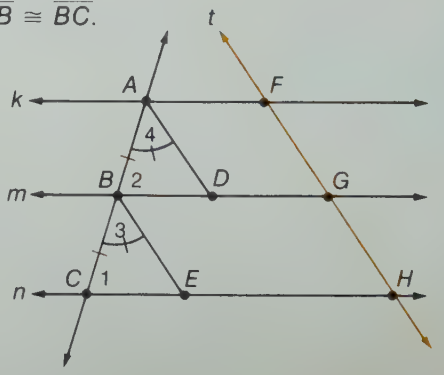


> If parallel lines cut off congruent segments on one **transversal**, then they cut off congruent segments on all transversals.

Explain how this theorem justifies the results of the Investigation.

2. The following exercise may help you understand why the theorem above is true.

Suppose that k, m, and n are parallel, and $\overline{AB} \cong \overline{BC}$.



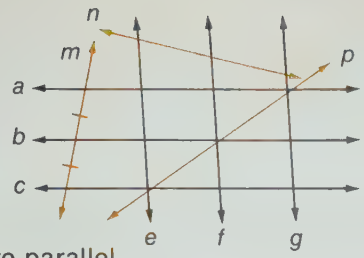
- **A** How does $\angle 1$ compare with $\angle 2$? Explain your answer.
- **B** Suppose that we draw $\overline{AD}$ and $\overline{BE}$ so that $\angle 3 \cong \angle 4$. Are $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ parallel?
- **C** What can you say about $\triangle ABD$ and $\triangle BCE$? Explain your answer.
- **D** Is it true that $\overline{AD} \cong \overline{BE}$? Explain.
- **E** Suppose that $\overleftrightarrow{FH}$ is a transversal parallel to $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$. What kind of figures are $ADGF$ and $BEHG$?
- **F** Do you think $\overline{FG}$ is congruent to $\overline{GH}$? Explain your answer.

Using the Ideas

1. In the figure, $a \parallel b \parallel c$ and $e \parallel f \parallel g$. The segments on line m are congruent.

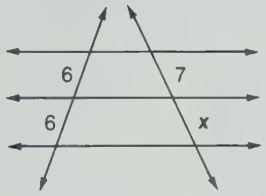
A Are the segments on line n congruent? Why?

B Are the segments on line p congruent? Why?

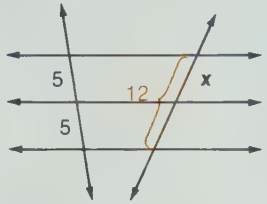


2. Find the length **x**. The horizontal lines are parallel.

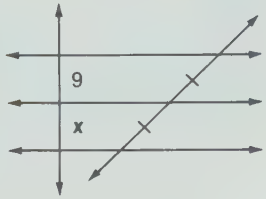
A



B

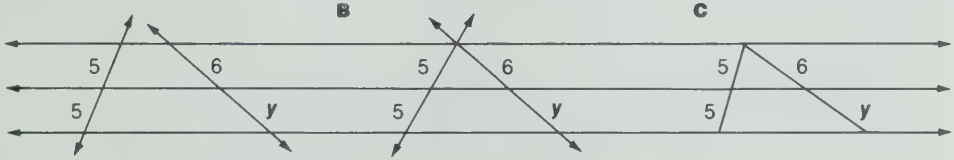


C



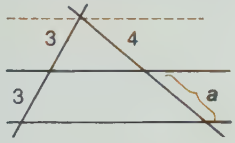
3. Find the length **y**. The horizontal lines are parallel.

A **B** **C**



4. Find the length **a**. The horizontal lines are parallel.

A



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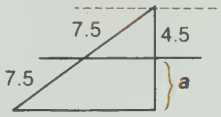

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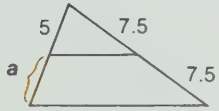
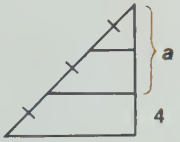
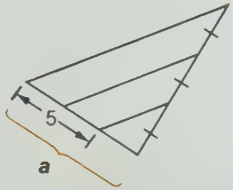


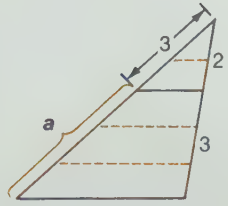
D



E

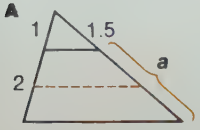


F

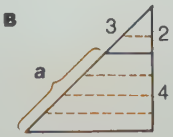


5. Sometimes it is helpful to sketch parallels as shown in parts **A**, **B**, and **C**. Assume that the horizontal lines are parallel and find the length **a**.

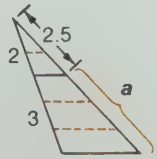
A



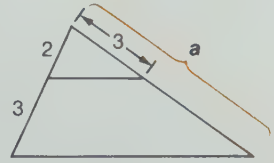
B



C



D



UTILIZATION

Exercises 2, 3, 4, and 5 are carefully programmed to help the student apply the theorem presented in the Discussion. Note the gradual omission of various parts of the drawing until, in parts of Exercises 4 and 5, only triangles are left.

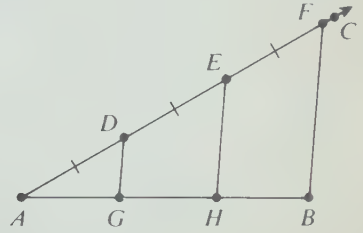
EXTENSION

Additional appropriate exercises are provided on page 85 of the *Workbook*.

Enrichment: Show students a ruler-and-compass construction method for dividing a given segment into any number of congruent segments:

Draw any segment AB. Then suppose that we wish to divide it into thirds, for example.

1. Draw any ray AC.
2. On $\overrightarrow{AC}$ mark off any three congruent segments: $\overline{AD} \cong \overline{DE} \cong \overline{EF}$.
3. Draw $\overline{FB}$.
4. Construct $\overline{DG}$ and $\overline{EH}$ parallel to $\overline{FB}$.



Assignments

Minimum 2–4. Average 1–4. Maximum 1–5.

Objective

The student can apply the definition of similar polygons to appropriate pairs of polygons.

PREPARATION

Materials: Graph paper (*Duplicator Masters*, page 83).

Review the fact that congruent figures have the same size and shape. Then tell the students that in this lesson they will study the properties of figures that have the same shape but are not necessarily the same size.

INVESTIGATION

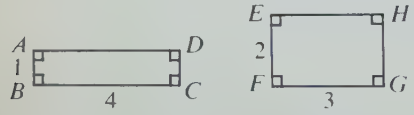
Suggest that students reproduce the triangle, quadrilateral, and pentagon on their graph paper. Then allow time for them to draw larger or smaller figures that have the same shape.

DISCUSSION

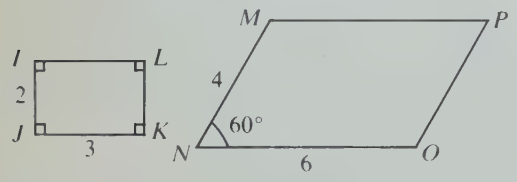
The definition of similar polygons in Exercise 1 contains three conditions which must be met:

1. There must be a one-to-one correspondence between the vertices of the pair of triangles such that
2. corresponding angles are congruent and
3. the lengths of corresponding sides are proportional.

To emphasize the definition, you might present some nonsimilar figures such as these.



Corresponding angles are congruent, but corresponding sides are not proportional.



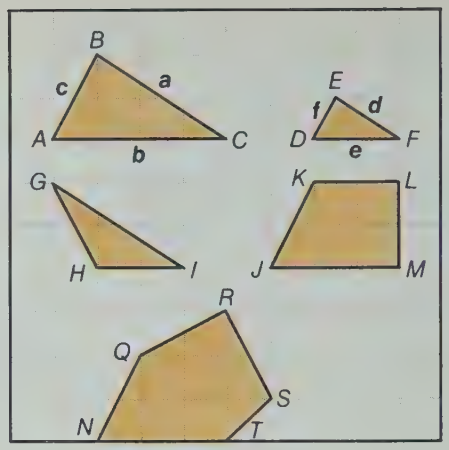
Corresponding sides are proportional, but corresponding angles are not congruent.

Similar Polygons

Investigating the Ideas

$\triangle ABC$ and $\triangle DEF$ have the same shape but not the same size.

Draw the other three polygons on a sheet of graph paper.



Can you draw polygons that have the same shape as these three, but not the same size?

Discussing the Ideas

1. **Similar polygons** have the same shape but not necessarily the same size. The formal definition follows.

> Two polygons are similar if there is a one-to-one correspondence between the vertices of the polygons so that corresponding angles are congruent and the lengths of corresponding sides are proportional.

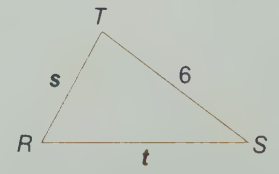
For $\triangle ABC$ and $\triangle DEF$ above, we say:

"$\triangle ABC$ is similar to $\triangle DEF$" and we write $\triangle ABC \quad \triangle DEF$.

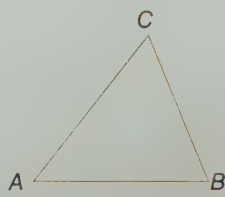
This means: $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$ and $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$.

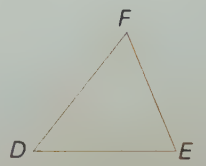
Label the vertices and sides of the other polygons you drew. Show which pairs of angles are congruent and which pairs of sides have the same ratio.

2. If $\triangle RST \sim \triangle UVW$,
   - **A** which vertices must correspond?
   - **B** which pairs of angles are congruent?
   - **C** what is the ratio of t to w? s to v?



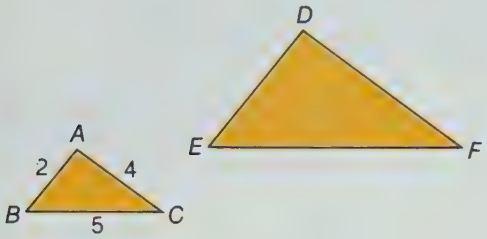
3. **A** If $\triangle ABC \sim \triangle DEF$ and $m\angle C = 62°$, what is $m\angle F$? Why?
   B If the ratio of the length of $\overline{FE}$ to the length of $\overline{CB}$ is 5 to 6, what is the ratio of the length of $\overline{DF}$ to the length of $\overline{AC}$?



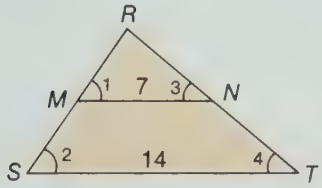


Using the Ideas

1. $\triangle ABC \sim \triangle DEF$. The length of $\overline{EF}$ is three times the length of $\overline{BC}$. What are the lengths of the sides of $\triangle DEF$?



2. In the figure, $\overline{MN} \parallel \overline{ST}$, M is the midpoint of $\overline{RS}$, and N is the midpoint of $\overline{RT}$.



A Is $\angle 1 \cong \angle 2$? Why?

B Is $\angle 3 \cong \angle 4$? Why?

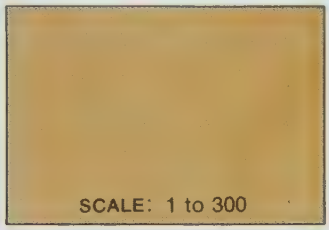
C Is $\triangle RMN \sim \triangle RST$?

3. If two triangles are equilateral, are they similar to each other? Why?

4. Suppose you know that $\triangle GHI$ is an equilateral triangle. You also know that $\triangle XYZ \sim \triangle GHI$. What is the measure of $\angle X$?

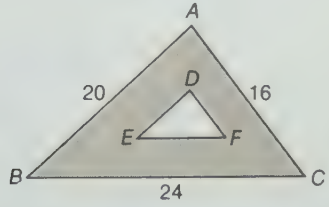
5. Suppose $\triangle JKL \sim \triangle PQR$ and $m\angle J = 27°$ and $m\angle L = 63°$. What are the measures of each of the angles of $\triangle PQR$?

6. Aretha made a scale drawing of her mathematics classroom using a scale of 1 to 300. Her drawing was 4.7 centimetres long and 3.2 centimetres wide. What were the actual dimensions of the classroom in metres?



7. A photograph of the surface of the planet Mars shows a crater 8 millimetres in diameter. If the photograph is $\frac{1}{7\,500\,000}$ actual size, about how many kilometres in diameter is the crater?

*** 8.** $\triangle ABC \sim \triangle DEF$, but the perimeter of $\triangle DEF$ is only $\frac{3}{4}$ of the perimeter of $\triangle ABC$. What is the length of each side of $\triangle DEF$? What is the ratio of the lengths of any two corresponding sides?



Think

Arrange nine dots in

1. eight rows of three each.
2. nine rows of three each.
3. ten rows of three each.

UTILIZATION

Solution, Exercise 2

A. Yes. $\overleftrightarrow{SR}$ is a transversal of $\overline{MN}$ and $\overline{ST}$. Since $\overline{MN} \parallel \overline{ST}$, $\angle 1 \cong \angle 2$.

B. Yes. $\overleftrightarrow{RT}$ is a transversal of $\overline{MN}$ and $\overline{ST}$. Since $\overline{MN} \parallel \overline{ST}$, $\angle 3 \cong \angle 4$.

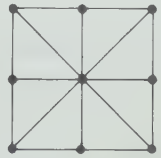
C. Yes. $\triangle RMN \sim \triangle RST$, because $\angle R \cong \angle R$, $\angle RMN \cong \angle RST$, $\angle RNM \cong \angle RTS$, and the ratio of any pair of corresponding sides is $\frac{1}{2}$.

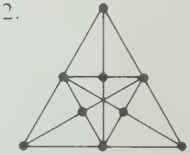
EXTENSION

Enrichment: You may extend the idea of Exercise 4 by having students make scale drawings of the floor plan of the classroom or other objects. The amount of detail, such as doors, windows, blackboards, etc., that should be included can be left to the students' discretion. When correctly drawn, a scale drawing is, of course, similar to the object represented.

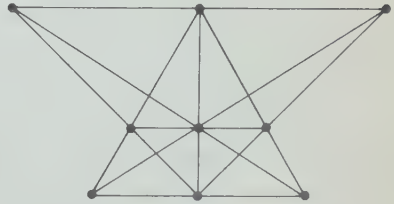
This would also be an opportune time to recommend that interested students undertake Research Project C on page G-71, which concerns construction and use of a pantograph.

Think Solution

1. 

2. 

3. A possible solution is shown.



Assignments

Minimum 1–5. Average 1–6. Maximum 1–8.

Objective

Given a pair of similar triangles, the student can write and solve a proportion to find an unknown side.

PREPARATION

Materials: Centimetre rulers, compass, protractor.

Review the ruler-and-compass method of constructing triangles given the length of three sides. This construction is required for the Investigation.

INVESTIGATION

The Investigation serves to introduce the student to a theorem that is sometimes called the Side-Side-Side Similarity Theorem. This theorem states that if the lengths of the corresponding sides of two triangles are proportional, then the triangles are similar. This theorem is usually proved in high school geometry courses, but will be accepted here without formal proof. If some students finish the construction quickly, you might suggest that they construct a triangle whose sides are only half as long or one third as long as the sides of the given triangle.

DISCUSSION

Exercise 2 suggests another construction activity to illustrate a second similarity theorem for triangles. This theorem, sometimes referred to as the Angle-Angle-Angle Similarity Theorem, is stated as follows:

> If the corresponding angles of two triangles are congruent, then the triangles are similar.

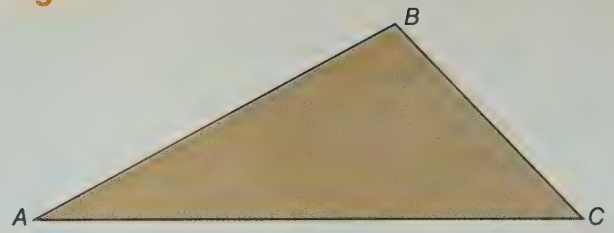
Use Exercise 3 to reinforce the idea that, if two triangles meet either of the stated conditions, one can conclude that the two triangles are similar.

Exercise 4 should be discussed in detail and extra examples provided, if necessary, before the exercises on page G-61 are assigned.

Conditions for Similar Triangles

Investigating the Ideas

Make a copy of $\triangle ABC$ on your paper.



Can you draw another triangle whose sides are twice as long as the sides of $\triangle ABC$?

Discussing the Ideas

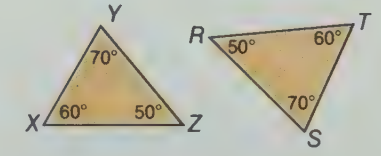
1. Do you think that the two triangles are similar to each other? Give your reasons.

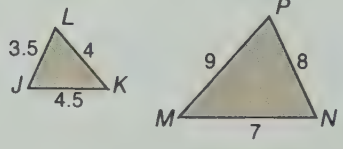
2. Measure each angle of $\triangle ABC$ with a protractor. Draw a triangle whose angles have the same measure as the angles of $\triangle ABC$ but whose sides are different in length from $\triangle ABC$. Is this triangle similar to $\triangle ABC$?

3. If two triangles are similar, then
   - [1] corresponding angles are congruent.
   - [2] corresponding sides are proportional.

   In later geometry courses, you will be able to prove that the existence of either condition *alone* is sufficient to prove that the triangles are similar.

   A Is $\triangle XYZ \sim \triangle TSR$? Why? **B** Is $\triangle JKL \sim \triangle MPN$? Why?

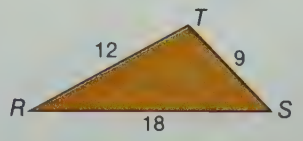


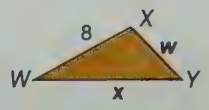


4. Suppose $\triangle RST \sim \triangle WXY$. The proportion below can be used to find length w.

$$\frac{w}{9} = \frac{8}{12}$$
$$12w = 72$$
$$w = 6$$

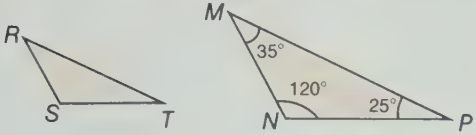
Write and solve a proportion to find length x.





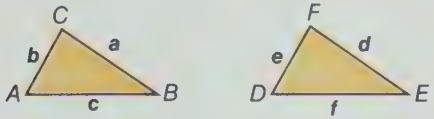
Using the Ideas

1. Suppose that $\triangle MNP \sim \triangle RST$. Give the measure of each angle of $\triangle RST$.



2. Suppose that $\triangle ABC \sim \triangle EDF$. If length $\overline{AB} = 2$, length $\overline{BC} = 3$, and $\overline{DE}$ is twice as long as $\overline{AB}$, how long is $\overline{DF}$?

In Exercises 3 through 11, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. The letters ***a***, ***b***, ***c***, ***d***, ***e***, and ***f*** represent the lengths of the sides.



EXAMPLE: Find ***a*** if $b = 3$, $d = 4$, $e = 12$

SOLUTION: $\frac{a}{d} = \frac{b}{e}$ or $\frac{a}{4} = \frac{3}{12}$

$12a = 12$, hence $a = 1$

3. Find ***f*** if $a = 6$, $d = 15$, $c = 2$.

4. Find ***e*** if $b = 5$, $c = 2$, $f = 3$.

5. Find ***a*** if $d = 4$, $b = 2$, $e = 7$.

6. Find ***c*** if $f = 4\frac{1}{2}$, $d = 5$, $a = 6$.

7. Find ***a*** if $b = e$, $d = 1.32$.

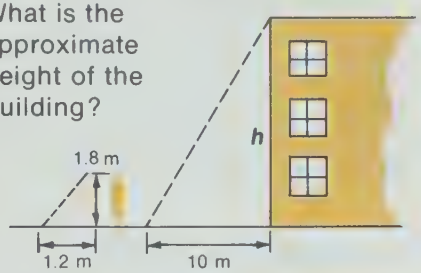
8. Find ***b*** if $d = 3$, $a = 2$, $e = 1$.

9. Find ***d*** if $c = 2$, $a = 2.1$, $f = 3.4$.

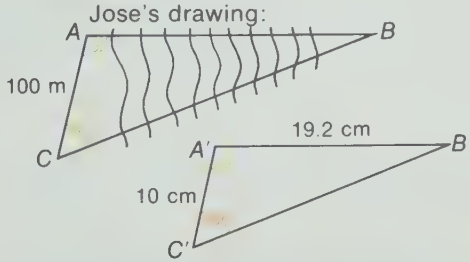
10. Find ***a*** if $d = 2\frac{1}{2}$, $c = 3\frac{1}{3}$, $f = 4\frac{1}{4}$.

More practice, page S-25, Set 48

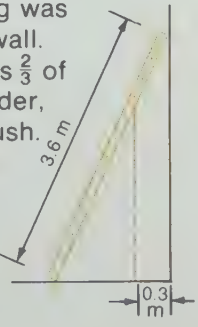
11. A woman 1.8 metres tall casts a shadow 1.2 metres long at the same time that a building casts a shadow 10 metres long. What is the approximate height of the building?



12. To measure the distance between points *A* and B on opposite sides of a river, Jose went downstream 100 m to a point *C*. Then he measured $\angle A$ and $\angle C$ and drew $\triangle A'B'C'$ similar to $\triangle ABC$, making the distance of $\overline{A'C'}$ 10 cm. Then he measured $\overline{A'B'}$ and found it to be 19.2 cm. What was the distance from *A* to *B*?



13. A ladder 3.6 m long was leaning against a wall. When a painter was $\frac{2}{3}$ of the way up the ladder, he dropped his brush. If the brush hit the ground 0.3 m from the wall, about how far was the base of the ladder from the wall?



UTILIZATION

Allow time for solutions of selected exercises to be placed on the chalkboard and discussed. The work with proportions in similar triangles provides an opportunity to review arithmetic and algebraic skills.

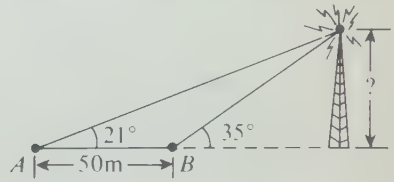
EXTENSION

To provide suitable additional practice, make selective assignments from Supplementary Exercise Set 48 on page S-25, *Workbook* page 86, and *Duplicator Masters* page 61.

Enrichment: This would be an appropriate time to call students' attention to Research Project A on page G-71.

Some students might enjoy working on these challenge problems:

1. The scale of a map is 1 to 1 000 000. What distance in kilometres does 1 cm on the map represent? (*Answer*: 10 km)
2. A man 1.6 m tall casts a shadow 60 cm long. At the same time, a house casts a shadow 4.2 m long. How tall is the house? (*Answer*: 11.2 m)
3. Solve this problem by making a suitable scale drawing. From point *A*, the angle formed by the ground and a line to the top of a radio tower is 21°. When the line is moved 50 m closer to the tower, the angle is 35°. What is the approximate height of the tower? (*Answer*: approximately 42.5 m)



Assignments

Minimum 1–5. Average 1–10. Maximum 1–13.

Objective

The student can solve problems involving pairs of similar right triangles.

PREPARATION

Review the two conditions for similar triangles (Discussion Exercise 3, page G–60). In this lesson these ideas will be extended to similar right triangles.

INVESTIGATION

The Investigation should be worked on independently by each student. Some students may feel that the given information pertaining to the triangles is insufficient.

DISCUSSION

The chief function of the Discussion Exercises is to convince students that the ratio of a leg of a right triangle to the hypotenuse depends on the size of the angle opposite that leg and not on the size of the triangle involved. In Discussion Exercise 3, observe that $\frac{a_1}{a_2} = \frac{c_1}{c_2}$ and, hence, that $\frac{a_1}{c_1}$ will be equal to the ratio $\frac{a_2}{c_2}$. The same result is found for a different set of similar triangles in Discussion Exercise 5.

Solution, Discussion Exercise 1

$\triangle ABC \sim \triangle ONM$ because corresponding angles are congruent (the two right angles are congruent).

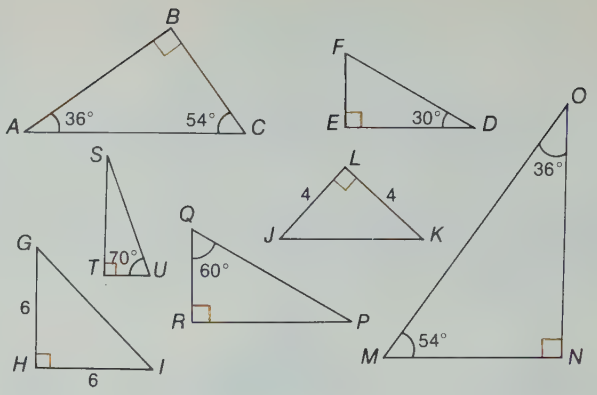
$\triangle GHI \sim \triangle JLK$. These two triangles are isosceles right triangles. Hence, both base angles in each triangle must have measures of 45°. Therefore corresponding angles are congruent.

$\triangle DEF \sim \triangle PRQ$. $m\angle R = m\angle E = 90°$; $m\angle P = 30°$, since $m\angle Q = 60°$; and $m\angle F = 60°$, since $m\angle D = 30°$. Therefore corresponding angles are congruent.

Similar Right Triangles

Investigating the Ideas

Several right triangles are shown.

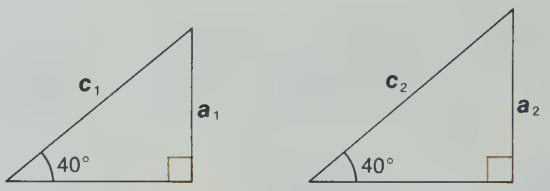


How many pairs of similar right triangles can you find?

Discussing the Ideas

1. Explain how you found the pairs of similar right triangles.

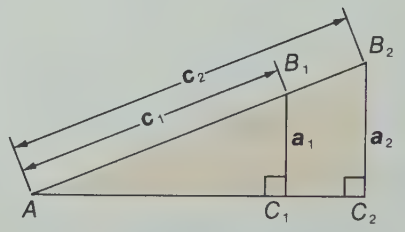
2. Explain why you can be sure that the two triangles are similar.



3. Since the two triangles above are similar, complete the proportions $\frac{a_1}{a_2} = \frac{c_1}{c_2}$ and $\frac{a_1}{c_1} = \frac{\blacksquare}{\blacksquare}$ and explain why they are true.

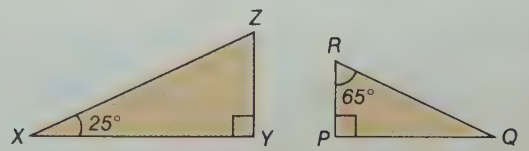
4. Explain why you can be sure that $\triangle AB_1C_1 \sim \triangle AB_2C_2$.

5. Complete the following proportion, constructed from the figure in Exercise 4. $\frac{a_1}{c_1} = \frac{\blacksquare}{\blacksquare}$



6. **A** Are the triangles similar? Why?

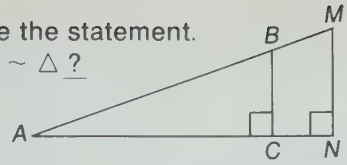
   B Complete the statement. $\triangle XYZ \sim \triangle \underline{?}$.



Using the Ideas

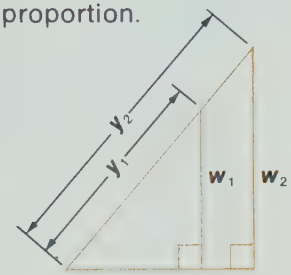
1. Complete the statement.

$\triangle ABC \sim \triangle$?



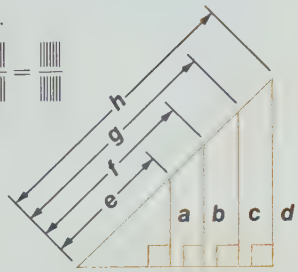
2. Complete the proportion.

$\frac{w_1}{y_1} = \frac{\blacksquare}{\blacksquare}$



3. Give the missing proportions.

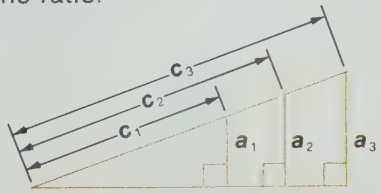
$\frac{a}{e} = \frac{b}{\blacksquare} = \frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare}$



4. Suppose you know that $\frac{a_1}{c_1} = \frac{1}{2}$. Give the ratio.

A $\frac{a_2}{c_2}$

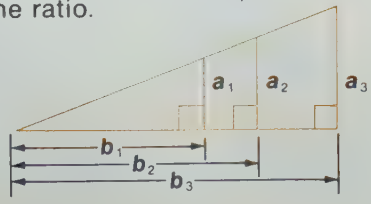
B $\frac{a_3}{c_3}$



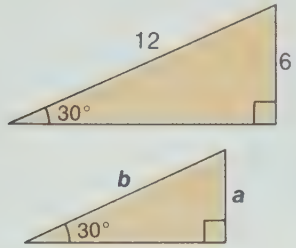
5. Suppose you know that $\frac{a_1}{b_1} = \frac{4}{5}$. Give the ratio.

A $\frac{a_2}{b_2}$

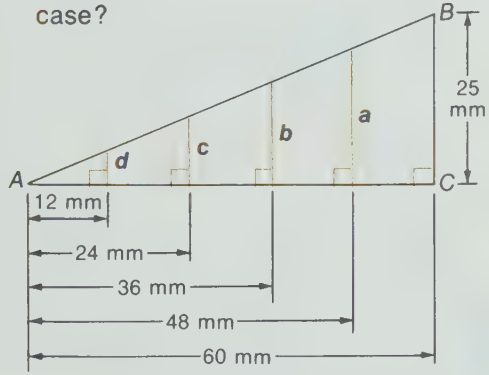
B $\frac{a_3}{b_3}$



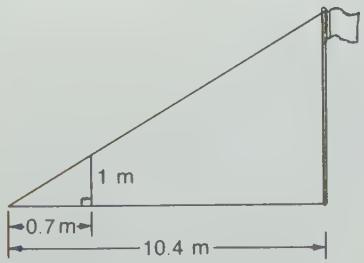
6. What is the ratio $\frac{a}{b}$?



7. Measure the lengths ***a***, ***b***, ***c***, and ***d*** in the figure to the nearest millimetre. Then use proportions to compute ***a***, ***b***, ***c***, and ***d***. Did you get the same numbers in each case?



8. How tall is the flagpole in the drawing?



More practice, page S-26, Set 49

UTILIZATION

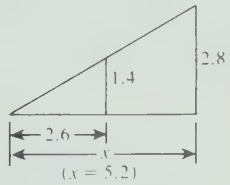
You may find it helpful to work through and discuss some of these exercises with your students. They constitute important background for the material in the lessons that follow.

EXTENSION

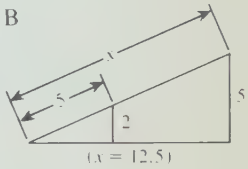
Supplementary Exercise Set 49 on page S-26 and *Workbook* page 87 would be suitable for extra practice assignments.

Remedial: Ask students to give the length x for each drawing.

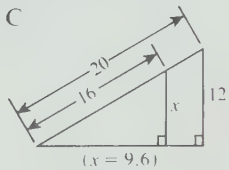
A



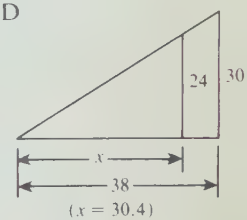
B



C



D



Assignments

Minimum 1–3, oral; 4; 5.
Average 1–3, oral; 4–7.
Maximum 1–3, oral; 4–8.

Objective

Given a right triangle and the measure of an acute angle of the triangle, the student can find the lengths of the sides of the triangle by using ratios.

PREPARATION

Materials: Compass, protractor, graph paper (*Duplicator Masters*, page 83).

The previous lesson should have prepared the student to accept the fact that the ratio of the lengths of two sides of a right triangle depends upon the measure of one of its acute angles, rather than upon the lengths of the side of the triangle. You may want to review this idea before beginning the Investigation.

INVESTIGATION

You may need to give considerable guidance in getting students started on the Investigation. If the angles are carefully drawn and lengths are estimated to the nearest tenth, you can expect about 2-digit accuracy in the ratio.

DISCUSSION

Before assigning the exercises on page G–65, you may want to have students correct their charts for the ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$. This will ensure that all students will be using the same ratios for the exercises. (See the annotated chart in the Investigation.)

Solution, Discussion Exercises 1 and 2B

1. $\frac{b}{c} \approx 0.42$, $b = 12$

   Therefore, $\frac{12}{c} = 0.42$

   and $c = \frac{12}{0.42} \approx 28.6$

2B. Since $\frac{a}{b} = 1.28 \approx 1.3$, $m\angle A \approx 52°$ according to the table.

Ratios for Right Triangles

Investigating the Ideas

Right $\triangle ABC$ has hypotenuse $c = 10$ units and $m\angle A = 30°$.
The lengths of sides **a** and **b** can be estimated as $a \approx 5.0$ and $b \approx 8.7$.
Then the ratios shown in the table can be found.

m∠A			
10°			
20°			
22°			
30°	0.5	0.87	0.57
45°			
52°			
60°			
65°			
70°			

On graph paper, draw seven more right triangles with $c = 10$ units but with angles at *A* having the other measures shown in the table.

? Can you find the ratios for each of the angles in the table?

Discussing the Ideas

1. For $m\angle A = 65°$, the ratio $\frac{a}{b} \approx 2.1$.

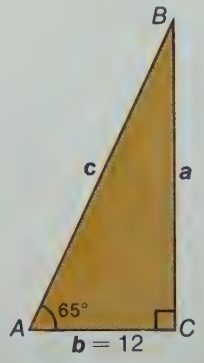
For $\triangle ABC$ at the right,

we have $\frac{a}{12} = 2.1$.

Hence, $a = 12 \cdot 2.1$

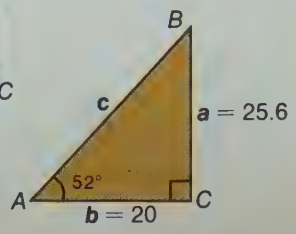
$a = 25.2$

Show how the ratio $\frac{b}{c}$ for 65° can be used to find length **c** in $\triangle ABC$.



2. A Find the ratio $\frac{a}{b}$ to the nearest hundredth.

B How can this ratio and the table above help you give the approximate measure of $\angle A$?



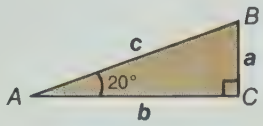
Using the Ideas

For these problems, you can use the table completed in the Investigation on the opposite page.

1. **A** What is $\frac{a}{b}$?

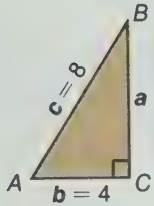
B What is $\frac{a}{c}$?

C What is $\frac{b}{c}$?



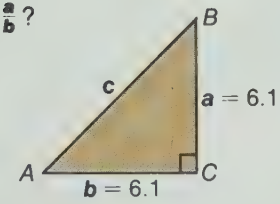
2. **A** What is $\frac{b}{c}$?

B What is $m\angle A$?



3. **A** What is $\frac{a}{b}$?

B What is $m\angle B$?

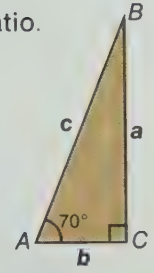


4. Give each ratio.

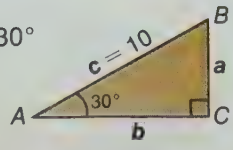
A $\frac{a}{c}$

B $\frac{b}{c}$

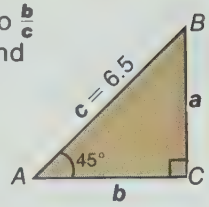
C $\frac{a}{b}$



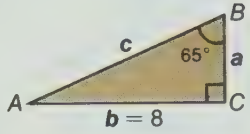
5. Use the ratio $\frac{a}{c}$ for 30° from the table to find length ***a*** in the triangle.



6. Use the ratio $\frac{b}{c}$ for 45° to find length ***b***.



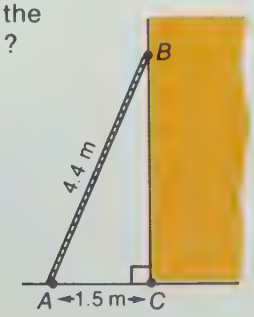
7. Find the length ***a*** for the triangle.



★ **8.** A ladder 4.4 metres long is leaned against a wall so that the base of the ladder is 1.5 metres from the wall.

A What is the measure of the angle the ladder makes with the ground ($m\angle A$)?

B How high up on the wall will the ladder reach?



UTILIZATION

Students will need the tables completed in the Investigation for reference as they do the exercises. Note that each right triangle has been consistently labelled so that $\angle C$ is the right angle.

EXTENSION

Appropriate additional practice exercises are provided on page 88 of the *Workbook*.

Enrichment: Some students may be interested in extending the table of ratios completed in the Investigation to include angle measures at 5° intervals from 5° to 85°. Then, when the table of trigonometric ratios is studied, they can compare their ratios with the more precise ratios given in the table.

Assignments

Minimum 1–5. Average 1–7. Maximum 1–8.

Objectives

Given an appropriate trigonometric ratio and the length of one of the sides of a right triangle, the student can compute the length of another side of the triangle.

Given the lengths of the sides of a right triangle, the student can compute the sine, cosine, and tangent ratios for an acute angle of the triangle.

PREPARATION

Materials: Centimetre rulers.

The previous lesson should have provided adequate preparation for this lesson.

INVESTIGATION

Have the students read the material in the Investigation. Point out that, because the special ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$ have many practical and mathematical applications, they are given special names. Encourage your students to remember these ratios, their names, and the symbols used to denote them.

The Investigation gives the student an opportunity to compute the sine, cosine, and tangent ratios for a given acute angle.

DISCUSSION

Exercise 2 introduces the terms *adjacent side*, *opposite side*, and *hypotenuse* in reference to $\angle A$. This agreement on language may make the trigonometric ratios easier for some students to remember.

Work through Exercise 3 with the class to show how the trigonometric ratios may be applied to particular triangles. You can extend this exercise by supposing that $\cos A \approx 0.92$ is given and then showing how the hypotenuse c may be found.

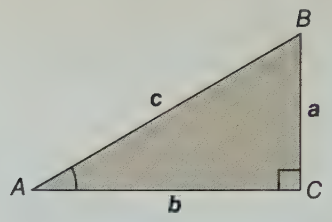
Solution: $b = 40$, $\cos A \approx 0.92 = \frac{b}{c}$

Therefore, $\frac{40}{c} = 0.92$

$0.92c = 40$ and $c = \frac{40}{0.92} \approx 43.5$

Trigonometric Ratios

Investigating the Ideas

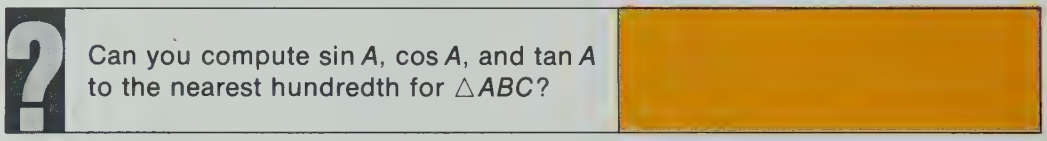


In $\triangle ABC$, if $\angle C$ is a right angle, the ratio $\frac{a}{c}$ is called the **sine** (pronounced like "sign") of $\angle A$. We abbreviate and write: $\sin A = \frac{a}{c}$

The other ratios $\frac{b}{c}$ and $\frac{a}{b}$ are called **cosine** and **tangent** respectively.

We abbreviate these and write: $\cos A = \frac{b}{c}$, $\tan A = \frac{a}{b}$

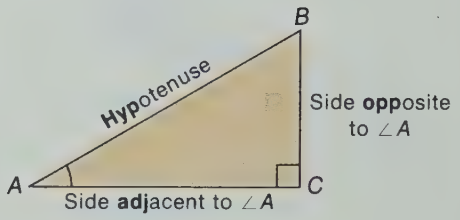
Measure each side of $\triangle ABC$ to the nearest millimetre.



Discussing the Ideas

1. The sine, cosine, and tangent ratios are called **trigonometric ratios** and can be computed with great accuracy by advanced mathematical methods. Using the table in the previous lesson, what is the approximate measure of $\angle A$?

2. It will be helpful to give names to the sides of the right triangle relative to $\angle A$. The following abbreviations may help you remember the ratios:

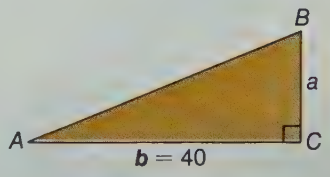


$\sin A = \frac{\text{opp}}{\text{hyp}}$ $\cos A = \frac{\text{adj}}{\text{hyp}}$ $\tan A = \frac{\text{opp}}{\text{adj}}$

   A Which ratio (sin, cos, or tan) can you find if you know the lengths of the opposite side and the hypotenuse?

   B Which ratio can you find if you know the lengths of the adjacent side and the opposite side?

3. If $\tan A \approx 0.43$ and $b = 40$,



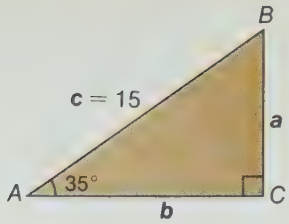
then $\tan A = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$

and $0.43 = \frac{a}{40}$.

Complete the solution to find a.

Using the Ideas

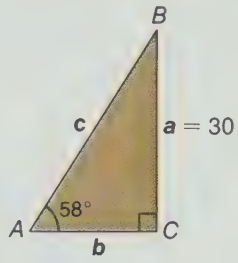
1. In right $\triangle ABC$, $c = 15$, $\sin 35° \approx 0.57$, and $\cos 35° \approx 0.82$. Complete the solution to find a and b.



A $\sin 35° = \frac{a}{15}$, therefore $\frac{a}{15} \approx 0.57$, and $a \approx$ ▨.

B $\cos 35° = \frac{b}{15}$, therefore $\frac{b}{15} \approx 0.82$, and $b \approx$ ▨.

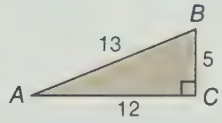
2. In the right triangle, $a = 30$, $\tan 58° \approx 1.6$, and $\sin 58° \approx 0.85$. Complete the solution to find b and c.



A $\tan 58° = \frac{30}{b}$, therefore $\frac{30}{b} \approx 1.6$, and $b \approx$ ▨.

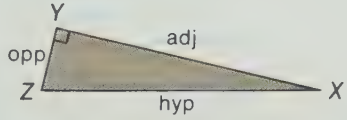
B $\sin 58° = \frac{30}{c}$, therefore $0.85 \approx \frac{30}{c}$, and $c \approx$ ▨.

3. Use the figure at the right to compute the ratios for $\angle A$ to the nearest hundredth.



A $\tan A$ **B** $\sin A$ **C** $\cos A$

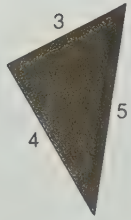
4. In right $\triangle XYZ$, $\angle Y$ is the right angle and the sides are designated as shown.



A Compute $\tan X$ if $\overline{XY}$ has a length of 40 cm and $\overline{YZ}$ has a length of 9 cm.

B Compute $\sin X$ if the length of the hypotenuse is 41 cm.

★ **5.** A right triangle has sides of lengths 3 units, 4 units, and 5 units.



A Which side is the hypotenuse?

B Let the side that is 3 units long be the side opposite $\angle A$. What is $\sin A$?

C What is $\cos A$?

D What is $\tan A$?

More practice, page S-26, Set 50

UTILIZATION

Note that Exercises 1 and 2 are arranged so that the students will need to solve only the equation given. In discussing these exercises, you should discuss how the equation was written from the given information.

Exercise 4 illustrates a departure from the standard labelling of a right triangle and thus uses the adjacent-opposite-hypotenuse terminology.

EXTENSION

Supplementary Exercise Set 50 on page S-26 and *Workbook* page 89 are suitable for use as further practice assignments.

Now would also be an appropriate time to call the students' attention to Research Project B on page G-71.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can interpret and apply a table of trigonometric ratios.

PREPARATION

If the students have retained the tables of ratios they constructed in the Investigation on page G–64, have them compare their ratios with the appropriate ones in this table.

DISCUSSION

Plan to devote some time to a discussion of the table, giving students the opportunity to locate ratios for various measures of angles. Have the students notice that the sine and cosine ratios are always numbers from 0 to 1. The tangent ratio becomes very large as the measure of the acute angle approaches 90°. The tangent ratio is not defined for a 90° angle, because the denominator of the tangent ratio would be 0 for 90°.

A TABLE OF TRIGONOMETRIC RATIOS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	0.000	1.000	0.000	45°	0.707	0.707	1.000
1°	0.017	1.000	0.017	46°	0.719	0.695	1.036
2°	0.035	0.999	0.035	47°	0.731	0.682	1.072
3°	0.052	0.999	0.052	48°	0.743	0.669	1.111
4°	0.070	0.998	0.070	49°	0.755	0.656	1.150
5°	0.087	0.996	0.087	50°	0.766	0.643	1.192
6°	0.105	0.995	0.105	51°	0.777	0.629	1.235
7°	0.122	0.993	0.123	52°	0.788	0.616	1.280
8°	0.139	0.990	0.141	53°	0.799	0.602	1.327
9°	0.156	0.988	0.158	54°	0.809	0.588	1.376
10°	0.174	0.985	0.176	55°	0.819	0.574	1.428
11°	0.191	0.982	0.194	56°	0.829	0.559	1.483
12°	0.208	0.978	0.213	57°	0.839	0.545	1.540
13°	0.225	0.974	0.231	58°	0.848	0.530	1.600
14°	0.242	0.970	0.249	59°	0.857	0.515	1.664
15°	0.259	0.966	0.268	60°	0.866	0.500	1.732
16°	0.276	0.961	0.287	61°	0.875	0.485	1.804
17°	0.292	0.956	0.306	62°	0.883	0.469	1.881
18°	0.309	0.951	0.325	63°	0.891	0.454	1.963
19°	0.326	0.946	0.344	64°	0.899	0.438	2.050
20°	0.342	0.940	0.364	65°	0.906	0.423	2.145
21°	0.358	0.934	0.384	66°	0.914	0.407	2.246
22°	0.375	0.927	0.404	67°	0.921	0.391	2.356
23°	0.391	0.921	0.424	68°	0.927	0.375	2.475
24°	0.407	0.914	0.445	69°	0.934	0.358	2.605
25°	0.423	0.906	0.466	70°	0.940	0.342	2.747
26°	0.438	0.899	0.488	71°	0.946	0.326	2.904
27°	0.454	0.891	0.510	72°	0.951	0.309	3.078
28°	0.469	0.883	0.532	73°	0.956	0.292	3.271
29°	0.485	0.875	0.554	74°	0.961	0.276	3.487
30°	0.500	0.866	0.577	75°	0.966	0.259	3.732
31°	0.515	0.857	0.601	76°	0.970	0.242	4.011
32°	0.530	0.848	0.625	77°	0.974	0.225	4.332
33°	0.545	0.839	0.649	78°	0.978	0.208	4.705
34°	0.559	0.829	0.675	79°	0.982	0.191	5.145
35°	0.574	0.819	0.700	80°	0.985	0.174	5.671
36°	0.588	0.809	0.727	81°	0.988	0.156	6.314
37°	0.602	0.799	0.754	82°	0.990	0.139	7.115
38°	0.616	0.788	0.781	83°	0.993	0.122	8.144
39°	0.629	0.777	0.810	84°	0.995	0.105	9.514
40°	0.643	0.766	0.839	85°	0.996	0.087	11.430
41°	0.656	0.755	0.869	86°	0.998	0.070	14.301
42°	0.669	0.743	0.900	87°	0.999	0.052	19.081
43°	0.682	0.731	0.933	88°	0.999	0.035	28.636
44°	0.695	0.719	0.966	89°	1.000	0.017	57.290
45°	0.707	0.707	1.000	90°	1.000	0.000	—

G–68

Using the Trigonometric Table

The table on the opposite page gives the sine, cosine, and tangent for angles whose measures are a whole number of degrees from 0° through 90°. The numbers in the table are rounded to nearest thousandth. Thus, when you read that "sin 27° ≈ 0.454," this means 0.4535 < sin 27° < 0.4545. The trigonometric ratios are useful in solving problems concerning right triangles. As you continue your study of mathematics, you will learn to use these ratios in many other situations.

1. Use the table of ratios to find the following:

A sin 24° **E** tan 81°
B tan 46° **F** sin 27°
C cos 55° **G** cos 88°
D sin 37° **H** sin 35°

2. Use the table to find the degree measures for **x** in each equation.

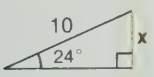
A $\sin 15° = \cos x°$ **C** $\cos 20° = \sin x°$
B $\sin 63° = \cos x°$ **D** $\sin 86° = \cos x°$

3. Complete this sentence.

If $\sin y° = \cos x°$, then $x + y =$ ▨.

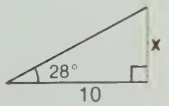
4. Find the length **x**.

A

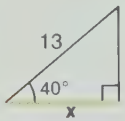


B

C

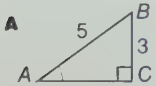


D

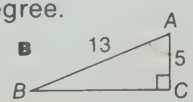


5. In each triangle, find $m\angle A$ to the nearest degree.

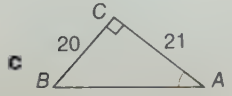
A



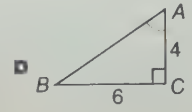
B



C



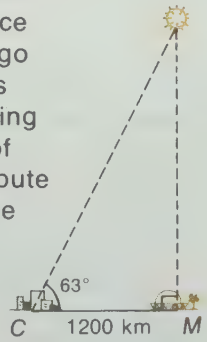
D



6. In traveling one kilometre along a road, suppose a person goes up a gradual incline of 50 metres.

What is the approximate degree measure of the upward slope of the road?

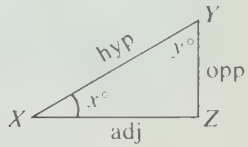
7. A satellite was sighted in Chicago at the time that it was directly over Montreal. The angle determined by the line of sight from Chicago and the ground was about 63°. The distance between Chicago and Montreal is 1200 km. Ignoring the curvature of the earth, compute the approximate altitude of the satellite.



UTILIZATION

Exercises 1, 2, and 3 involve reading of the Table of Trigonometric Ratios. Students may find Exercise 3 a surprise. If the sine of a given angle ($\angle X$) is found, then there is a corresponding angle ($\angle Y$) such that $\sin X = \cos Y$ and, furthermore, $\angle X$ and $\angle Y$ are always complementary. This can be explained as follows:

If X and Y are the two acute angles in a right triangle, they are complementary angles. Let $m\angle X = x°$ and $m\angle Y = y°$. $\sin x° = \frac{\text{opp}}{\text{hyp}} = \cos y°$.



Solution, Exercise 6

Let $\angle A$ be the angle opposite the 50-m side of the triangle. Then $\sin A = \frac{50\ m}{1\ \text{km}} = \frac{50\ \text{m}}{1000\ \text{m}} = 0.050$. Using the table, $m\angle A \approx 3°$.

Solution, Exercise 7

Let h = height of satellite.

$$\frac{h}{1200\ \text{km}} = \tan 63° \approx 1.963$$

$$h = 1.963 \times 1200\ \text{km}$$

$$h \approx 2355.6\ \text{km}$$

EXTENSION

Appropriate additional practice exercises are provided on page 62 of the *Duplicator Masters*.

Enrichment: As a follow-up to this lesson, you may find it useful to do some field work, such as finding the height of a building or a tree. To do this, you will need some type of instrument to measure the angle of inclination. A number of these instruments are available commercially and would prove a great asset in this lesson. You will also need a tape measure in order to measure the distance from the vertex of the angle to the base of the object being measured.

Assignments

Minimum 1, 2, 4. Average 1–5. Maximum 1–7.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

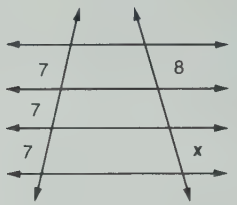
UTILIZATION

The review questions are arranged in the same sequence as the lessons of the module. This will help you in making assignments of the review exercises. Exercises 1 and 2 review parallel lines and transversals. Exercises 3 through 7 cover the work on similar triangles. Exercises 8 through 12 cover material presented in the trigonometry lessons.

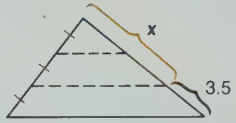
Page 90 of the *Workbook* provides additional exercises that would be suitable for use in reviewing the ideas presented in this module.

REVIEWING THE IDEAS

1. In each figure, the horizontal lines are parallel. Find length **x**.



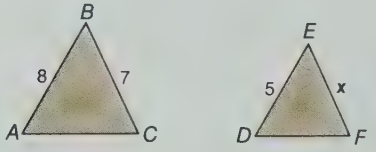
2. Find **x**. The horizontal lines are parallel.



3. $\triangle JKL \sim \triangle PQR$

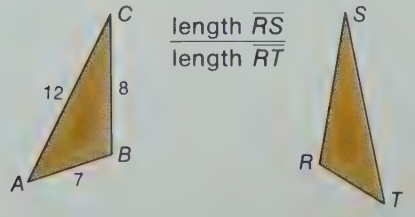
- **A** What pairs of angles are congruent?
- **B** What pairs of sides must correspond?

4. $\triangle ABC \sim \triangle DEF$. What is the measure **x**?

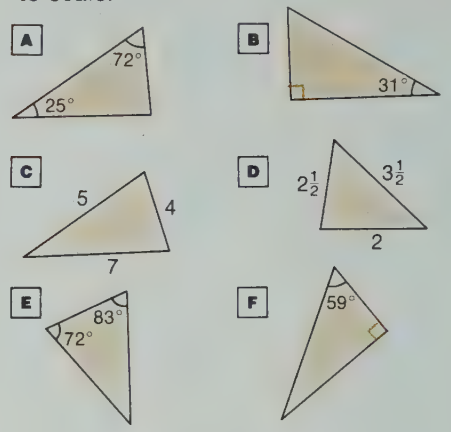


5. If $\triangle BAD \sim \triangle RSV$, then $\angle A$ is congruent to what angle of $\triangle RSV$?

6. $\triangle ABC \sim \triangle TRS$. What is the ratio?

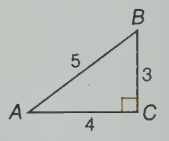


7. Which pairs of triangles are similar? The figures are not necessarily drawn to scale.



8. Using the 3-4-5 right triangle, give the name (sin, cos, or tan) for each ratio.

- **A** ? $A = \frac{3}{5}$
- **B** ? $A = \frac{4}{5}$
- **C** ? $A = \frac{3}{4}$



9. For the triangle in Exercise 8, use the table of trigonometric ratios to estimate the measure of $\angle A$ to the nearest degree.

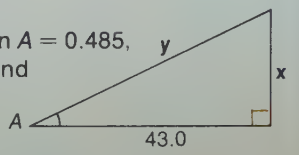
10. Use the table of trigonometric ratios on page G-68 to find each of these.

- **A** sin 27°
- **B** cos 38°
- **C** tan 19°
- **D** cos 71°
- **E** tan 48°
- **F** sin 83°

11. Use the table of trigonometric ratios to give the degree measure of $\angle A$.

- **A** $\sin A = 0.940$
- **B** $\cos A = 0.988$
- **C** $\sin A = 0.500$
- **D** $\cos A = 0.500$

12. Suppose that $\sin A = 0.485$, $\cos A = 0.875$, and $\tan A = 0.554$. Find **x** and **y**.

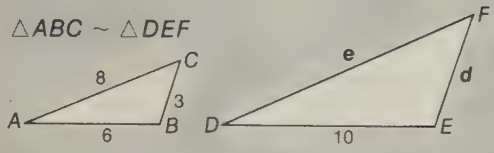


TEST YOURSELF

1. Find length **x**. The horizontal lines are parallel.

2. If $\triangle PQR \sim \triangle JKL$, what angle is congruent to $\angle R$?

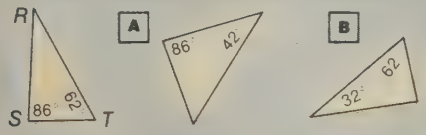
3. $\triangle ABC \sim \triangle DEF$



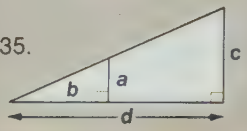
A What is the ratio $\frac{d}{e}$?

B Find length ***d***.

4. Which triangle is similar to $\triangle RST$?

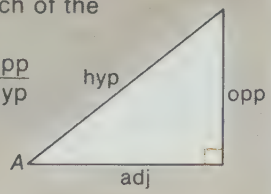


5. In the figure, $\frac{a}{b} = 0.35$. What is $\frac{c}{d}$?



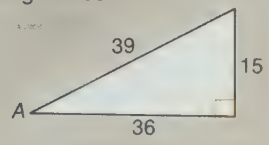
6. Give the name (sin *A*, cos *A*, or tan *A*) for each of the following:

A $\frac{\text{opp}}{\text{adj}}$ B $\frac{\text{opp}}{\text{hyp}}$ C $\frac{\text{adj}}{\text{hyp}}$

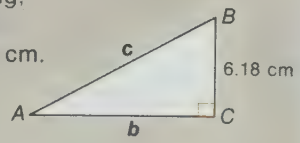


7. Give the following ratios:

A sin *A*
B cos *A*
C tan *A*

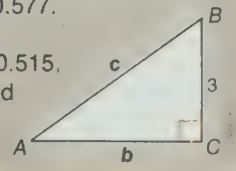


8. In the drawing, $\sin A = 0.500$ and $a = 6.18$ cm. What is length ***c***?



9. Using the triangle in Exercise 8, find ***b*** if $\tan A = 0.577$.

10. Find ***b*** if $\sin A = 0.515$, $\cos A = 0.857$, and $\tan A = 0.601$.

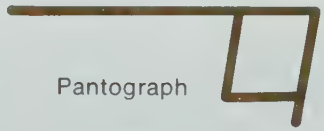


RESEARCH PROJECTS

A Find how you can use similar triangles to determine the heights of trees, buildings, and so on, by measuring their shadows. (See *Mathematics: A Man's Key to Progress* by Richard A. Denholm; Pasadena, California: Franklin Publications, 1968, pp. 53-55; Grolier Ltd.)

B Find out about the origins of trig-, onometry. (See *The Wonderful World of Mathematics* by Lancelot Hogben; Garden City, New York: Doubleday, 1955.)

C A **pantograph** is a mechanical linkage that can be used to produce drawings that are similar to a given drawing.



Find out how pantographs are made. Try making your own model of a pantograph and demonstrate its use. (See *The World Book Encyclopedia*; Chicago: Field Enterprises Educational Corporation, 1974, vol. 15, p. 112.)

TEST YOURSELF

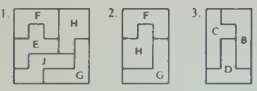
You may need to adjust the assignment of these exercises to conform to your coverage of the topics in this module. For example, if you did not cover any of the lessons on trigonometry, the students should not be expected to answer questions 6 through 10.

Note that the annotated student text page at the left indicates the correlations between the test items and module objectives.

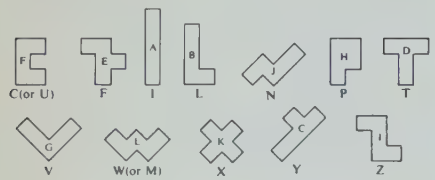
RESEARCH PROJECTS

Each of these projects has been suggested for use in the commentary for specific lessons of this module; however, if the students have not yet had the time or the materials to complete their work, encourage them to do so at this time.

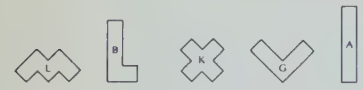
1–3.



4.

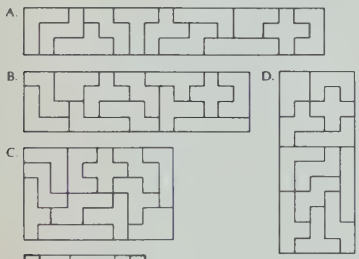


5.

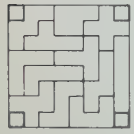


6. Answers will vary. Among the many possibilities: LIT, PIN, TIN, ICY, WIN, DIP, PUN, CITY.

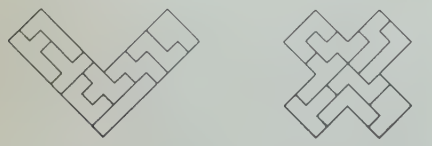
7.



8.



9. Triplication of the V pentomino and the X pentomino:

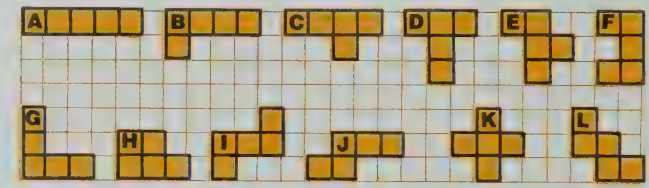


10. 128 square units. (For a solution, see Martin Gardner's "Mathematical Games," *Scientific American*, June 1973.

MATHEMATICAL RECREATIONS

PENTOMINOS

Pentominos are regions formed by five squares connected edge to edge. There are exactly 12 different pentominos.



Cut out the 12 pentominos from graph paper. Then try the problems below. The pieces may be turned over or rotated.

1. Make a square using pieces E, F, G, H, and J.
2. Form a 3 by 5 rectangle using pieces F, G, and H.
3. Form another 3 by 5 rectangle using pieces B, C, and D.
4. Each of the 12 pieces has the approximate shape of one or more letters of the alphabet. What are they?
5. Form a Roman numeral for 1066 using the appropriate pieces.
6. How many words can you spell with the pieces?
7. All 12 pieces can be used to form the following rectangles:
   A 3 by 20 **B** 4 by 15 **C** 6 by 10 **D** 5 by 12
   How many of these can you form?
8. Cover all of an 8 by 8 square except the four corner squares with the pentominos.
9. Choose one of the pentominos. Then use nine other pentominos to form a region whose dimensions are three times those of the pentomino selected.
10. Determine the size of the largest "field" you can enclose with the 12 pieces. Any two pieces must touch along an entire side of a square.

TEST YOURSELF Answers

1. $x = 12$ **2.** $\angle L$ **3.** **A** $\frac{3}{8}$; **B** $d = 5$ **4.** **B** **5.** $\frac{c}{d} = 0.35$
6. **A** $\tan A$; **B** $\sin A$; **C** $\cos A$ **7.** **A** $\frac{15}{39}$; **B** $\frac{36}{39}$; **C** $\frac{15}{36}$ **8.** $c = 12.36$
9. $b = 10.71$ **10.** $b = 4.99$

UNIT G: GEOMETRY

Module 4: Circles, Cylinders, Cones, and Spheres

General Objectives

To review the definitions and terminology associated with circles, cylinders, cones, and spheres.
To provide intuitive proofs of the circumference and area formulas for circles.
To provide applications of circumference and area of circles.
To present volume and lateral area formulas for cylinders and cones.
To offer applications of volume and surface area of spheres.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities	
97	**RED** Given appropriate examples, the student can identify basic parts of circles and spheres, such as radius, chord, great circle, and so on.	G-74, 75 G-76, 77	**WB** 91, 92 **DM** 63		**SWM 3** 38-39
98	**RED** Given appropriate examples, the student can identify right and oblique cylinders and cones and their respective parts.	G-78, 79			
99	**RED** Given the radius or diameter of a circle, the student can find its circumference.	G-82, 83 (G-86)	**WB** 93, 94 **DM** 64	**ASC** AP-5 **PD** b-3	
100	**RED** Given the radius or diameter of a circle, the student can find its area.	G-84, 85 (G-86)	**WB** 95 **DM** 65	**ASC** AP-6 **PD** e-3	**SWM 3** 289
101	**YELLOW** Given the dimensions of a cylinder or a cone, the student can find the volume and surface area of the figure.	G-88, 89	**WB** 96 **DM** 66	**ASC** AP-14, 15 **PD** e-4	**SWM 3** 308-309
102	***GREEN*** *Given the radius of a sphere, the student can find its surface area and volume.*	G-90, 91	**WB** 97 **DM** 67	**ASC** AP-16	
	Reviewing the Ideas	G-92	**WB** 98		

MATHEMATICS

Rigorous proofs of the formulas presented in the module are beyond the range of most students, but sufficient intuitive experience is provided to secure student acceptance of the formulas.

Circumference of a circle is defined to be the limit of the perimeters of the inscribed regular polygons in the circle. Area of circles is defined as the limit of the areas of the regular inscribed polygons.

Particular emphasis is given to understanding the definition of π as the ratio of the circumference of a circle to its diameter.

The work with circles is extended to space geometry. Formulas for surface area and volume of circular cylinders and cones are developed based on previous work with prisms and pyramids, in Module 2 of this unit.

Finally, formulas for volume and surface areas of spheres are presented without proofs.

TEACHING THE MODULE

Materials

Rulers, compasses, balls or other models of spheres, tagboard or cardboard, scissors, cellophane tape, models of cones and cylinders, string, metre sticks or metric tapes, modeling clay.

Vocabulary

arc
cardioid
chord
circle
circular cone
circular cylinder
circumference
deltoid
diameter
exterior of a circle
geodesy
great circle
height
hemisphere
interior of a circle
lateral surface
limaçon
major arc
minor arc
nephroid
oblique circular cone
oblique circular cylinder
perimeter
pi (π)
radius
right circular cone

right circular cylinder
semicircle
small circle
sphere
spherical cap
spherical triangle
spiral
tangent line
tangent plane

The material on circles will not be new to most students, but the development of the circumference and area formulas will be slightly more sophisticated than they have encountered before. Therefore you should proceed slowly through this material. As many students as possible should be involved in the construction and measurement activities inasmuch as these activities are essential to the development of an intuitive understanding of the mathematical formulas.

The work on cylinders, cones, and spheres will probably be new material for most students. It is not expected that all students will master this material or memorize the formulas presented in the lessons. The degree of mastery should be determined by the ability of your students.

Lesson Schedule

Your lesson schedule will depend on the emphasis you give to the various topics. Almost all of the lessons could be extended for more than one class period if you wish to pursue the topic in depth.

A minimum of 10 days should be allowed for complete coverage of the module, but with extension work, projects, and evaluation, a few additional days might be consumed.

Evaluation

Evaluation of the student's work in this module should include concepts, terminology, and the ability to use the various formulas that have been developed for circumference, area, and volume. Most students should be expected to remember the formulas for circumference and area of circles, but other formulas need not be committed to memory. Rather, the emphasis should be placed on applying the formulas to problem-solving situations. The Reviewing the Ideas page and the Test Yourself questions can serve as models for teacher-made tests covering the module. Note also that the Cumulative Review exercises cover all the modules of this learning unit and should be completed before a unit test is administered.

RESOURCES FOR ACTIVE LEARNING

Activities

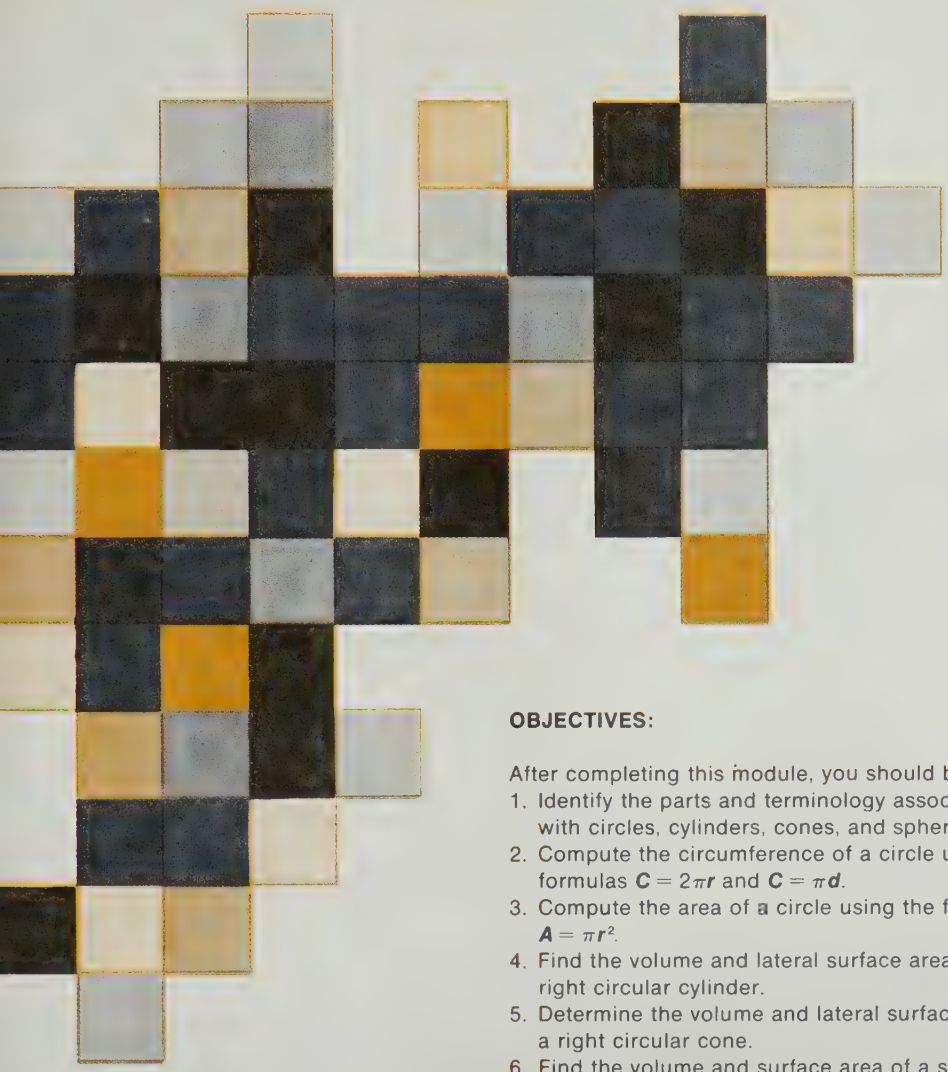
Aftermath 4, p. 76, Creative Publications
Circles, Cuisenaire Co.
Cloudburst, Vol 5 (shapes), Nos. 6100 series, Nos. 7100 series, Midwest Publications
Creative Geometry Experiments, "The Nine-point Circle," No. 26, Midwest Publications
Developmental Math Cards (cycloid), K18, 20, Addison-Wesley
Experiments in Mathematics, Stage 1 (curves) pp. 30–41; Stage 2 (envelopes), pp. 34–37; Stage 3, pp. 28–31, 44–47, Houghton Mifflin
Geoboard Activity Cards (circles), Geocards 120–125, 129–233, Cuisenaire Co.
Mathematics: Man's Key to Progress, Book A, pp. 29–35, 88–91, Rand McNally
Mira Activities for Junior High, Activities 103–105, Creative Publications
Paper Folding, "Circle Relationships . . .," pp. 8–10; "Conic Sections," pp. 20–22, NCTM
Sourcebook for Substitutes, p. 84, Addison-Wesley

Manipulative Devices

Circle Geoboards (Scott Resources; Selective Educational Equipment)
Circle Master Compass (Creative Publications)
Curve stitching (LaPine; Selective Educational Equipment)
Ellipses template (Yoder Instruments)
French Curves template (Sargent-Welch; Yoder Instruments)
Geoboards (Addison-Wesley)
Geodesic Dome Kit (Cuisenaire Co.; Edmund Scientific)
Geometric figures and solids (Creative Publications; Geyer; LaPine; Sigma)
Metric Aids (Addison-Wesley; Balla; Dick Blick; Creative Publications; Educational Teaching Aids)
Mira (Creative Publications; Cuisenaire Co.)
Multi-model Geometric Construction Set (Yoder Instruments)
Orbit Material for Mathematics (Webster, McGraw-Hill)
Perspectograph template (Yoder Instruments)
Protractor (Geyer; Selective Educational Equipment; Sigma)
Sage Kit (LaPine; Sigma)
Space Spider (Geyer; Mind/Matter)
Spheres, cones, and conic sections (Geyer; LaPine)
Thread Sculpture (LaPine; Selective Educational Equipment)

Games and Puzzles

Polyhedron Rummy (Scott, Foresman)



OBJECTIVES:

After completing this module, you should be able to:

1. Identify the parts and terminology associated with circles, cylinders, cones, and spheres.
2. Compute the circumference of a circle using the formulas $C = 2\pi r$ and $C = \pi d$.
3. Compute the area of a circle using the formula $A = \pi r^2$.
4. Find the volume and lateral surface area of a right circular cylinder.
5. Determine the volume and lateral surface area of a right circular cone.
6. Find the volume and surface area of a sphere.

Objective

The student can identify the following terms associated with circles: radius, diameter, chord, arc, tangent, interior, exterior.

PREPARATION

Materials: Rulers and compasses.

See that students have paper, rulers, and compasses; then have them begin work on the Investigation.

INVESTIGATION

Suggest that students draw the circle with centre 0 with a radius of about 3 cm. It should be located centrally on the paper so that the other circles that are to be drawn will fit on the paper. Emphasize that the centres of all the other circles drawn must lie on the circle and that the circles must contain point *P*.

For part A, there are two circles that can be drawn. The centres of these circles lie on opposite sides of $\overleftrightarrow{OP}$.

Many circles that meet the requirements for parts B and C can be drawn.

The largest of all the circles (part D) is the circle whose centre is diametrically opposite point *P* and therefore has a radius twice that of the given circle.

DISCUSSION

Allow students sufficient time to construct the additional circles suggested in Exercise 1. A careful drawing will result in a family of circles whose envelope is a *cardioid* (see Exercise 4, page G-87). It is not necessary to emphasize this word at this time.

For Exercise 2, encourage students to express in their own words the idea that a *circle* is the set of all points in a plane that are at a fixed distance from a given point.

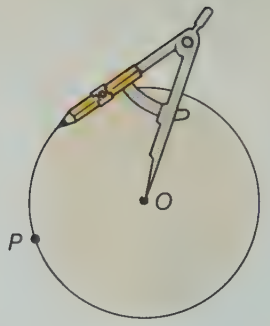
Exercises 3 and 4 review terms associated with circles. No formal definitions of these terms are given, but students should be able to identify various parts of circles in drawings.

A *chord* is a line segment with its endpoints on the circle. A *radius* is any segment from the centre of a circle to a point on the circle. A *diameter* is any chord of a circle that contains the centre.

Circles

Investigating the Ideas

Draw a circle with centre *O* and mark a point *P* on the circle.



Can you draw the circles described below?

Each of the circles must have its **centre on the circle** and must also **pass through point *P***.

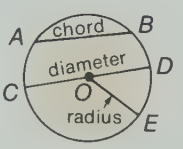
- **A** A circle that is the same size as the given circle.
- **B** A circle smaller than the given circle.
- **C** A circle larger than the given circle.
- **D** The largest of all such circles.

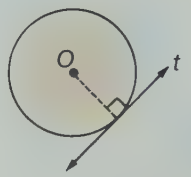
Discussing the Ideas

1. The curve encompassing all the circles which have centres on the given circle and pass through point *P* is called a **cardioid**.
   - **A** Draw several more circles of that type.
   - **B** Look up the word cardioid in a dictionary to see why this curve has this name.

2. Give your own definition of **circle**.

3. The figures below may help you to review some terms associated with circles.



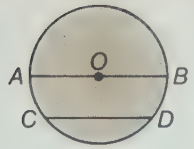


   - **A** Give a definition of **chord**, **radius**, and **diameter**.
   - **B** The set of points on the circle shown in color is minor arc *ABC* (SYMBOL: $\overset{\frown}{ABC}$). Why is it a minor arc? What points on the circle could be a **major arc**?
   - **C** Line *t* is **tangent** to the circle. How are *t* and the circle related?

4. **A** What is meant by the **interior** of a circle?
   B What is the **exterior** of a circle?

Using the Ideas

1. The circle with centre O has a diameter of 2 centimetres.



- **A** How long is a radius?
- **B** What is $\overline{CD}$ called?
- **C** Which is a minor arc of the circle, $\overset{\frown}{ABD}$ or $\overset{\frown}{ACD}$?

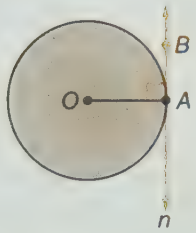
2. A circle has a radius of 5 centimetres with centre at point P. Which word (**interior**, **exterior**, or **on**) describes the location of each of the following points of the circle?

- **A** Point X is 3.7 centimetres from P.
- **B** Point Y is 50 millimetres from P.
- **C** Point Z is 6.09 centimetres from P.

3. Extend the table at the right up through 10 points on a circle.

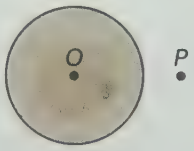
Number of points	Number of chords
2	1
3	3
5	

4. Suppose that a line n is perpendicular to the radius $\overline{OA}$ at point A. Let B be any other point on line n.



- **A** Which is longer, $\overline{OB}$ or $\overline{OA}$?
- **B** Does line n touch the circle in more than one point?
- **C** Is line n tangent to the circle?

★ **5.** **A** Draw a circle with centre O and a radius of 2 centimetres. Mark a point P in the exterior of the circle 3 centimetres from O.



- **B** Draw at least 10 circles whose centres are on the circle and which pass through point P. The curve encompassing all such circles (the *envelope* of the circles) is called a **limaçon** (pronounced "lee'ma·son").

UTILIZATION

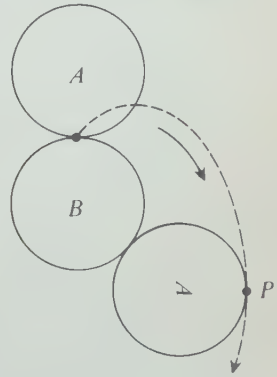
Exercises 1 through 4 can be assigned to all students. In completing Exercise 3, students are likely to discover a pattern in the sequence of the number of chords in a circle. They can check this pattern by actually constructing the chords. There are $\frac{n(n-1)}{2}$ chords determined by n distinct points in a circle.

Encourage as many students as possible to try the construction activity suggested in Exercise 5. Some beautiful designs can be formed by shading or coloring the regions formed by the intersecting circles.

EXTENSION

See *Workbook* page 91 for additional practice exercises.

Enrichment: A cardioid may be constructed in the following manner. Obtain two circular disks, A and B, of equal diameter. Mark a point P on the periphery of disk A. Keeping disk B fixed, mark the path of point P as disk A is rolled, *without slipping*, around disk B. The path of point P will be a cardioid.



Assignments

Minimum 1–4. Average 1–4. Maximum 1–5.

Objective

The student can identify the significant terms associated with spheres.

PREPARATION

Materials: Several balls (or models of spheres) of different sizes, cardboard or tagboard, scissors, rulers, compasses.

It would be desirable for students to work in pairs or small groups on this Investigation. This would reduce the quantity of materials needed for the activity.

INVESTIGATION

The Investigation will require that the student devise some method for determining the diameter of the sphere (ball) he is using. Allow students to choose and use their own methods. Students who cut the hole too large should repeat the activity. You might have students move from station to station, repeating the activity with balls of different sizes.

DISCUSSION

Use the balls and the cardboard with holes that students have made to illustrate the terms associated with a sphere.

In discussing Exercise 1, ask some students to give a definition of a sphere. Try to elicit the idea that a *sphere* is the set of all points, in space, at a fixed distance from a given point. Point out that this definition is analogous to that of a circle except that we are now considering points in space rather than in a plane. Other related definitions are presented for your convenience:

A *chord* is a line segment with its endpoints on the sphere. A *radius* is any segment from the centre of a sphere to a point on the sphere. A *diameter* is any chord of a sphere that contains the centre. A *small circle* is a circle formed by the intersection of a sphere and a plane that does not contain the centre of the sphere. A *great circle* is a circle formed by the intersection of a sphere and a plane that contains the centre of the sphere. A *hemisphere* is one of the two portions of a sphere divided by a great circle.

Spheres

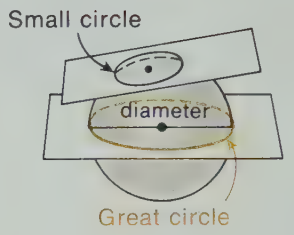
Investigating the Ideas

Choose a small ball such as a marble, golf ball, tennis ball, baseball, or handball.

? Can you cut a circular hole in a piece of cardboard so that the ball you chose will just slip through the hole?

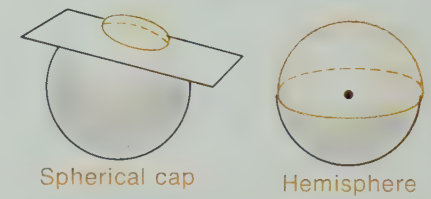
Discussing the Ideas

1. A ball is a model of a **sphere**. If the ball just slips through the circle that you cut in the cardboard, then the circle is the size of the **great circle** on the sphere.
   - A What is the diameter of the sphere that you chose?
   - B If the ball is too big for the hole, is the circle a **small circle** on the sphere?



2. A How many points of intersection do two great circles on a sphere have?
   - B How are the points of intersection related?

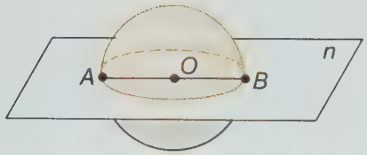
3. A plane cuts from a sphere a **spherical cap**. When is a spherical cap a **hemisphere** (half a sphere)?



4. If a plane and a sphere have only one point in common, then the plane is tangent to the sphere.
   - A What physical examples of a plane tangent to a sphere can you find?
   - B If a plane intersects a sphere in more than one point, what is their intersection?

Using the Ideas

1. In the figure, plane n contains point O, the centre of the sphere.



- **A** What is the intersection of plane n and the sphere?
- **B** What is $\overline{OB}$ called?
- **C** If P is a point such that $\overline{OB} \cong \overline{OP}$, where is point P?
- **D** What is the set of points on the sphere on one side of plane n called?

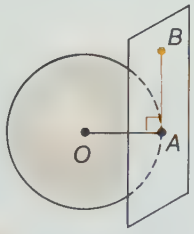
2. Can three points on a sphere all lie on one line? Explain.

3. Point A is on the sphere with centre O. What is the set of all points B such that $\overline{OB} < \overline{OA}$?

4. **A** What part of a sphere is related to a whole sphere as a semicircle is related to a whole circle?

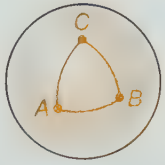
- **B** An arc is related to a circle in the same way as what part of a sphere is related to a sphere?

5. Suppose that a plane is perpendicular to the radius $\overline{OA}$ of a sphere at point A.

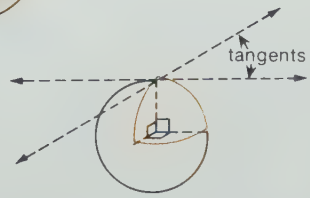


- **A** Which is longer, $\overline{OB}$ or $\overline{OA}$?
- **B** Does the plane touch the sphere at more than one point?
- **C** Is the plane tangent to the sphere?

★**6.** Three arcs of great circles on a sphere form what is called a **spherical triangle**. The second figure shows a special spherical triangle. It encloses one eighth of the sphere's surface. Suppose you define the *measure of an angle of a spherical triangle* as the measure of the angle between the tangents of the circles at the great circle intersections.



Spherical triangle



- **A** What would be the measure of each angle of this special triangle?
- **B** What is true for the sum of the angles of a plane triangle that is not true for a spherical triangle?

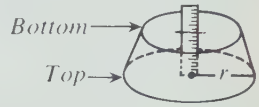
UTILIZATION

If the exercises are assigned as independent work, plan to allow time afterwards for discussion of student answers. You may wish to discuss these exercises rather than make a written assignment.

EXTENSION

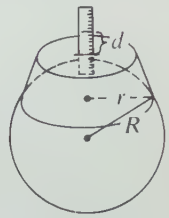
To provide appropriate further practice, make assignments from *Workbook* page 92 and *Duplicator Masters* page 63.

Enrichment: A *spherometer* is the device for measuring the diameter of a sphere. Students might be interested in making a model of one from a plastic margarine cup. Cut a slot in the bottom of a margarine cup that will accommodate a small plastic centimetre ruler.



Place the inverted cup on a flat surface with the ruler in the slot and with a felt pen mark the point on the ruler that is even with the bottom of the cup. Carefully measure the radius r of the open top of the cup.

Place the cup on a sphere whose radius is unknown. Determine the distance d between the mark on the ruler and the bottom of the cup.



The radius R of the sphere can then be computed by the formula $R = \frac{r^2 + d^2}{2d}$.

Since r is a constant for a given cup, it is also possible to devise a scale on the ruler that will give R directly.

Assignments

Minimum 1–4. Average 1–5. Maximum 1–6.

Objective

The student can identify the following terms associated with cylinders and cones: radius, height, axis, base, apex, slant height.

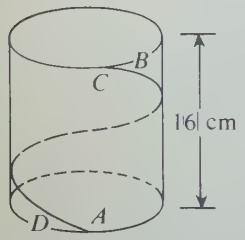
PREPARATION

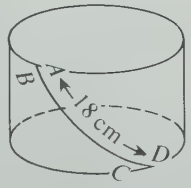
Materials: Scissors, paper, cellophane tape, models of cylinders and cones.

Display a variety of cylindrical food containers. Show that the labels are, ordinarily, rectangles which are wrapped around the containers. Then refer students to the Investigation.

INVESTIGATION

The Investigation presents students with a challenging construction problem. Some students may recall seeing labels from canned biscuits or cookies which are in the shape of parallelograms. Large sheets of paper will be needed for the parallelograms, or students can make the parallelograms by taping together two right triangles whose legs are 18 cm and 16 cm respectively. Two different shapes of cylinders can be made from the Parallelograms. As shown below, $\overline{AD}$ and $\overline{BC}$ can form the bases or $\overline{AB}$ and $\overline{DC}$ can form the bases. Have students tape the two parallelograms to show both possibilities.



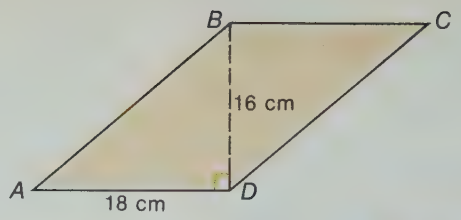


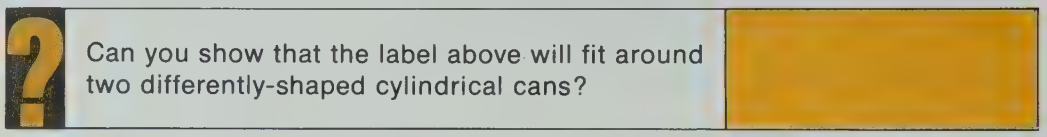
DISCUSSION

Use the models that students made in the Investigation to illustrate the parts of a cylinder. Allow time for students to construct a model of a circular cone in Exercise 3. Point out that the radius of the circular sector becomes the slant height of the cone.

Investigating the Ideas

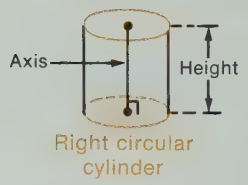
Suppose a label from a cylindrical food can is the size and shape of this parallelogram. Cut out two parallelograms like this one.



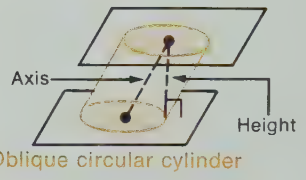


Discussing the Ideas

1. The parallelograms in the Investigation were used to form models of **right circular cylinders**. Which of the two cylinders you made do you think has the greater capacity?

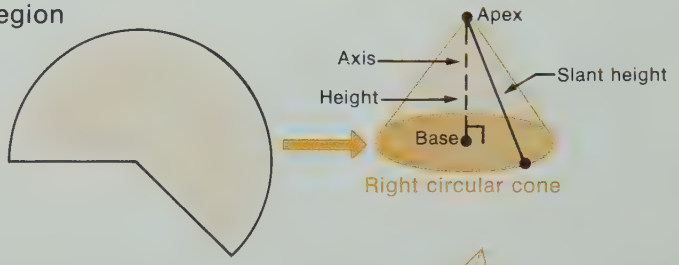


2. What is the difference between an **oblique cylinder** and a right cylinder?



3. A portion of a circular region can be used to make a model of a **cone**.

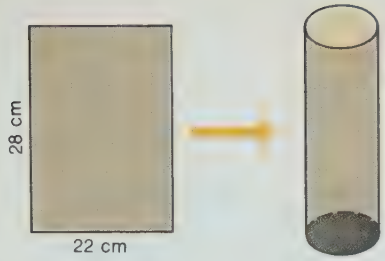
   Make a model of a cone which has a slant height of 10 centimetres.



4. What do you think is the difference between an **oblique circular cone** and a right circular cone?

Using the Ideas

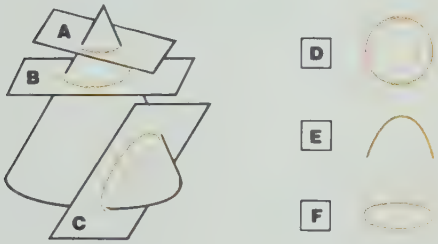
1. The rectangular sheet of paper was used to form a model of a right circular cylinder.
   - **A** What is the height of the cylinder?
   - **B** Give the height of another cylinder that could be formed from this rectangular sheet of paper.



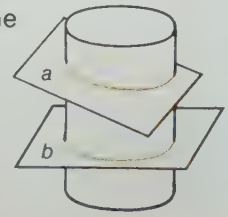
2. Patterns for models of right circular cones are shown below.

   - **A** What is the slant height of each cone?
   - **B** Which cone will have the greatest height?
   - **C** Which cone's base will have the greatest diameter?

3. Three planes intersect this cone. Match each plane **A**, **B**, and **C** with the correct intersection **D**, **E**, or **F**.



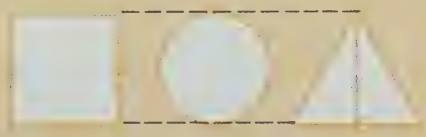
4. **A** Sketch the intersection of plane *a* and the circular cylinder.
   B Sketch the intersection of plane *b* and the circular cylinder.



think

By turning it to the proper position, a solid of a certain shape can be passed through each of these three holes. At some stage of its passage through each hole, the solid will completely fill the hole.

Draw a picture or make a model of the solid.



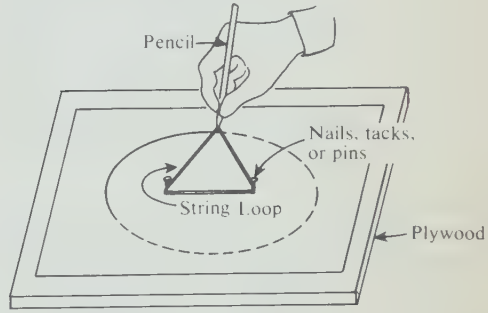
UTILIZATION

If the exercises are assigned as independent work, time should be allowed for discussion afterward. Models of cones made from the patterns in Exercise 2 will aid the discussion.

You may want to give the technical names for the curves in Exercise 3. The curve for plane A is an ellipse and for plane C a parabola. Students need not memorize these names.

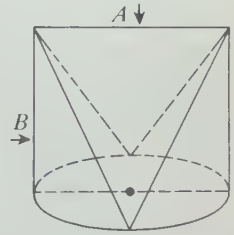
EXTENSION

Enrichment: The illustration below shows a method of constructing an ellipse that students might enjoy trying. The string loop should be kept taut while the pencil is moved around the nails to form the ellipse.



Think Solution

A solid which will pass through the holes is illustrated below.



You can easily see the square shape in the illustration. To see the circular shape, you must view the solid from direction A. To see the triangular shape, you must view the solid from direction B.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–4.

Objective

The student can demonstrate an understanding of the fact that the perimeters of the inscribed regular polygons in circles approach the circumference of the circles as the number of sides of the polygons increase.

PREPARATION

Materials: Centimetre rulers and compasses.

Some time and effort could be saved if duplicated copies of the four inscribed polygons were available for student use in the Investigation. This Investigation will require somewhat more time than most Investigations, so have students begin work on it immediately.

INVESTIGATION

You may need to give students some guidance in the construction of the inscribed regular polygons. Since students will use their constructions for measurement purposes, encourage careful work.

DISCUSSION

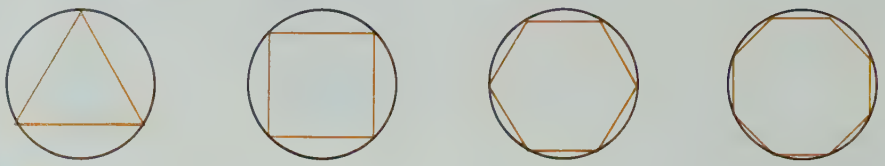
The main objective in this section is for students to gain an understanding of the fact that as the number of sides increases, the perimeters of inscribed polygons approach some number which represents the distance around the circle. Since *length* has generally been used in reference to linear measure, some special definition must be made in order to discuss the distance around a circle. Therefore the *limit of the perimeters* of inscribed regular polygons in a circle is defined to be the *circumference* of the circle.

In discussing Exercise 1, you may expect slight variations in the measurement of lengths and perimeters of the polygons. This should not detract from the main objective of the discussion.

Circumference

Investigating the Ideas

Draw four circles each with a radius of 10 centimetres.



Can you inscribe regular polygons of 3, 4, 6, and 8 sides in the circles?

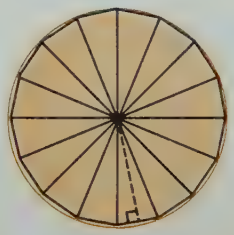
Discussing the Ideas

1. Find the measure of the sides of each inscribed polygon to the nearest tenth of a centimetre. Then record the length of one side and the perimeter of each polygon.

Polygon	Length of one side	Perimeter

2. **A** Which inscribed polygon has the longest sides?
   B Which inscribed polygon has the greatest perimeter?
   C Do the perimeters of the inscribed polygons increase as the number of sides increase?

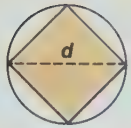
3. Consider regular inscribed polygons with a large number of sides.
   A What happens to the length of the sides as more sides are added?
   B What happens to the perimeter as more sides are added?



It is possible to prove that as the number of sides of the regular polygons increase, their **perimeters approach a definite number**. This number is the **circumference** of the circle.

Using the Ideas

1. **A** Draw circles with diameters of 4, 6, and 8 centimetres



B Inscribe squares in each circle.

C Measure the sides of each square to the nearest tenth of a centimetre and find the perimeter of each square.

D Compute the ratio $\frac{P}{d}$, when P is the perimeter of the square and d is the diameter of the circle.

E What do you notice about the ratios $\frac{P}{d}$ for the circles?

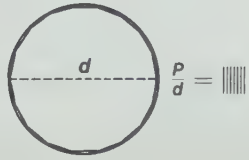
d	Length of side	P	$\frac{P}{d}$
4 cm			
6 cm			
8 cm			

2. Repeat Exercise 1, but inscribe regular hexagons in the circles.

3. Repeat Exercise 1, inscribing regular octagons in the circles.

4. Compare the ratios $\frac{P}{d}$ found in Exercises 1, 2, and 3. Does the ratio $\frac{P}{d}$ increase as the number of sides of the inscribed polygons increase?

5. It can be proved that as the number of sides of the regular inscribed polygons increase, the ratios $\frac{P}{d}$ **approach a definite number**. This number is the ratio of the circumference of a circle to its diameter. What is your estimate of this number?



think

The eight small equilateral triangles can be formed using 16 toothpicks. Can you remove four of the toothpicks so that only four small equilateral triangles will remain? No partial triangles allowed!

UTILIZATION

The exercises have a definite purpose: as students carry out the suggested constructions and measurements, they are expected to discover that the ratio of the perimeter of inscribed regular polygons to the diameter increases as the number of sides of the polygons increase. This sequence of ratios has a limit which is the ratio of the circumference of the circle to its diameter.

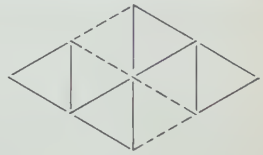
You might assign Exercise 1 to one group of students, Exercise 2 to another group, and Exercise 3 to a third group. Each group could report their findings, and the results could be summarized when discussing Exercises 4 and 5. In discussing Exercise 5, some students will probably mention π, which is the topic of the next lesson.

EXTENSION

Appropriate additional practice exercises are provided on page 93 of the *Workbook*.

Think Solution

Remove the 4 toothpicks represented by the dotted segments in the drawing below.



Assignments

Minimum 1, 2, 5. Average 1–3, 5. Maximum 1–5.

Objective

The student will be able to find the circumference of a circle by using the formula $C = \pi d$ or $C = 2\pi r$.

PREPARATION

Materials: Several cylindrical objects (such as cans or paper rolls), string, metric tapes or metre sticks.

You may find it helpful to establish four to six stations in the classroom with materials available for carrying out the Investigation.

INVESTIGATION

If feasible, arrange for students to work together in small groups on the Investigation. Ask students to take special care to see that the measurements are sufficiently accurate to yield reasonable numbers for the ratio $\frac{C}{d}$. Record various students' results for the ratio $\frac{C}{d}$ on the chalkboard for use in the Discussion.

DISCUSSION

It should be interesting to students to see how near the average of their Investigation findings for the ratio $\frac{C}{d}$ is to the commonly used approximation for π (3.14).

Students should know and be able to use either the formula $C = \pi d$ or $C = 2\pi r$ to find circumferences of circles.

The Number π

Investigating the Ideas

Choose several cylindrical objects with different diameters. The illustration below suggests a way of using a string to find the circumference of a circle.

Can you complete a table like the one at the right for the circles you chose?

Circumference (C)	Diameter (d)	$\frac{C}{d}$
62.4 cm	19.9 cm	3.135
27.5 cm	8.8 cm	3.125

Discussing the Ideas

1. A Is the last column in the table approximately the same number for all the circular objects?
   B Find the average of the numbers in the last column.

2. The ratio $\frac{C}{d}$, which is the same for all circles, is given a special name, π (pronounced "pie"). The table at the right gives various approximations for π.
   A What is π to the nearest hundredth?
   B What is π to the nearest thousandth?

HISTORICAL APPROXIMATIONS FOR π

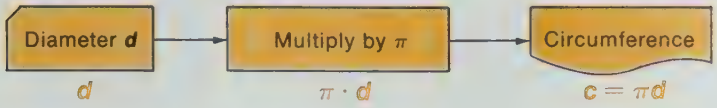
Babylonia, 2000 B.C., $\pi = 3$
Egypt, 1700 B.C., $\pi = \left(\frac{16}{9}\right)^2$
China, 1000 B.C., $\pi = 3$
100 A.D., $\pi = \sqrt{10}$
300 A.D., $\pi = 3\frac{1}{8}$
500 A.D., π computed to six decimal places
Greece, Archimedes,
250 B.C., $3\frac{10}{71} < \pi < 3\frac{1}{7}$
Europe, Middle Ages, $\pi = 3, 3\frac{1}{7}, \sqrt{10}$
Today, with digital computers, π has been computed to 100 000 decimal places.
$\pi = 3.141592653589\ldots$

3. Since $\frac{C}{d} = \pi$, you can find the circumference of any circle by the formula $C = \pi d$. What is the approximate circumference of a circle whose diameter is 10 centimetres? (Use $\pi \approx 3.14$.)

4. If r is the radius of a circle, then the diameter $d = 2r$. Therefore, $C = \pi \cdot d = \pi \cdot 2r = 2\pi r$. What is the approximate circumference of a circle whose radius is 4 centimetres?

Using the Ideas

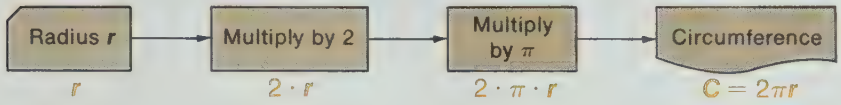
1. Find the circumference of each circle with the given diameter. The flow chart may help you.



A $d = 8$ cm **B** $d = 15$ cm **C** $d = 6.2$ cm **D** $d = 120$ mm **E** $d = 0.71$ m **F** $d = 1.3$ m **G** $d = 1$ dm **H** $d = 2.4$ cm

2. A circle has a circumference of 83.21 centimetres. What is its diameter? (Use $\pi \approx 3.14$.)

3. Compute the circumference of each circle with the given radius. The flow chart may help you.



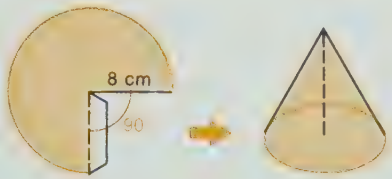
A $r = 5$ cm **B** $r = 6.8$ mm **C** $r = 37$ mm **D** $r = 0.8$ m

4. The Earth is spherical with a radius of about 6400 kilometres. What is the approximate circumference at the equator to the nearest thousand kilometres?

★**5.** Suppose a model of a cone is made from the pattern shown.

A What is the circumference of the base of the cone?

B What is the diameter of the base of the cone?



★**6.** A bicycle wheel has a diameter of 65 centimetres.

A About how many revolutions will it make per kilometre?

B At 30 km/h, about how many revolutions will the wheel make per minute?

★**7.** The best approximations for π are computed by using **infinite series**. The following is one way of getting π from an infinite series:

First compute $\alpha = \frac{1}{2} - \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{5} \cdot \left(\frac{1}{2}\right)^5 - \frac{1}{7} \cdot \left(\frac{1}{2}\right)^7 + \cdots$

Then compute $\beta = \frac{1}{3} - \frac{1}{3} \cdot \left(\frac{1}{3}\right)^3 + \frac{1}{5} \cdot \left(\frac{1}{3}\right)^5 - \frac{1}{7} \cdot \left(\frac{1}{3}\right)^7 + \cdots$

Then compute $4 \cdot (\alpha + \beta)$ to get an approximation for π.

UTILIZATION

Have the solutions to selected exercises placed on the chalkboard and discussed. Exercise 5 will be more interesting if a model is constructed from the suggested pattern.

EXTENSION

To provide further practice, make assignments from *Workbook* page 94, *Duplicator Masters* page 64, and *Arithmetic Skill Card* AP-5.

Enrichment: Students may be amused by the following mnemonic device for remembering the decimal approximation for π. The number of letters in each word of the poem below represents a digit in the decimal expansion of π.

"Sir, I bear a rhyme
excelling
In mystic force and magic
spelling
Celestial sprites elucidate
All my own striving can't
relate."

Solution, Exercise 7

The computation is shown using decimals when possible.

$$\alpha = 0.50000 - \left(\tfrac{0.125}{5}\right) + 0.2(0.03125) - \left(\tfrac{0.0078125}{7}\right)$$
$$\approx (0.50000 + 0.00625) - (0.041667 + 0.001116)$$
$$\approx 0.506250 - 0.042783$$
$$\approx 0.463467$$
$$\beta = 0.333333\ldots - \left(\tfrac{0.037\ldots}{3}\right) + 0.2(0.004074) - \left(\tfrac{0.000457}{7}\right)$$
$$\approx (0.333333\ldots + 0.000815) - (0.01233\ldots + 0.000065)$$
$$\approx 0.334148 - 0.012398 = 0.321750$$
$$4(\alpha + \beta) \approx 4(0.463467 + 0.321750)$$
$$\approx 4(0.785217)$$
$$\approx 3.140868$$
$$\approx \pi$$

Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Objective

The student can use the formula $A = \pi r^2$ to find areas of circles.

PREPARATION

You may want to discuss the following ideas before beginning the Investigation. Defining the area of a circle poses the same kind of problem as defining circumference, because area has only been defined in terms of unit squares. From an intuitive point of view, it is clear that a circular disk or a circular region has an area. The chief task in this section is to use what is known about the area of triangles and parallelograms to develop a formula for computing the area of a circular region.

INVESTIGATION

Extimates of the areas of the inscribed polygons will vary. (Square: 32 sq units; hexagon: 40–42 sq units; octagon: 44–46 sq units) The purpose of the Investigation is to enable the student to observe that the area of the inscribed polygons in a circle increases as the number of sides increases, but has an upper bound or limit.

DISCUSSION

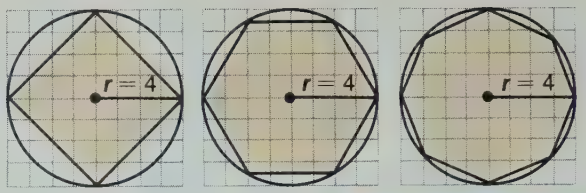
Area of circles is defined in Exercise 2. It is not necessary for students to be able to state the definition.

It will be helpful to have a circular disk cut up as indicated in the figure for Discussion Exercise 3. If you divide the circle into semicircles and tape the pie-shaped pieces of each half together along the outer edge of the circle, you can then "open" the semicircles and arrange them as in the bottom figure. Your students should be able to see that, as they consider polygons with more and more sides, this figure will look more and more like a parallelogram. Since the area of this parallelogram would be its length (which is about one half the circumference, or $\pi \cdot r$), times its height (which is nearly r), it is quite reasonable to assume that the area of a circular region is πr^2.

Area of Circles

Investigating the Ideas

Regular polygons of 4, 6, and 8 sides are inscribed in circles whose radii are 4 units.



Can you estimate the area of each regular polygon by counting squares?

Discussing the Ideas

1. As the number of sides of the polygon increased, did the area also increase?

2. It can be proved that as the number of sides increase the areas of the inscribed regular polygons in a given circle approach a definite number, depending, of course, on the radius of the circle. This number is defined to be the **area of the circle**.

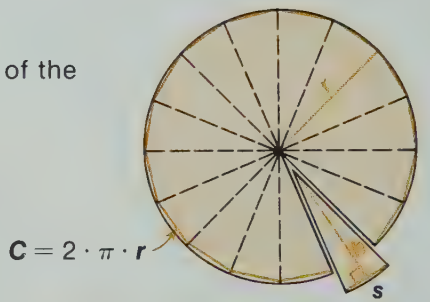
   What are your estimates for the areas of the circles in the Investigation?

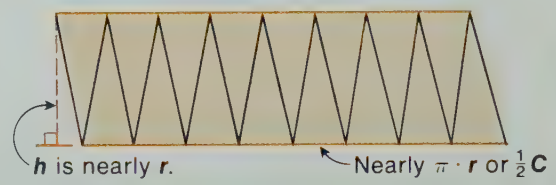
3. The figure at the right is a regular polygon inscribed in a circle of radius r. Think of cutting the polygonal region to form the parallelogram below.

   A Explain why the base of the parallelogram is nearly $\pi \cdot r$.

   B Why is the height h nearly r?

   C Explain why the area of the parallelogram is almost $\pi \cdot r \cdot r$ or $\pi \cdot r^2$.





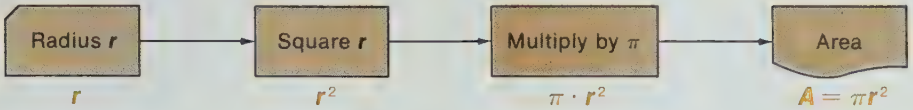
4. The area A of any circle can be found by the formula $A = \pi r^2$ where r is the radius of the circle.

   A What are the areas of the circles in the Investigation?

   B How close is this to your estimate in Exercise 2?

Using the Ideas

1. Find the area of each circle with the given radius. The flow chart may help you.



A $r = 7$ cm **B** $r = 10$ cm **C** $r = 1.2$ m **D** $r = 0.07$ mm

2. The area of a circle is 78.5 square centimetres. What is its radius? (Use $\pi \approx 3.14$.)

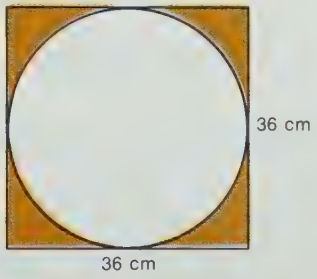
3. Copy and complete the table of data about circles. (Use $\pi \approx 3.14$.)

	Radius	Diameter	Circumference	Area
A	8.0 cm			
B		5 m		
C				12.56 cm²
D		20 km		
E			14.13 cm	
F	0.12 cm			

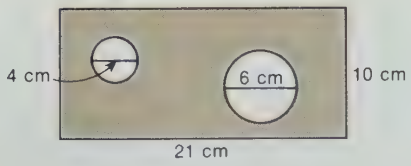
4. A circular mirror is cut from a square piece of glass 36 centimetres on each side.

A If the diameter of the mirror is as large as possible, what is its area?

B What is the area of the glass outside the circle?



★ **5.** Two circular holes are cut in a rectangular piece of sheet metal. The holes are 4 centimetres and 6 centimetres in diameter. The sheet of metal is 21 centimetres by 10 centimetres. What is the remaining area of the rectangular piece after the holes are cut?



UTILIZATION

Assign the exercises as independent work. When the exercises have been completed, students can show some of their solutions on the chalkboard (particularly those for Exercises 4 and 5).

EXTENSION

Appropriate additional practice exercises are provided on *Workbook* page 95, *Duplicator Masters* page 65, and *Arithmetic Skill Card* AP-6.

Enrichment: It is sometimes convenient to express the area of a circle in terms of the diameter of the circle. Since $A = \pi r^2$ and $r = \frac{d}{2}$, we have

$$A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4} = \frac{\pi}{4} \cdot d^2$$

Using $\pi = 3.14$, $\frac{\pi}{4} = 0.785$.

$$\text{Hence, } A = \frac{\pi}{4} \cdot d^2 = 0.785d^2$$

This formula would be useful if one were using a desk calculator or hand-held minicomputer.

Assignments

Minimum 1, 2, 3A–D. Average 1–4. Maximum 1–5.

Objective

The student can solve problems dealing with applications of area and circumference of circles.

UTILIZATION

The exercises can be assigned selectively as desired. The approximation for π is not given for these exercises. You might want to remind students that the approximation 3.14 for π is adequate for most problems except those which require a high degree of accuracy.

Solutions, Exercises 4 and 7

4. The two ends of the track are semicircles with diameter 63.7m. Together they make a complete circle with circumference equal to $\pi \cdot 63.7\text{m} \approx 200\text{m}$. This distance plus the two straight sides gives a distance of 400m.

7A. Area (smallest semicircle) $= \pi \cdot (1.5)^2 = 2.25\pi$
Area (middle semicircle) $= \pi \cdot 2^2 = 4\pi$
Area (large semicircle) $= \pi \cdot (2.5)^2 = 6.25\pi$

B. $2.25\pi + 4\pi = 6.25\pi$

Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Circumference and Area Problems

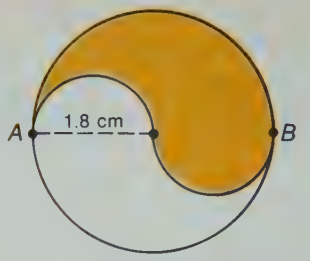
1. **A** What is the circumference of the large circular region?

B What is the distance along the S-shaped curve from *A* to *B*?

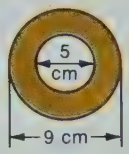
C What is the distance from *A* to *B* around the large circle?

D What is the area of the large circle?

E What is the area of the shaded portion of the circle?

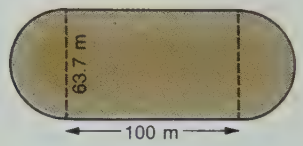


2. Suppose that the ring is stamped out of sheet metal weighing 13 grams per square centimetre. What is its mass?



3. A giant sequoia tree has a circumference of 30.9 metres. What is the diameter of the tree to the nearest tenth of a metre?

4. An oval track has two straight sides 100 metres in length. The ends of the track are semicircles with diameters of 63.7 metres. What is the distance, to the nearest metre, around the track?

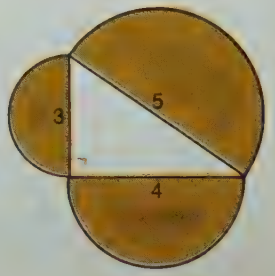


5. What is the area enclosed by the oval track in Exercise 4?

6. The Santa Maria Del Tule Tree, a Montezuma cypress in Oaxaca, Mexico, is reported to have the greatest circumference of any tree in the world. At a height of 150 centimetres above the ground, the circumference is 45.7 metres. What is the diameter of the tree to the nearest tenth of a metre?

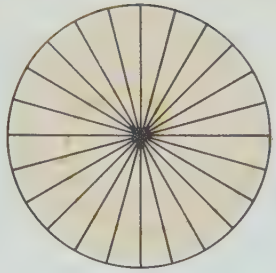
7. **A** Find the area of the semicircle on each side of the right triangle.

B Show that the sum of the areas of the two smaller semicircles is equal to the area of the semicircle on the hypotenuse.

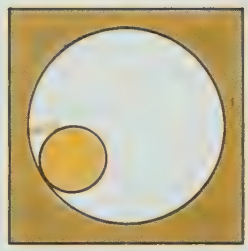


SPECIAL CURVES

1. Draw a circle with a 10-centimetre diameter. Draw a radius every 15°. Mark a dot at the centre of the circle, then measure 0.4 centimetres out from the centre and mark a dot on a radius. Then put a dot 0.8 centimetres from the centre on the next radius, a dot 1.2 centimetres from the center on the next radius, and so on, until you have put a dot on each radius. Connect the dots with a smooth curve to form a **spiral**.

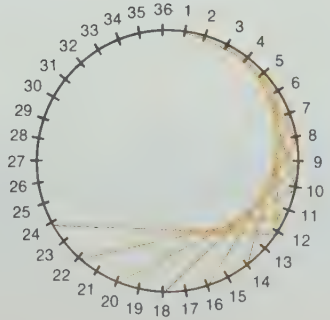


2. Cut a circular hole 12 centimetres in diameter in a piece of cardboard. Cut a circular disk 4 centimetres in diameter from the cardboard. Mark a point *P* on the edge of the disk. Roll the disk along the circular hole (without slipping) and mark the path of point *P* with dots. Connect the dots with a smooth curve. This curve is called a **deltoid**.



3. Draw a circle with centre *O* and a line ℓ containing *O*. Now construct many circles all with their centres on the first circle and tangent to line ℓ. The envelope of all these circles will be a kidney-shaped curve called a **nephroid**.

4. Use a protractor to divide a circle into 36 congruent arcs by marking points every 10° around the circle. Number the points successively from 1 through 36. Join point 1 with 2 by a segment, point 2 with 4, point 3 with 6, in general, point ***n*** with 2***n***. For points numbered more than 18, you will have to repeat. Thus, 19 will be joined with 2, 20 with 4, and so on. The curve outlined by these segments is a **cardioid**.



Objective

The student can construct some special mathematical curves.

UTILIZATION

This set of problems should be regarded as optional material. However, because of their motivational nature, you will probably want several of your students to try them. Students may enjoy a change of pace from purely computational problems.

Enlarged duplicated copies of the circles for Exercises 1 and 4 will enable the student to save time in making the constructions.

EXTENSION

Enrichment: Students who are especially interested in the curves suggested in the exercises may be encouraged to try one or more of the Research Projects on page G–93.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

Given the radius of the base and the height of circular cylinders or cones, the student can compute the volume or lateral surface area by using the appropriate formula.

PREPARATION

Materials: Models of cylinders and cones.

Review the formulas for the volume of prisms and pyramids.

DISCUSSION

In this section, formulas are developed for the volume and lateral surface area of cylinders and cones. This is done by thinking of inscribed prisms and inscribed pyramids, for which the volume formulas are already known. As the number of sides of the prisms and pyramids is increased, their volumes approach the volumes of the cylinders and cones in which they are inscribed. The same is true with regard to surface area.

Answers, Discussion Exercises 1C and 3A–C

1C. You can think of the cylinder as a prism with infinitely many sides. Since the base is a circular closed region, its area is πr^2. Thus, the area of the base (πr^2) times the height (h) will give us the volume (V).

3A. The volume of a pyramid is $\frac{1}{3}Bh$. As the number of sides of the base increases, the area of the base approaches the area of the circle. Then, the volume of the cone is $\frac{1}{3}Bh$ or $\frac{1}{3}(\pi r^2)h$.

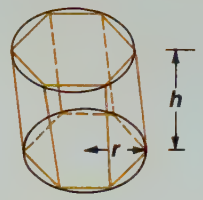
3B. The lateral surface area of a right pyramid is $\frac{1}{2}$ times the perimeter of the base times altitude of each face. As the number of sides increases, the surface area of the pyramid approaches the surface area of the cone. Thus, the surface area of the cone is $\frac{1}{2}$ times circumference times slant height; surface area $= \frac{1}{2} \cdot C \cdot s = \frac{1}{2} \cdot (2\pi r) \cdot s = \pi rs$.

3C. Since s, r, and h are sides of a right triangle, then the Pythagorean Theorem applies.

Volume and Surface Area of Cylinders and Cones

Discussing the Ideas

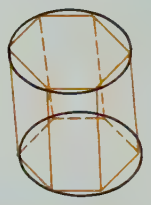
1. Think about a prism inscribed in a cylinder (as in the figure). We know that the volume of the prism is the area of the base times the altitude to the base.



Volume $= \pi \cdot r^2 \cdot h$

   - **A** How does the volume of the prism compare with the volume of the cylinder?
   - **B** As the number of sides of the prism is increased, how does the resulting volume compare to that of the cylinder?
   - **C** Explain why the volume of the cylinder is
     V = area of base · height $= \pi \cdot r^2 \cdot h$

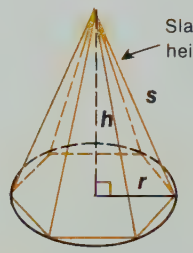
2. In the figure, a right prism is inscribed in a right circular cylinder.



Lateral surface area $= 2 \cdot \pi \cdot r \cdot h$

   - **A** How does the lateral surface area (surface area not including the two bases) of the prism compare to that of the cylinder?
   - **B** As the number of sides of the prism is increased, how does the resulting lateral surface area of the prism compare to that of the cylinder?
   - **C** The lateral surface area of a right prism is the perimeter of the base times the height. Explain why the lateral surface area of the cylinder is $2 \cdot \pi \cdot r \cdot h$.

3. In the figure, a right pyramid is inscribed in a right circular cone.

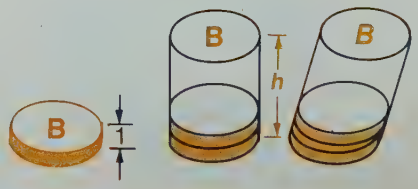


Volume $= \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$

Lateral surface area $= \pi \cdot r \cdot s$

   - **A** Explain why the volume of the cone is $\frac{1}{3} \cdot \pi \cdot r^2 \cdot h$.
   - **B** Explain why the lateral surface area of the cone is half the circumference times the slant height, or $\pi \cdot r \cdot s$.
   - **C** Explain why $s^2 = r^2 + h^2$.
   - **D** If $r = 3$ and $h = 4$, what is s?

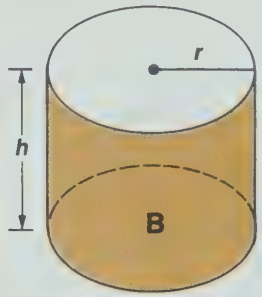
4. If a cylinder is 1 unit thick, what is its volume if the area of the base is B? Explain the formula for the volume of a cylinder in terms of counting slabs of a unit thickness.



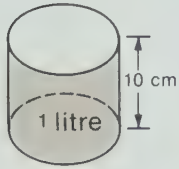
Using the Ideas

1. Copy and complete the table for right circular cylinders. (Use $\pi \approx 3.14$.)

	Radius *r*	Height *h*	Area of base *b*	Volume *V*	Lateral surface
A	6 cm	10 cm			
B	12 cm	6 cm			
C	20 cm	60 cm			
D	0.5 cm	0.25 cm			
E		8 cm		220 cm³	



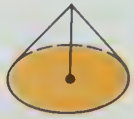
2. A cylindrical can is designed to hold one litre (1000 cm³). The height of the can is 10 centimetres. What is the diameter of the base to the nearest tenth of a centimetre?



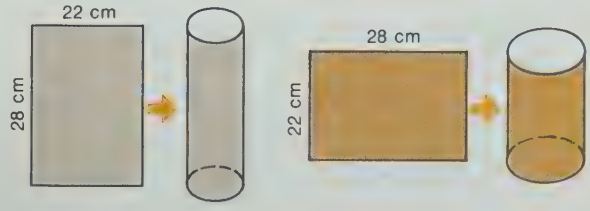
3. Copy and complete the table. The dimensions are for right circular cones. (Use $\pi \approx 3.14$.)

	Radius *r*	Height *h*	Slant height *s*	Area of base *B*	Circumference of base	Volume *V*	Lateral surface
A	6 cm	8 cm					
B	1.2 dm	1.6 dm					
C	5 m		13 m				

4. About how many cubic metres of sand are in a conical sand pile 2.5 metres in diameter and 1.2 metres high?



★ **5.** Two rectangular sheets of typing paper, each 22 by 28 centimetres, are used to form models of cylinders.



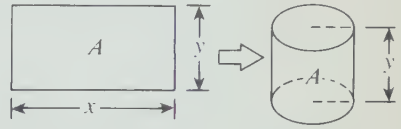
A What is the lateral surface area of each cylinder?

B Which cylinder has the greater volume?

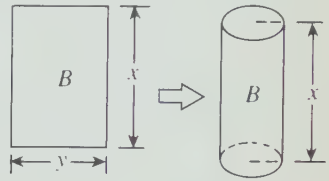
UTILIZATION

Note that in completing the tables for Exercises 1 and 3 there is much computing using the volume and lateral surface formulas. Therefore, you should adjust your assignments according to the needs of your students.

Exercise 5 illustrates that if two cylinders have equal lateral surface area, the cylinder having the larger base will have the greater volume. Consider models of circular cylinders made from rectangular sheets of paper x units long and y units wide, with $x > y$.



radius of base $= \frac{x}{2\pi}$; $V_{(A)} = \pi\left(\frac{x}{2\pi}\right)^2 \cdot y = \frac{x^2 y}{4\pi}$



radius of base $= \frac{y}{2\pi}$; $V_{(B)} = \pi\left(\frac{y}{2\pi}\right)^2 \cdot x = \frac{y^2 x}{4\pi}$

Since $x > y$, $\frac{x^2 y}{4\pi} > \frac{y^2 x}{4\pi}$ or $V_{(A)} > V_{(B)}$

EXTENSION

Further practice exercises are provided on *Workbook* page 96, *Duplicator Masters* page 66, and *Arithmetic Skill Cards* AP-14 and AP-15.

Enrichment: Have some students investigate this question: "If you pour sand from a bucket into two conical piles, will they be similar in shape?" Encourage them to carry out some experiments with sand to see if there is a relation between the base and the height of the cone. The experiment might be varied to include other materials such as salt or sugar.

Assignments

Minimum 1A and B, 3A and B.
Average 1–3. Maximum 1–5.

Objective

The student can use appropriate formulas to compute volume and surface area of a sphere.

PREPARATION

Materials: Modelling clay.

Prepare several balls from modelling clay with diameters of approximately 5 cm. These will be used by students in the Investigation.

INVESTIGATION

Students can work in small groups on this Investigation, but estimates for each part of the Investigation should be made by each student. The ball of modelling clay can be reformed approximately into a cube that has the same volume, by pressing the clay on a flat surface. It is quite likely that such a cube will have slightly rounded corners, but you should not expect great accuracy in this crude method of determining volume of a sphere. In general, students should find that the edge of the cube is about 0.8 of the diameter, or 4 cm. Thus the volume of the cube is about 4^3 cm^3, or 64 cm^3. The estimated surface area should be about 100 cm^2.

DISCUSSION

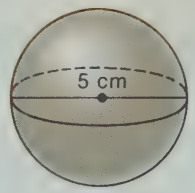
Exercises 1, 2, and 3 relate to the Investigation. Most students can probably see intuitively the surface area of the sphere is less than the surface area of the cube.

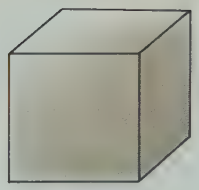
The formulas for surface area and volume of spheres must be accepted without proof since the proofs require more mathematics than the students have had. Be sure to point out that the surface area of a sphere is just 4 times the area of one of its great circles.

Volume and Surface Area of Spheres

Investigating the Ideas

Suppose you made a model of a sphere with a diameter of 5 centimetres from modelling clay. Then you formed a cube from the clay sphere.





What is your estimate for

A the length of an edge of the cube?

B the volume of the cube?

C the surface area of the cube?

Find some ways to check your estimates.

Discussing the Ideas

1. Will the length of one edge of the cube be longer or shorter than the diameter of the sphere?

2. The volume of the cube is the same as the volume of the sphere. What is your estimate for the volume of the sphere?

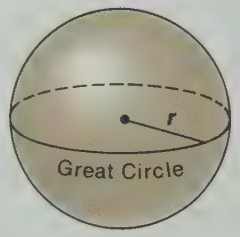
3. Do you think the surface area of the cube is the same as the surface area of the sphere? If not, which has the greater surface area?

4. We shall accept, without proof, the formula for the surface area of a sphere with radius r to be:

   Surface area $= 4\pi r^2$

   A How does the surface area of a sphere compare with the area of a great circle of a sphere?

   B What is the surface area of the sphere in the Investigation?



Surface area $= 4 \cdot \pi \cdot r^2$

Volume $= \frac{4}{3} \cdot \pi \cdot r^3$

5. The formula for the volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$

   A Use this formula to compute the volume of the sphere in the Investigation.

   B How close was your estimate for the volume of the cube to the calculated volume of the sphere?

Using the Ideas

1. Copy and complete the table for spheres. (Use $\pi \approx 3.14$.)

	Radius	Volume	Surface Area
A	3 cm		
B	6 mm		
C	11.4 cm		
D	8 cm		
E	1 m		

2. What is the mass of a solid iron ball with a 10-centimetre diameter if one cubic centimetre of iron has a mass of 7.8 grams?

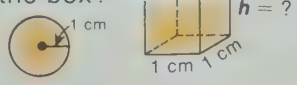
3. The Earth is nearly a sphere with a radius of 6400 kilometres.

- A What is its surface area?
- B About 70.6% of the Earth's surface is covered by water. How many square kilometres is this?
- C What is the Earth's volume?

4. The radius of Earth's moon is about 1728 kilometres.

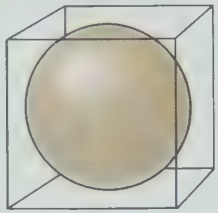
- A What is the surface area of the moon?
- B What is its volume?

5. A ball of wax has a radius of 1 centimetre. When it is softened, it is pushed into a box with a square base 1 centimetre on each side. About how high will the wax be in the box?



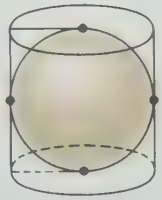
6. A basketball has a radius of approximately 12 centimetres.

- A What is the volume of the ball?
- B How much larger is the volume of a cube that will just hold the ball?



- C What is the surface area of the ball?
- D How much larger is the surface area of the cube than that of the ball?

7. Archimedes, the greatest of the Greek mathematicians, took pride in his proof that the ratio of the volume of the sphere to the volume of the touching cylinder is $\frac{2}{3}$. Show that this is so.



8. One sphere has twice the radius of another. How do their volumes compare? How do their surface areas compare?

9. What happens to surface area and volume if the radius of a sphere is tripled?

UTILIZATION

The exercises provide extensive practice in the use of area and volume formulas for spheres. It will be helpful if solutions to selected exercises are placed on the chalkboard.

Solutions, Exercises 7, 8, and 9

7. Note that the height of the cylinder is twice the radius of the sphere.

$$\frac{\text{Volume of sphere}}{\text{Volume of cylinder}} = \frac{\frac{4}{3} \cdot \pi \cdot r^3}{\pi \cdot r^2 \cdot 2r}$$
$$= \frac{\frac{4}{3}}{2}$$
$$= \frac{2}{3}$$

8. Volume: $V_1 = \frac{4}{3}\pi r^3$
$V_2 = \frac{4}{3}\pi (2r)^3$
$V_2 = 8(\frac{4}{3}\pi r^3)$
Thus, the volume of the sphere with its radius doubled is eight times greater.
Surface area: $S_1 = 4\pi r^2$
$S_2 = 4\pi (2r)^2$
$S_2 = 4(4\pi r^2)$
Hence, the surface area of the sphere with its radius doubled is four times greater.

9. Volume: $V_2 = \frac{4}{3}\pi (3r)^3$
$= 3^3(\frac{4}{3}\pi r^3)$
$= 27(\frac{4}{3}\pi r^3)$
Volume is increased 27 times.
Surface area: $S_2 = 4\pi (3r)^2$
$= 3^2(4\pi r^2)$
$= 9(4\pi r^2)$
Surface area is increased nine times.

EXTENSION

To provide additional practice, make selective assignments from *Workbook* page 97, *Duplicator Masters* page 67, and *Arithmetic Skill Card* AP-16.

Assignments

Minimum 1–3. Average 1–6. Maximum 1–9.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

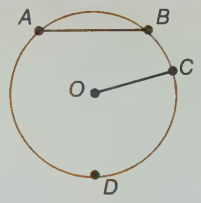
UTILIZATION

Adjust your assignment of the review exercises to the coverage of the module. Encourage students to ask questions and discuss any problems relative to the material of the module.

Page 98 of the *Workbook* provides additional exercises that would be suitable for use in reviewing the major ideas presented in this module.

REVIEWING THE IDEAS

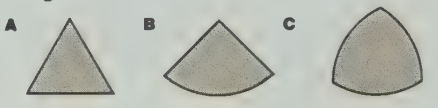
1. Name each of the following on the circle.



- **A** A chord.
- **B** A radius.
- **C** The centre.
- **D** A minor arc containing point *C*.

2. If a plane intersects a sphere and contains the centre of the sphere, what is this intersection called?

3. Which pattern will form a model for a right circular cone?



4. What is the ratio of the circumference of a circle to its diameter?

5. State the formula for the circumference of a circle.

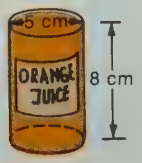
6. Find the circumference of a circle with a radius of 12.3 centimetres. (Use $\pi \approx 3.14$.)

7. If a circle has a circumference of 17.36 metres, what is the radius of the circle to the nearest tenth of a metre?

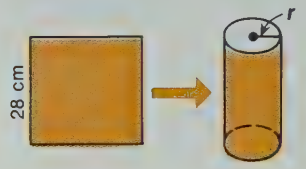
8. What is the formula for the area of a circle with radius r?

9. Find the area of a circle with a radius of 6.12 centimetres.

10. What is the area of a rectangular piece of paper that will just fit around the juice can?



11. A model of a right circular cylinder is made from a square sheet of paper 28 centimetres on each side.

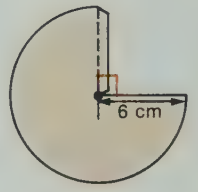


What is the lateral area of the cylinder?

12. What is the radius of the base of the cylinder in Exercise 11?

13. A model of a circular cone is made from $\frac{3}{4}$ of a circular region with a radius of 6 centimetres.

- **A** What is the slant height?
- **B** What is the circumference of the base?
- **C** What is the radius?



14. A cylindrical tank has an inside diameter of 2.1 metres and a height of 1.5 metres. What is the volume of the tank?

15. What is the volume of a cone whose base and height are the same as the cylinder in Exercise 14?

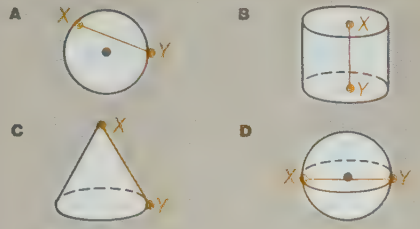
16. What is the lateral surface area of a right circular cone with a height of 12 centimetres and the diameter of the base 14 centimetres?

17. A golf ball has a radius of about 2.1 centimetres.

- **A** Find the volume.
- **B** Find the surface area.

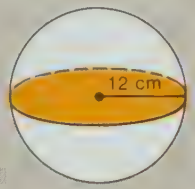
TEST YOURSELF

1. Give the name of $\overline{XY}$ for each figure.

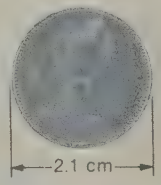


2. If a circle has a diameter of 20 centimetres, what is its circumference? (Use $\pi \approx 3.14$.)

3. A sphere has a radius of 12 centimetres. What is the circumference of a great circle of the sphere?



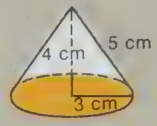
4. What is the area of one side of this coin to the nearest hundredth of a centimetre?



5. A cylinder has a radius of 4 centimetres and a height of 6.5 centimetres. What is the lateral surface area?

6. What is the volume of the cylinder in Exercise 5?

7. What is the lateral surface area of this circular cone?



8. What is the volume of the cone in Exercise 6?

9. What is the surface area of a sphere with a radius of 2 metres?

10. What is the volume of a ball whose radius is 7 centimetres? Round your answer to the nearest hundredth of a cubic centimetre.

RESEARCH PROJECTS

A Investigate several different kinds of mathematical curves such as the **astroid**, the **deltoid**, **parabola**, **ellipse**, and **hyperbola**. Show constructions of the curves. (See *Book of Curves* by E. H. Lockwood; Cambridge, Massachusetts: C.U.P., 1963; Macmillan.)

B Make a collection of ordinary objects that are examples of cylinders, cones, and spheres. Find the dimensions of each and then use the appropriate formulas to find the area and volume of each object. Make a display of them.

C Make a geometric pattern using colored threads and heavy paper. (See *Curves in Space* by Donovan A. Johnson; St. Louis: Webster, 1961; McGraw-Hill Ryerson.)

D Mosaic designs are often formed from polygons and other geometric figures. Using a straightedge and a compass, construct your own mosaic design. (See *Mathematical Recreations* by Maurice Kraitchik; New York: Dover Publications, 1953, pp. 193–213.) General Publishing.)

TEST YOURSELF

The items on this self-evaluation test are designed to give students an opportunity to discover for themselves how effectively they can work with the major ideas of this module. If they do well on this test, they may be expected to perform satisfactorily on an achievement test covering this material.

Encourage students to check their own answers against the key provided on page G–96 and return to the appropriate lessons of the module to review any concepts that gave them difficulty.

Note that correlations between test items and the module objectives are indicated by annotations on the student text page at the left.

RESEARCH PROJECTS

Additional references for Project A include: *Circles* by Josephine Mold, in the series *Topics in Mathematics*; New York: Cambridge University Press, 1967, distributed in Canada by Macmillian Company of Canada Ltd., Toronto; and *Mathematical Models*, by H. Martyn Cundy and A. P. Rollett, New York: Oxford University Press, 1954, pp. 28–38, distributed in Canada by Oxford Press Ltd., Don Mills, Ontario.

Research Project B could be participated in by all students as a class project. Each person could contribute one article for the display.

Students interested in design and art might find Projects C and D particularly interesting.

Objective

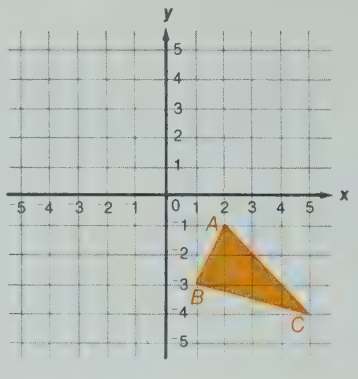
The student can demonstrate the ability to work with the major concepts presented in this learning unit.

UTILIZATION

The significant concepts of the entire learning unit are reviewed in the items included in the Cumulative Review. The items are sequenced so that, if you wish, you may make separate assignments of the exercises related to specific modules. Exercises 1 through 6 pertain to the ideas presented in Module 1; Exercises 7 through 12, to Module 2; Exercises 13 through 16, to Module 3; and Exercises 17 through 22, to Module 4.

CUMULATIVE REVIEW

For Exercises 1, 2, and 3, draw $\triangle ABC$ in the position shown on the grid.



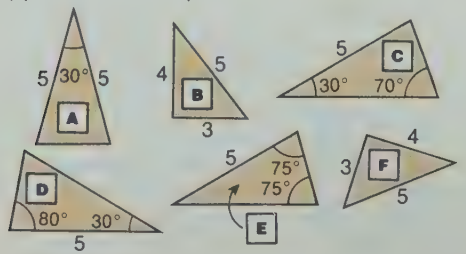
1. Show the slide image of $\triangle ABC$ after a translation of "left 3, up 5."

2. Show the reflection image of $\triangle ABC$ if the **x**-axis is the reflecting line.

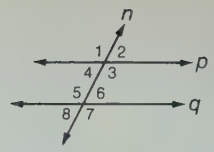
3. Using the origin (0,0) as a centre of rotation, a rotation image of point A is (1, 2). What are the rotation images of B and C for the same rotation?

4. Suppose $\triangle PQR \cong \triangle XYZ$.
   A $\angle Q \cong$ ▮▮▮ **B** $\overline{PQ} \cong$ ▮▮▮

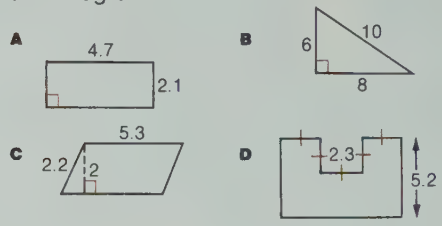
5. State which pairs of triangles are congruent and give the congruence theorem (SSS, SAS, or ASA) that applies for each pair.



6. Lines p and q are parallel. Name the angles congruent to $\angle 1$.

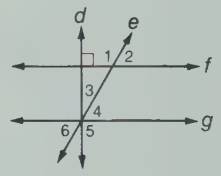


7. Find the area and perimeter of each region.

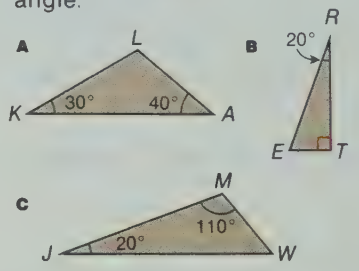


8. Suppose $f \parallel g$ and $m\angle 2 = 66°$.

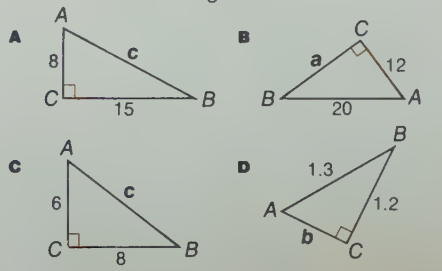
   What is the measure of each of the other angles?



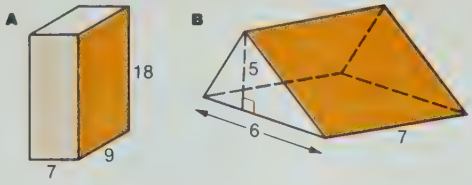
9. Find the measure of the third angle.



10. Find the length of the third side of each triangle.

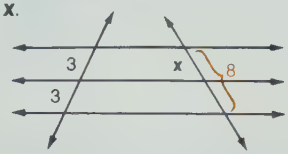


11. Find the volume of each solid.

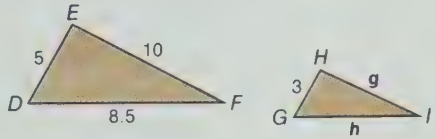


12. If a pyramid has a base area of 276 square metres and a height of 15 metres, what is the volume?

13. The horizontal lines are parallel. Find length **x**.



14. In the figure, $\triangle DEF \sim \triangle GHI$. Find the length of ***g*** and ***h***.

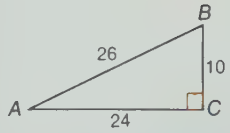


15. $\triangle ABC$ is a right triangle. Give each ratio.

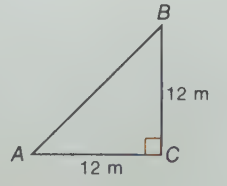
A $\sin A$

B $\cos A$

C $\tan A$



16. If $\cos A = 0.707$, what is the length of the hypotenuse to the nearest hundredth of a metre?



17. In every circle, what is the ratio of the circumference to the diameter?

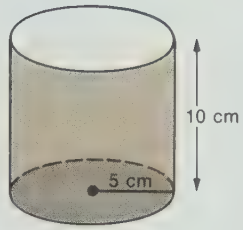
18. What is the circumference of a circle whose diameter is 10 centimetres?

19. What is the area of a circle whose radius is 6 metres?

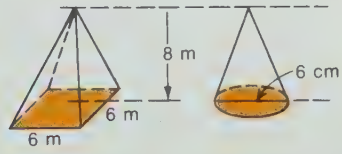
20. **A** What is the volume of this cylinder?

B What is its lateral surface area?

C What is its total surface area?



21. A pyramid has a square base 6 metres on each side and a height of 8 metres. A circular cone has a diameter of 6 metres and a height of 8 metres.



A Find the volume of the pyramid.

B Find the volume of the cone.

22. A sphere has a radius of 2 centimetres.

A Find the volume.

B Find the surface area.

MATHEMATICAL RECREATION

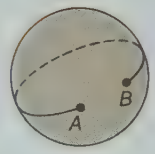
This Mathematical Recreation offers an unusual opportunity to relate an application of mathematics to the physical world. A world globe and a piece of string are all the equipment needed.

Encourage students to make up their own geodesic problems like these and see whether their classmates can solve the problems.

MATHEMATICAL RECREATIONS

GEODESICS

A **geodesic** is the shortest path between two points on the *surface* of a geometric figure. On a sphere, a geodesic is an arc of one of the great circles of the sphere. Airline pilots flying long distances generally fly along the geodesics of the Earth.



If you stretch a string tightly between two points on a globe, the string will fall along the geodesic for the two points. Use a piece of string and a globe to find the geodesic between each pair of cities in the first list. Then choose the name of the third city, from the second list, that also lies almost directly on the geodesic of the pair in the first list.

GEODESIC DETERMINED BY:	THIRD CITY ON GEODESIC
1. Vancouver, B.C. and Miami, Florida	A Bangalore, India
2. New York City and Berlin, Germany	B Berlin, Germany
3. Peking, China and London, England	C Calcutta, India
4. Mexico City and New York City	D Fairbanks, Alaska
5. London, England and Sidney, Australia	E Glasgow, Scotland
6. Rome, Italy and Chicago, Illinois	F Havana, Cuba
7. Toronto, Canada and Tokyo, Japan	G Helsinki, Finland
8. Portland, Oregon and Melbourne, Australia	H Honolulu, Hawaii
9. Addis Ababa, Ethiopia and Singapore, China	I Le Mans, France
10. Madrid, Spain and Seoul, Korea	J New Orleans, Louisiana
11. Copenhagen, Denmark and Denver, Colorado	K Reykjavik, Iceland
12. Detroit, Michigan and Lima, Peru	L Topeka, Kansas

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A chord; B axis; C slant height; D diameter
2. 62.8 cm
3. 75.36 cm
4. 3.46 cm^2
5. 163.28 cm^2
6. 326.56 cm^3
7. 47 1 cm^2
8. 37.68 cm^3
9. 50.24 m^2
10. 1436.03 cm^3

G-96

UNIT H: SOME SPECIAL TOPICS IN MATHEMATICS

Module 1: Probability

General Objectives

To review and extend simple probability concepts.
To introduce the idea of an event and the probability of an event.
To introduce permutations and combinations as a means of solving special counting problems.
To introduce the probability of the union and intersection of events.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
103	**RED** Given a probability experiment, the student can list the outcomes and give the probability of each outcome.	H-2, 3 H-4, 5	**WB** 99, 100	**SWM 3** 134-137
104	**RED** Given a probability experiment, the student can find the theoretical probability of a specified event.	H-6, 7 H-8, 9	**WB** 101, 102 **DM** 68	**SWM 3** 140-143
105	**RED** Given a certain number of elements, the student can find the number of permutations of the elements.	H-12, 13 H-14, 15	**WB** 104 **DM** 70	
106	**RED** Given an appropriate situation, the student can apply the Counting Principle: If an operation can be done in any one of n ways and a second operation can be done in any one of m ways, then the two operations can be performed in $n \cdot m$ ways.	H-16, 17		
107	**YELLOW** Given a probability experiment in which repeated events occur, the student can use a tree diagram to find the probability of each outcome.	H-10, 11	**WB** 103 **DM** 69	
108	**YELLOW** Given a certain number of elements, the student can find the number of combinations of n elements taken r at a time.	H-18, 19 (H-20, 21)	**WB** 105 **DM** 71	
109	***GREEN*** *Given two events of a probability experiment, the student can find the probability of the union or intersection of the events.*	H-22, 23		
	Reviewing the Ideas	H-24	**WB** 106	

MATHEMATICS

Probability theory is said to have begun in the seventeenth century with a correspondence between Blaise Pascal (1601–1665) and Pierre de Fermat (1623–1662) concerning a gambling problem. Since that time probability has grown to have its applications to game theory, statistics, and operations research.

To apply probability to real situations, one makes a *probability model* of the real situation. For example, in tossing a coin, the coin is usually assumed to be fair and will be as likely to come up heads as tails. Each event is then assigned the probability of $\frac{1}{2}$; however, this is just a guess, a model.

Consider a probability experiment with a set S of outcomes, $\{o_1, o_2, o_3, \ldots o_n.\}$ The set $S = \{o_1, o_2, o_3, \ldots o_n\}$ is called a *sample space* and any subset of S is called an *event.*

The set S is a *probability space* with probability function P if

(a) $0 \leq P(o_i) \leq 1$ for each $i = 1, \ldots, n$,

(b) $\sum_{i=1}^{n} P(o_i) = P(o_1) + P(o_2) + \ldots + P(o_n) = 1$

If each outcome in the sample space is equally likely, it follows that the probability of each outcome is $\frac{1}{n}$. The probability of an event is the sum of the probabilities of each outcome in that event.

The number of possible outcomes for certain probability problems is sometimes difficult to ascertain. Therefore, several lessons of the module are devoted to counting problems, emphasizing permutations and combinations.

A *permutation* of a set of elements is an arrangement of the elements in a certain order.

If a set contains n elements there are $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n = n!$ permutations or arrangments of the elements of the set.

A *combination* is a subset of a set. Given a set of 5 elements, we may consider all the different 3-element subsets of this set. To do this, we speak of the number of combinations (subsets) of 5 things (elements) taken 3 at a time. The number of combinations of n things taken r at a time is denoted by ${}_nC_r$. There are other symbols for ${}_nC_r$, such as $\binom{n}{r}$ and $C(n, r)$.

TEACHING THE MODULE

Materials

Beads, beans, wooden or plastic cubes, coins, dice, spinners (homemade), tagboard, scissors, cellophane tape, playing cards, rulers, protractors, thumbtacks, paper bags.

Vocabulary

chances
combinations
Counting Principle
equally likely outcomes
event
factorial
intersection
outcomes
permutations (arrangements)
probability
sample space
tree diagram
union

This module is designed to acquaint students with some elementary ideas of probability. Although no previous experience with probability is assumed, students who have studied about probability in Unit D, Module 2, will have a headstart. With such students, the first two lessons can be regarded as a review of basic ideas concerning probability.

In teaching the module it is important that you have available some simple probability materials, such as those listed above, for use in simple experiments. This module provides an excellent opportunity for student activity and investigation. Encourage students to perform probability experiments and compare the empirical results achieved through their experiments with theoretical results.

In carrying out experiments, combine the results of several students or even the whole class in order to get a large number of trials. There is a good opportunity for students to learn to organize their data in efficient and meaningful ways.

The material on permutations and combinations will be new to all students. The last two lessons of the module should probably be assigned to only your more able students. Consult the Long-Range Planning Chart for possible coverage of material.

Lesson Schedule

You could easily spend 12 to 15 days covering the material in the module. You will find that student involvement in the activities of this module will tend to slow down the coverage. Some experiments may require nearly a whole class period to carry out. In the long run, however, such concentrated effort may yield dividends in increased motivation and keener interest in mathematics. Adjust your lesson schedule and coverage of material to meet the needs and interests of your students.

Evaluation

The limited number of objectives for this module indicates that you should not expect a great number of specific student skills to be developed. Certainly, the student is expected to be able to give the theoretical probability of outcomes for simple probability experiments. However, it is also important that the students understand that probability theory can help them make reasonable estimates and predictions in certain kinds of probabilistic situations where no exact answers are possible.

RESOURCES FOR ACTIVE LEARNING

Activities

Cloudburst, Vol. 5 (sampling), Nos. 4118–4138, Midwest Publications

Developmental Math Cards (geometric progression), L21, Addison-Wesley

Experiments in Probability and Statistics, Midwest Publications

Let's Play Games in Mathematics, Vol. 7, "Probability in Color," p. 17, National Textbook Co.

Franklin Series: *Mathematics: Man's Key to Progress*, Book B, pp. 41–47, Rand McNally

Nuffield Project: *Probability and Statistics*, pp. 12–29, 46–50; *Problems* – Red Set, Nos. 15, 15A, 22–23A, 26, Wiley

Towards Probability, Cuisenaire Co.

1-Inch Color Cubes, "Permutations," and "Combinations," HR-4, 5, Midwest Publications.

Manipulative Devices

Binostat (General Learning Corp.; Sigma)

Centimetre and colored cubes (Addison-Wesley)

Hexstat (Gamco; LaPine)

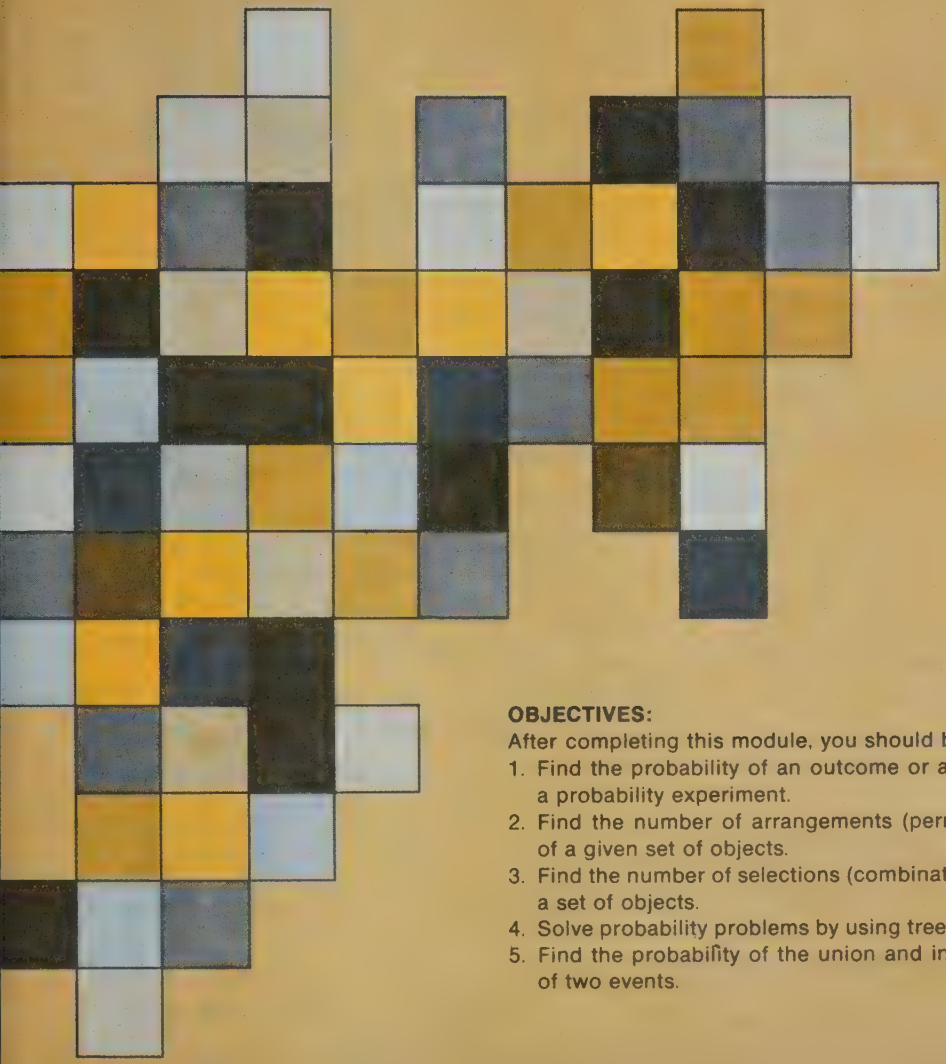
Probability Kit (Creative Publications; LaPine)

Protractor (Geyer; Selective Educational Equipment; Sigma)

Spinners and dice (Creative Publications)

UNIT H: *Some Special Topics in Mathematics*

MODULE 1: *Probability*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the probability of an outcome or an event in a probability experiment.
2. Find the number of arrangements (permutations) of a given set of objects.
3. Find the number of selections (combinations) from a set of objects.
4. Solve probability problems by using tree diagrams.
5. Find the probability of the union and intersection of two events.

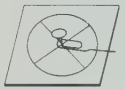
Objective

The student can give the probability of the outcomes of simple probability experiments such as a toss of a coin or a die.

PREPARATION

Materials: Beads or beans of two colors, dice, coins or facsimiles of coins, spinners.

Counting blocks can be used instead of dice; and checkers, washers, or cardboard disks (with opposite sides marked for identification) can be used instead of actual coins. Spinners like the one below can be made from cardboard, paper clips, and thumbtacks.



INVESTIGATION

Interest in the Investigation will be heightened if you prepare a jar of beads (or beans) like the one shown in the text. Have one student take a handful of beads from the jar and count them. Then ask the remainder of the class to guess how many beads of each color are in the handful. Finally, check the estimates by counting. If 350 are black and 650 are orange, then one would expect $\frac{350}{1000}$ or 35% of the beads to be black and 65% to be orange.

DISCUSSION

For Exercise 1, some students may reason that since there are about twice as many orange beads as black beads in the jar, the handful should contain about that same ratio of orange to black beads.

The term *probability* is introduced in Exercise 3 as a simple ratio. A more formal definition can be stated as follows:

> If, in a given experiment, there are n possible outcomes, all of which are equally likely, and x of these are regarded as favorable, then the probability of a favorable outcome is simply the number $\frac{x}{n}$. The chances of a favorable outcome are "x in n."

Probability Experiments

Investigating the Ideas

Suppose the jar at the right holds 650 orange beads and 350 black beads, and they are well mixed. Suppose you grab a handful of beads and find that you drew out 43 beads.

Can you predict how many orange beads and how many black beads you would get?

Discussing the Ideas

1. A What are your predictions of the number of beads of each color for some different-sized handfuls?
   B How did you make your predictions?

2. Do you think it is very likely that someone could grab a handful of beads that are all the same color? Give your reasons.

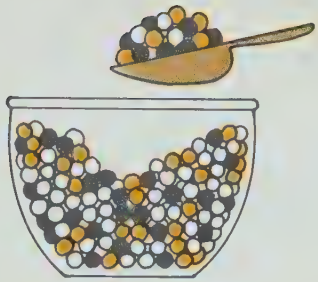
3. There are 650 *chances* in 1000 of drawing an orange bead. The **probability** of drawing an orange bead is $\frac{650}{1000}$. We write: $P(\text{orange bead}) = \frac{650}{1000} = 0.65$.
   A What are the chances of drawing a black bead?
   B What is $P(\text{black bead})$?

4. Suppose you had a container which would hold exactly 100 beads when full.
   A About how many orange beads would you expect to see if you poured out the beads?
   B About how many black beads?

5. The spinner dial is divided into four parts of equal size.
   A What are the chances of the arrow stopping in region 1?
   B What is $P(1)$?
   C About how many times would you expect the arrow to stop in region 3 in 200 spins?

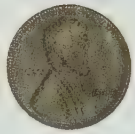
Using the Ideas

1. A container has 500 black beads, 300 orange beads, and 200 white beads. Suppose you can scoop out 100 beads at a time.



- **A** How many beads of each color would you expect in a scoop of 100 beads?
- **B** The chances of drawing a black bead are ? in 1000. P(black bead) = ?.
- **C** The chances of drawing an orange bead are ? in 1000. P(orange bead) = ?.
- **D** The chances of drawing a white bead are ? in 1000. P(white bead) = ?.

2. Suppose a coin is tossed 100 times.



- **A** About how many times would you expect it to come up heads?
- **B** P(Heads) = ? and P(Tails) = ?.

3. **A** When a die is tossed, there is 1 chance in ? that the face with 5 dots will come up.

- **B** What is the probability of getting a 5 with one toss of a die?

4. The spinner dial is divided into three parts of equal size.

- **A** When the arrow is spun, the chances of its stopping in region 2 are 1 in ?.
- **B** The probability of the arrow stopping in region 2 is ?.
- **C** How many times would you expect the arrow to stop in region 3 in 60 spins?
- **D** How many times would you expect the arrow to stop in region 1 in 300 spins?

think

Arrange the nine prime numbers 5, 17, 29, 47, 59, 71, 89, 101, and 113 to form a three-by-three magic square.

UTILIZATION

The exercises may be assigned as independent work, but you should plan to discuss each problem to make certain that the concepts are clear.

EXTENSION

The exercises on page 99 of the *Workbook* would be appropriate for use as an additional practice assignment.

Enrichment: Exercises 2, 3, and 4 can be used as probability experiments. Students can work in teams of two or three to carry out each experiment. An especially valuable part of the experiment is the comparison of experimental results with theoretical expectations.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–4.

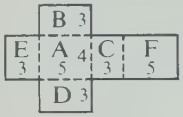
Objective

The student can give the sample space for a probability experiment and give the probability of each outcome in the sample space.

PREPARATION

Materials: Tagboard, scissors, cellophane tape. (Optional: coins, dice, spinners, cards.)

Introduce the Investigation by having students make a model of the block shown. Show the pattern for the block on the chalkboard, or, to save time, prepare and distribute duplicated copies of the pattern for students to use. One block for every 2 students will be sufficient if they carry out the Investigation in teams of 2.



INVESTIGATION

The Investigation introduces a probability experiment with 6 outcomes, but all of the outcomes are not equally likely. Suggest to the students that, when they conduct the experiments, they toss or drop the blocks from a distance of 1 metre or more from the floor in order to lessen the likelihood of biased results.

DISCUSSION

Because of the symmetry of the block, the faces on opposite sides are equally likely to come up when the block is tossed. Students should notice that since sides A and F have the greatest area, those are more likely to come up.

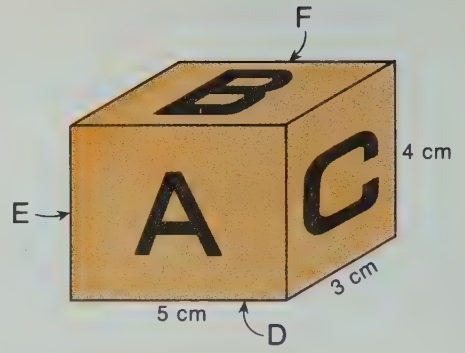
Students can compare their results with the results of their classmates and with the chart given in Exercise 3. Combine all the student results to obtain a better approximation of the probability of each face coming up.

When discussing Exercise 4, emphasize that a die, unlike the block discussed in Exercises 1, 2, and 3, has 6 equally likely outcomes. Each outcome is assigned a probability of $\frac{1}{6}$. In general, if there are n equally likely outcomes, each outcome has a probability of $\frac{1}{n}$.

Sample Spaces and Outcomes

Investigating the Ideas

Suppose you had a rectangular block as shown and labelled the six faces A, B, C, D, E, and F. Suppose you tossed the block and recorded the letter of the face that came up. How many times would you predict that each letter would come up in 100 tosses?



Can you check your prediction by trying the experiment?

Discussing the Ideas

1. Should the letters that are on the opposite faces, such as A and F, come up about the same number of times? Why?

2. The set of all possible outcomes for a probability experiment is called a **sample space** for the experiment. The sample space for the experiment in the Investigation could be designated as {A, B, C, D, E, F}.
   If two outcomes have the same chances of occurring, the outcomes are **equally likely outcomes.**
   - A Which outcomes do you think are equally likely for the experiment?
   - B Do you think outcomes A and C are equally likely? Why or why not?

3. The table shows the results of 100 trials of the experiment above.
   - A Complete the table.
   - B If the experiment were repeated, do you think the experimental probabilities would be different?

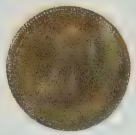
Outcome	Occurrences	Experimental Probability of the Outcome
A	38	$\frac{38}{100}$
B	12	$\frac{12}{100}$
C	3	
D	10	
E	5	
F	32	

4. - A Is each of the outcomes for tossing a die equally likely?
   - B Do equally likely outcomes have equal probabilities?

Using the Ideas

1. A sample space for a probability experiment is given. Give the probability of each outcome in the sample space.

A Toss a coin.



SAMPLE SPACE: $\{H, T\}$

PROBABILITY: $\frac{1}{2}$ $\frac{1}{2}$

B Toss a die.

SAMPLE SPACE: $\{$ ⚀ ⚁ ⚂ ⚃ ⚄ ⚅ $\}$

PROBABILITY: $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

C Spin the arrow.

SAMPLE SPACE: {A, B, C}

PROBABILITY: $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

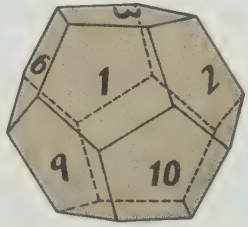
D Draw a card.

SAMPLE SPACE: {J, Q, K, A}

PROBABILITY: $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

2. What is the sum of the probabilities of the outcomes in a sample space?

3. A regular dodecahedron has its faces labelled 1 through 12. It is rolled and the number on the top face is recorded.



A What is a sample space for the experiment?

B Are the outcomes equally likely?

C What is the probability of each outcome?

4. The table shows the results of an experiment with three outcomes.

Outcome	Tally of Occurrences
A	𝍸 𝍸 𝍸 |||
B	𝍸 |
C	|

A How many trials does the table show?

B Give the experimental probability for each outcome.

C On the basis of the experiment, how many times would you expect each outcome to occur in 100 trials of the experiment?

UTILIZATION

When discussing these exercises, you should point out that the particular sample space selected for a given experiment may depend on the purposes of the experiment or on how the results are to be reported or utilized. Some experiments may have many different possible sample spaces. For example, there are many sample spaces for the experiment of drawing a card from a standard deck of playing cards. Two different sample spaces for this experiment are {ace, non-ace} and {red suit, black suit}, but obviously, a great many others could be chosen.

EXTENSION

Page 100 of the *Workbook* may be used to provide further appropriate practice.

Remedial: Ask oral questions like these:

1. Suppose that a coin is tossed 50 times. About how many times would you expect it to come up heads? (about 25)
2. An ordinary die is to be rolled 1200 times. What do you think is the best estimate of
   A. the number of times a 5 will turn up? (200)
   B. the number of times an even number will turn up? (600)
   C. the number times a number less than 3 will turn up? (400)
   D. the number of times a number larger than 3 will turn up? (600)

Enrichment: Have students design a spinner with several outcomes, not all of which are equally likely. Carry out an experiment to compare the actual and experimental probabilities for each outcome on the spinner.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can give the probability of an event in a sample space.

PREPARATION

Materials: Tagboard, scissors, rulers, protractors, thumbtacks.

Guide the students in making the spinner for the Investigation. Each central angle of the spinner is 72°. You may suggest that the spinner be made in the same way as that shown in the Preparation section on page H–2.

INVESTIGATION

The Investigation can be carried out by pairs of students or small groups of students. Since 3 out of 5 of the numbers are odd numbers, most students will predict that in 100 spins they will get about 60 odd numbers.

DISCUSSION

This lesson builds upon the concepts introduced in the previous lesson. Once students learn how to determine the probability of each simple outcome in a sample space, they will be able to understand what is meant by an *event* and the *probability of an event*.

In Exercise 3, emphasize the definition of an event as *any subset of a sample space* as well as the fact that the probability of an event is *the sum of the probabilities of the outcomes of the event*. In particular, the probability of the event that is impossible (the empty set) is zero and the sum of the probabilities of all the outcomes in a sample space is 1.

There are several ways of designating events. For example, in the experiment of tossing a die with sample space {1, 2, 3, 4, 5, 6}, a given event might be designated in any of these ways:

(1) Verbally: The event of getting a number less than 3

(2) Set notation: {1, 2}

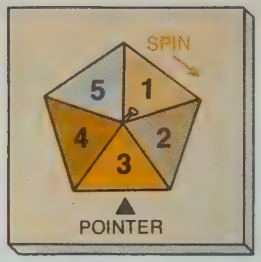
(3) Algebraic or open sentence notation: $x < 3$

Students should become familiar with all three ways of referring to events.

Probability of an Event

Investigating the Ideas

Suppose a probability experiment consists of spinning the pentagon and then recording the number of the region indicated by the pointer.



Can you predict about how many times in 100 spins the pointer would show an odd-numbered region?	Try the experiment to check your prediction.

Discussing the Ideas

1. The **event** of getting an odd-numbered outcome has occurred if any number in the set {1, 3, 5} is an outcome of the experiment. The probability of an event is the sum of the probabilities of each outcome in the event. What is the probability of an odd-numbered outcome?

Event: {1, 3, 5}

P(Event): $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} =$ ▮

2. A What subset of the sample space for the spinner experiment is the event for an even-numbered outcome?

B What is the probability of the event?

3. The examples above illustrate three important ideas:

1 Any subset of a sample space is an **event.**

2 The **probability of the event** is the sum of the probabilities of the outcomes in that event.

3 The sum of the probabilities of the outcomes in a sample space is 1.

Suppose $F = \{1, 2\}$. Is F a subset of the sample space of the spinner experiment above? What is $P(F)$?

4. If an event is *certain* to happen, its probability is 1. Is the event of getting a number greater than zero in the spinner experiment certain?

5. If an event can *never* happen, the probability of the event is 0. The event of getting a number greater than 5 is the empty set ϕ. What is the probability of the event?

Using the Ideas

1. Complete the table for the spinner experiment shown.

Experiment: Spin the arrow.
Sample space: {1, 2, 3, 4}
Probability: $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

	Description of event	Subset of the sample space (Event)	Probability of the event
A	The event of getting an odd number as an outcome.	$A = \{1, 3\}$	$P(A) =$
B	The event of getting 1 as the outcome.	$B = \{1\}$	$P(B) =$
C	The event of getting a number between 1 and 4.	$C = \{?\}$	$P(C) =$
D	The event of getting a number greater than 0.	$D = \{?\}$	$P(D) =$
E	The event of getting a number greater than 4.	$E = \{?\}$	$P(E) =$

2. An experiment of tossing a die has a sample space {1, 2, 3, 4, 5, 6}. Give the probability that each event below will occur.

A {1, 3, 5} B {1, 4} C {6} D {1, 2, 3, 4, 5, 6} E {1, 2, 4, 6}

3. The example below shows how algebraic symbolism can be used to describe events.

Example (using the experiment in Exercise 2): $P(x > 3) = P(\{4, 5, 6\}) = \frac{1}{2}$

Give the probability.

A $P(x = 1 \text{ or } x = 4)$
B $P(x \text{ is even or } x > 3)$
C $P(x \neq 1)$
D $P(x \text{ is even and } x > 3)$
E $P(x = 1 \text{ or } x = 4 \text{ or } x = 6)$
F $P(x \neq 1 \text{ and } x \neq 2)$

4. The numerals 1 through 9 are written on slips of paper and placed in a container. An experiment consists of drawing one of the slips from the container. For each event described, give the subset of the sample space and the probability of the event. Part A is worked as an example.

A An odd number.
SOLUTION: $P(\{1, 3, 5, 7, 9\}) = \frac{5}{9}$
B An even number.
C A multiple of 4.
D A factor of 4.
E A square number.
F A prime number.
G A factor of 6.
H $P(x > 7)$
I $P(x \geq 7)$
J $P(x < 7)$
K $P(x \leq 7)$
★ L $P(x \text{ is even or } x < 7)$
★ M $P(x \text{ is even and } x < 7)$
★ N $P(x < 5 \text{ and } x \neq 3)$

More practice, page S-28, Set 54

UTILIZATION

Exercise 3 introduces algebraic symbolism to describe events. Also, open sentences involving *and* and *or* occur throughout Exercises 3 and 4. Since these words have been used in many instances before, they should require little emphasis in this lesson; however, some students might benefit from a brief review of them before doing the exercises.

EXTENSION

Additional appropriate practice exercises are provided by Supplementary Exercise Set 54 on page S-28, *Workbook* page 101, and *Duplicator Masters* page 68.

Enrichment: Call the students' attention to Buffon's Needle problem, on Mathematical Recreation page H-26. This would be an opportune time to encourage students to try this experiment. Also, you might want to suggest that some students undertake Project B on page H-25.

Assignments

Minimum 1-2. Average 1-3. Maximum 1-4.

Objective

The student can give the probability of an event when the outcomes are ordered pairs.

PREPARATION

Materials: Coins or coin substitutes, dice, spinners.

Since the Investigation will consume several minutes, you should proceed with it immediately.

INVESTIGATION

The Investigation lends itself to grouping students in pairs. One student should toss the coins and the other keep a tally of the outcomes. Students unfamiliar with the experiment may think that each event in the table is equally likely and be surprised by the relative frequency of the "one head, one tail" event.

DISCUSSION

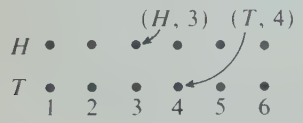
Discuss Exercise 2 carefully, pointing out that there is more than one way to think about a sample space for an experiment.

Ordered pair outcomes: $\{\underbrace{(H,H)}, \underbrace{(H,T)\,(T,H)}, \underbrace{(T,T)}\}$

Events: Two heads; One head, one tail; Two tails

Since there are two outcomes in the event {one head, one tail}, each with a probability of $\frac{1}{4}$, then $P(\{\text{one head, one tail}\}) = \frac{1}{2}$.

To help students understand why there are 12 outcomes for Exercise 3, you can show a lattice system as depicted below. Each point in the lattice represents an ordered pair outcome in the sample space.



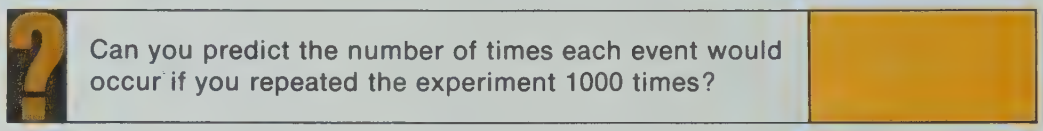
Ordered Pair Outcomes

Investigating the Ideas

Suppose you toss a penny and a nickel 100 times. How many times do you think you would get two heads?

Try the experiment and keep a tally of the events in a table like the one shown.

EVENT	TALLY
Two heads	
One head, one tail	
Two tails	



Discussing the Ideas

1. Steve thought that each of the three outcomes was equally likely, so he predicted that both coins would come up heads about 33 times in 100 tosses. Do your results agree with Steve's prediction?

2. Paula said that Steve was not correct and used the following chart to explain her reasoning:

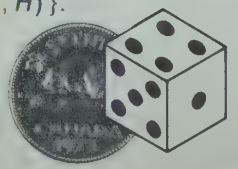
Penny	Nickel	Sample Space	
H	H	$\{(H,H)$	The probability of both coins being heads is $\frac{1}{4}$.
H	T	(H,T)	
T	H	(T,H)	We write:
T	T	$(T,T)\}$	$P(H,H) = \frac{1}{4}$
Each of the **four** pairs is equally likely.		The set of four equally likely outcomes	

   A Each outcome in the sample space is denoted by an ordered pair. What is the probability of each outcome?

   B The event {one head, one tail} is the subset $\{(H,T), (T,H)\}$. Why are there two ordered pairs in this event?

   C What is the probability of this event?

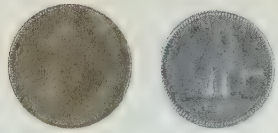
3. How many outcomes are there in the sample space for tossing a coin and a die at the time?



Outcome: $(T, 4)$

Using the Ideas

1. An experiment consists of tossing a penny and a dime. Two of the elements in a sample space for this experiment are (H, T) and (H, H) where the first letter in each pair indicates the outcome for the penny and the second letter the outcome for the dime.



Outcome: (H, T)

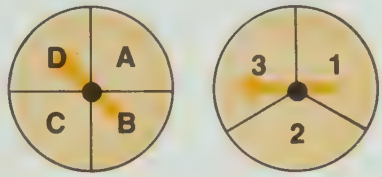
- **A** Give the complete sample space for the experiment.
- **B** What is the probability of each outcome in the sample space?

2. Using the coin-tossing experiment in Exercise 1, give the subset of the sample space (event) described in each part.

- **A** Penny heads and dime tails.
- **B** Penny comes up heads.
- **C** Dime comes up tails.
- **D** At least one head comes up.
- **E** One head and one tail come up.
- **F** The two coins match.

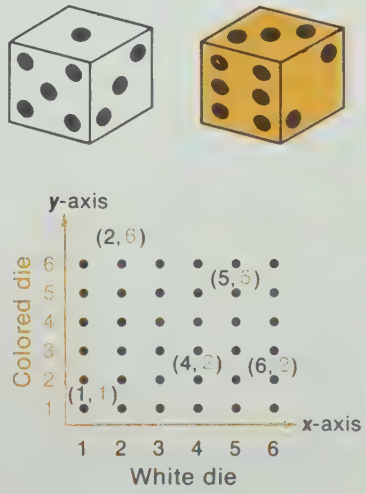
3. Give the probability of each event in Exercise 2.

4. Two spinners like those shown are used in a probability experiment. Each outcome results in a letter and a numeral.



- **A** What outcome is shown on the spinners?
- **B** List all the outcomes as ordered pairs. How many are there in all?
- **C** What is $P(\mathbf{C}, 2)$?

5. Two dice, one white and one colored, are tossed. The co-ordinate system shown at the right suggests how the sample space for the experiment can be found. Suppose **x** denotes the number on the white die and **y** denotes the number on the colored die. Compute the probabilities.



- **A** $P(x + y = 2)$
- **B** $P(x + y = 5)$
- **C** $P(x + y = 10)$
- **D** $P(x + y \neq 7)$
- **E** $P(x + y = 8)$
- **F** $P(x + y = 7)$
- **G** $P(x + y = 12)$
- **H** $P(x > y)$

UTILIZATION

Allow sufficient time to discuss the exercises when students have completed them. It may be helpful to have demonstration spinners like those shown in Exercise 4 to illustrate the outcomes in part B.

EXTENSION

The exercises on page 102 of the *Workbook* would be appropriate for use in providing further practice.

Remedial: Have students compute the probability of each possible sum for tossing a pair of dice. They might set up a lattice system as shown in Exercise 5.

Enrichment: Students may enjoy playing a dice game called "Chuck-A-Luck." Two or more students can play the game using three dice and a set of checkers, counters, or poker chips. One person is designated as banker. Each player pays the banker 1 counter to play each game. Each player chooses a number from 1 to 6. The three dice are rolled. If a player's number comes up on one die, the banker pays him 2 counters. If a player's number comes up on two dice, the banker pays the player 3 counters. If a player's number comes up on three dice, the banker pays the player 4 counters. The odds are in the banker's favor. It can be shown that for each 216 counters paid by the players to play the game, the banker will return only 199. Thus a player loses about 7.8% of the amount he pays to play.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

The student can construct and use a tree diagram to find probabilities of outcomes resulting from repeated trials.

PREPARATION

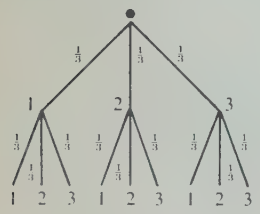
Ask students questions like the following: "Do you think it is very likely that you could toss a penny and have it come up heads three times in a row?" Or, "Suppose you tossed a penny and got heads 5 times in succession. Now, do you think you are more likely to get tails on the next toss than heads?" (The answer is that the probability of getting heads is still $\frac{1}{2}$ regardless of what has taken place before.)

DISCUSSION

Discuss the construction of the probability tree in Exercise 1. Each branch of the tree is labeled "$\frac{1}{2}$" because the probability of each outcome (head or tail) is $\frac{1}{2}$. Notice that to reach point *A* in the tree, one must toss heads on each of the three tosses. Outcome *A* is denoted by the symbol *HHH*; outcome *B*, by the symbol *HHT*.

Tree diagrams are useful in many probability problems in which repeated tosses occur. There is no standard way of drawing the trees. The branches may "grow" upward, downward, or sideward. The text, however, usually shows them "growing downward," since this representation allows the eye to travel downward along the branches in a natural reading sequence.

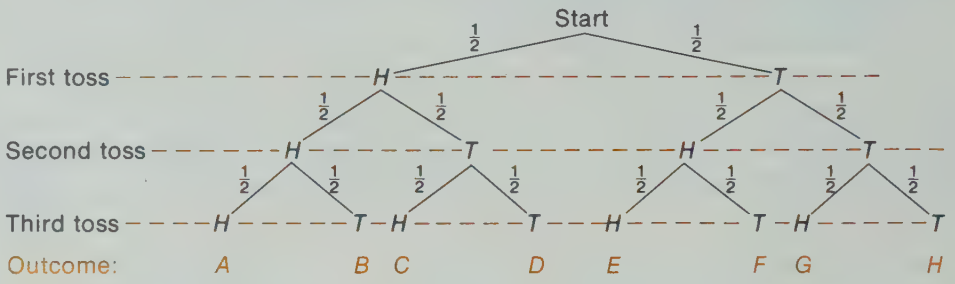
If another example is needed, consider repeated flips of a spinner divided into three equal regions. Two stages of a tree for this experiment are shown.



Using Tree Diagrams in Probability

Discussing the Ideas

1. **Tree diagrams** are helpful in counting outcomes and computing probabilities of certain events. The tree diagram below illustrates all the outcomes when a coin is tossed three times in succession.



   - **A** How many different outcomes are possible if a coin is tossed three times in succession?
   - **B** Outcome *C* can be denoted by (*HTH*). What does the symbol stand for?
   - **C** Describe outcome *A*.

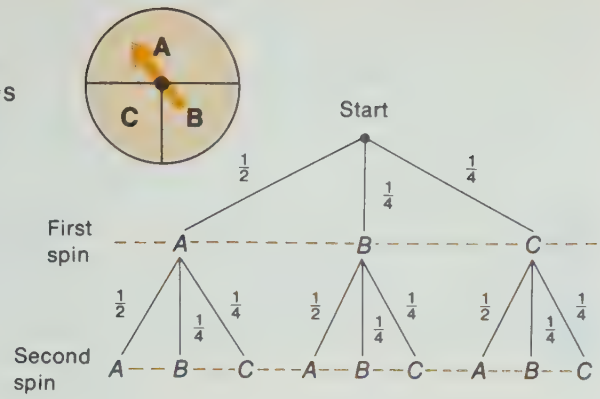
2. Each branch of the tree is labelled "$\frac{1}{2}$" because the probability of tossing either a head or a tail is $\frac{1}{2}$. The tree diagram shows that there are 8 outcomes for three tosses of a coin. Since each outcome is equally likely, $P(A) = P(HHH) = \frac{1}{8}$ and $P(B) = P(HHT) = \frac{1}{8}$. Note that the probability of each outcome can be found by multiplying the probabilities along each branch of the tree. Thus $P(C) = P(HTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.
   - **A** Which point in the tree diagram is associated with the outcome *THT*?
   - **B** What is $P(THT)$?

3. **A** Explain how the tree diagram could be extended to show the outcomes for four tosses of a coin.
   - **B** How many outcomes would there be for four tosses of a coin?
   - **C** What is the probability of each outcome?

4. **A** How many outcomes would there be for five tosses of a coin?
   - **B** What would be the probability of each outcome?

Using the Ideas

1. A spinner and a tree diagram for the experiment of flipping the spinner is shown at the right. The sample space for the experiment is $\{A, B, C\}$. $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$, and $P(C) = \frac{1}{4}$. A symbol like AC represents the event of getting A on the first spin and C on the second spin. Give the probability of each event.



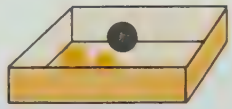
- **A** AA
- **B** AB
- **C** AC
- **D** BA
- **E** BB
- **F** BC
- **G** CA
- **H** CB
- **I** CC

2. Two hockey teams are to play a series of 3 games. The teams are evenly matched, so for each game the probability that each team will win is $\frac{1}{2}$. Make a tree diagram for the three games, using W to represent the outcome of the home team winning and using L to represent the outcome of the home team losing. Label each branch with the probability $\frac{1}{2}$.

3. Using the tree diagram of Exercise 2, give the probability of each event.

- **A** WW: the home team wins the first two games.
- **B** WWL: the home team loses only the third game.
- **C** WLW
- **D** LWW
- **E** The home team loses one game of the three.
- **F** The visiting team wins the first and third games.

4. Three marbles, two colored and one black, are placed in a box. An experiment consists of drawing one marble from the box, recording its color, replacing the marble, drawing again and recording the color, and so on. Make a tree diagram for three draws of this experiment.



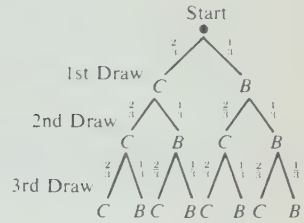
5. Use the tree diagram in Exercise 4 to find the probability of each event.

- **A** BC
- **B** BB
- **C** CC
- **D** CCB
- **E** CBC
- **F** CCC
- **G** BBB
- **H** BCC

UTILIZATION

Tree diagrams are utilized in most of the exercises. Exercise 4 involves an experiment in which not all outcomes are equally likely.

Answer, Exercise 4



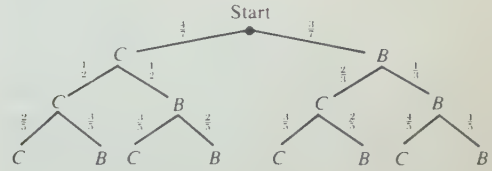
EXTENSION

To provide further practice, make assignments from *Workbook* page 103 and *Duplicator Masters* page 69.

Enrichment: Ask students to construct the tree diagram described below.

A box contains 4 colored marbles (C) and 3 black marbles (B). Draw a tree diagram for the experiment of drawing three marbles *without* replacing them. Show the probabilities along each branch of the tree.

Solution:



After students draw the tree, ask them to find the probability of each of these events.

1. The first two marbles are colored. ($\frac{3}{14}$)
2. The first two marbles are black. ($\frac{3}{21}$)
3. The first marble is colored and the second marble is black. ($\frac{4}{14}$)
4. CCB ($\frac{6}{35}$)
5. CBC ($\frac{6}{35}$)
6. BCB ($\frac{4}{35}$)

Assignments

Minimum 1–3. Average 1–5. Maximum 1–5.

Objective

The student can count the number of arrangements of the elements in a given set of objects.

PREPARATION

You might introduce the lesson by pointing out that often the most difficult part of analyzing an experiment and computing the probabilities associated with the experiment is counting the number of possible outcomes. In this lesson and the following ones, some methods of handling some difficult counting problems will be studied.

INVESTIGATION

Allow students to use their own techniques to find the number of arrangements of the four students. If students try to list all the different ways, they will find it convenient to use symbols: A for Amy, B for Bill, C for Chuck, and D for Diane. A few students may be able to reason that any one of 4 students could sit in the first desk, then any one of the 3 remaining students could sit in the second desk; any one of the 2 remaining, in the third desk; and the last remaining student, in the fourth desk. This will give 24 different arrangements.

DISCUSSION

The tree diagram in Exercise 1 shows that there are 6 different arrangements when Amy sits in the first desk. Since there are three other trees like this one for the cases where each of the other students sits in the first desk (parts B, C, D), $4 \cdot 6$, or 24, different arrangements are possible.

Some Counting Problems

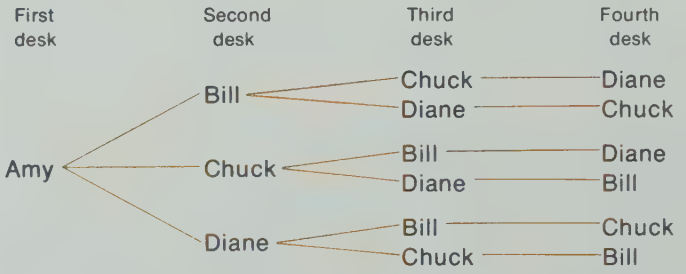
Investigating the Ideas

Amy, Bill, Chuck, and Diane are to be seated at four desks in one row.

? How many different ways can you find for the four students to arrange themselves in a row?

Discussing the Ideas

1. One way of counting the number of arrangements for the students in the desks is to construct a tree diagram. Part of a tree diagram is shown below in which Amy occupies the first desk.



   A. How many seating arrangements does the tree show?
   B. Make a tree that shows Bill in the first desk.
   C. Make a tree that shows Chuck in the first desk.
   D. Make a tree that shows Diane in the first desk.

2. How can you use the tree diagrams in Exercise 1 to find the total number of arrangements of the four students at the four desks?

Using the Ideas

1. Three students, Jim, Sarah, and Michelle, are running for the offices of president, vice president, and secretary of a club. The person with the most votes becomes president, the person with next highest number of votes becomes vice president, and the third person becomes secretary.
   - **A** Make a tree diagram that shows all the possible arrangements of the three students for the three offices.
   - **B** How many arrangements are possible?

2. How many three-digit numerals can you form using each of the digits 2, 7, and 4 once in each numeral?

3. Four girls, Maria, Helen, Irene, and Jessie, are on a relay team. Each person runs one lap of the relay. How many different arrangements of order are possible for the four runners?

4. A triangle has its vertices labelled *X*, *Y*, and *Z*. How many different ways could the triangle be named using the three letters?

△*YZX* is one name.

★5. The five cards are to be stacked on top of each other. How many different arrangements of the five cards are possible?

★6. The names of six students are written on slips of paper, put into a box, and mixed thoroughly. The names are then drawn from the box and recorded in order. How many different possibilities are there for the order of the names?

think

From the 5 coins shown, you can select any 3 of the coins. How many different amounts of money can be selected?

UTILIZATION

The student is expected to use tree diagrams or other intuitive methods to solve the problems. Do not expect proficiency in counting arrangments at this point. Further techniques will be developed in succeeding lessons.

EXTENSION

Remedial: To help students solve the problem of counting the number of arrangements of, say, the 4 letters A, B, C, and D, suggest that they use graph paper to make a box diagram like the one below and systematically list the different arrangements of the letters.

1	2	3	4
A	B	C	D
A	B	D	C
A	C	B	D
A	C	D	B
A	D	B	C
A	D	C	B
B	A	C	D
B	A	D	C
.	.	.	.
.	.	.	.
.	.	.	.

Think Solution

Students can solve these problems by listing the different choices.

1. 50¢, 25¢, 10¢ (85¢)
2. 50¢, 25¢, 5¢ (80¢)
3. 50¢, 25¢, 1¢ (76¢)
4. 50¢, 10¢, 5¢ (65¢)
5. 50¢, 10¢, 1¢ (61¢)
6. 50¢, 5¢, 1¢ (56¢)
7. 25¢, 10¢, 5¢ (40¢)
8. 25¢, 10¢, 1¢ (36¢)
9. 25¢, 5¢, 1¢ (31¢)
10. 10¢, 5¢, 1¢ (16¢)

Assignments

Minimum 1–4. Average 1–5. Maximum 1–6.

Objective

The student can find the number of permutations of the elements in a given set.

PREPARATION

Materials: Cards or squares of paper (optional).

Some students may find it helpful to write the 4 letters on cards or squares of paper and manipulate them to find the various arrangements of the letters; otherwise, this Investigation requires no special preparatory work.

INVESTIGATION

Have the students record the different arrangements of the 4 letters. Of the 24 possible arrangements the following meaningful words are possible:

STOP, POST, TOPS, POTS, SPOT, OPTS

DISCUSSION

The word *permutation* is introduced in Exercise 1. The table in Exercise 2 will help the student to see that the two-element set $\{A, B\}$, has two permutations or arrangements, AB and BA. A three-element set $\{A, B, C\}$ has six permutations: ABC, ACB, BAC, BCA, CAB, and CBA. Notice that the product $2 \cdot 1$ gives the number of permutations of a two-element set and $3 \cdot 2 \cdot 1$ gives the number of permutations of a three-element set. In general, the product

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$$

gives the number of permutations, or arrangements, of a set of n elements.

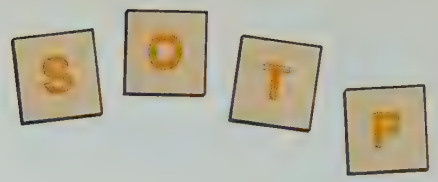
When discussing the exercises, you may want students actually to list the arrangements for a 4-element set and a 5-element set.

Factorials are introduced as a way of designating the number of permutations of a set of objects. Since the symbol $n!$ will be new to most students, provide some practice using the symbol before assigning the exercises. Encourage students to solve equations such as $\frac{8!}{6!} = n$ and $n \cdot 5! = 6!$ by thinking of the definition of factorials rather than by carrying out complicated computation.

Permutations

Investigating the Ideas

How many different 4-letter arrangements can you make using the letters on these cards?



How many meaningful words do your arrangements form?

Discussing the Ideas

1. An arrangement of the elements of a set in a certain order is called a **permutation** of the set. Thus, SOTP is one permutation of the set of letters above.

How many permutations did you find for the four letters in the Investigation?

2. Study the table below.

Set	Permutations of the Set	Number of Permutations
A	A	1
A, B	AB, BA	$1 \cdot 2 = 2$
A, B, C	$ABC, ACB, BAC,$ BCA, CAB, CBA	$1 \cdot 2 \cdot 3 = 6$

A Does the product $1 \cdot 2 \cdot 3 \cdot 4$ give the number of permutations of a 4-element set?

B How many permutations does a 5-element set have?

3. In general, the product $1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$ gives the number of permutations of a set of n elements.

A How many permutations for a set of 6 elements?

B How many permutations for a set of 7 elements?

4. The product of all the whole numbers from 1 through n inclusive is called **n factorial** and is denoted by $n!$.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$$

Explain how to compute each of the following:

A $3!$ **B** $5!$ **C** $10!$ **D** $3! \cdot 4!$ **E** $5! \div 4!$

Using the Ideas

1. Extend the table of factorials at the right to 12.

TABLE OF FACTORIALS

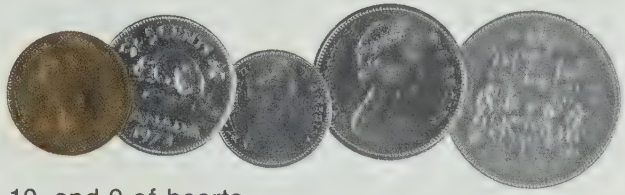
Number of objects	Number of arrangements	Symbol
1	1	1!
2	$1 \cdot 2 = 2$	2!
3	$1 \cdot 2 \cdot 3 = 6$	3!
4	$1 \cdot 2 \cdot 3 \cdot 4 = 24$	4!

2. How many different arrangements of the letters are possible for the word DEN?

3. Use the extended table of factorials to find the number of possible batting orders for a baseball team of nine players.

4. Seven students sit in a row of seven seats. In how many ways can these students be arranged in the seven seats?

5. How many ways can these coins be arranged in a row?



6. The ace, king, queen, jack, 10, and 9 of hearts are placed in a line on a table. In how many ways can they be placed in a line?

★ 7. Solve the equations, avoiding computation if you can.

EXAMPLE: $6 \cdot 5! = n$ $\quad n = 6!$

A $10 \cdot 9!$

B $7 \cdot 6 \cdot 5!$

C $12 \cdot n = 12!$

D $n \cdot 4! = 5!$

E $n \cdot 3! = 5!$

F $\frac{6!}{5!} = n$

G $\frac{10!}{8!} = n$

H $\frac{2! \cdot 6!}{4!} = n$

I $\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = n$

J $\frac{8!}{n} = 8$

K $\frac{6!}{n} = 30$

L $\frac{4!}{n!} = \frac{1}{30}$

M $7! \cdot n = 9!$

N $\frac{n!}{(n-1)!} = 5$

think

The 10 letters in the box are arranged to correspond to a well-known set of objects. Write the next 4 letters in the arrangement.

OTTFFSSENT

More practice, page S-29, Set 55

UTILIZATION

All students should complete Exercise 1. Then they can use the completed table of factorials as an aid in solving the remaining exercises.

Answers, Exercise 1

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$
$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$
$7! = 6! \cdot 7 = 5040$
$8! = 7! \cdot 8 = 40\,320$
$9! = 8! \cdot 9 = 362\,880$
$10! = 9! \cdot 10 = 3\,628\,800$
$11! = 10! \cdot 11 = 39\,916\,800$
$12! = 11! \cdot 12 = 479\,001\,600$

EXTENSION

To provide suitable further practice, make assignments from Supplementary Exercise Set 55 on page S-29, *Workbook* page 104, and *Duplicator Masters* page 70.

Enrichment: Challenge students to find the number of arrangements possible with a deck of 52 cards. The solution is, of course, 52! A table of factorials of numbers gives about 8.0658×10^{67} or about 80 decillion decillions for this very large number.

Think Solution

The letters in the set are the first letters of the first ten counting number words: one, two, three, four, five, six, seven, eight, nine, ten. Thus, the next four letters would be ETTF.

Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

The student can compute the number of permutations of n *things taken* r *at a time.*

PREPARATION

Materials: Cards or squares of paper (optional)

Review the method of finding permutations of sets of objects.

INVESTIGATION

Some students may find that it will be helpful to write the four numerals on cards and then manipulate the cards to form 2-digit numerals. The Investigation provides an opportunity for the student to think intuitively about the number of permutations of a set of four elements when two elements at a time are considered.

DISCUSSION

The *counting principle* introduced in Discussion Exercise 3 gives the student a simple way of finding the number of possible arrangements. This principle is then applied to the problem of computing the number of permutations of a set of n elements when r of the elements are taken at a time, when $r \leq n$. In discussing Exercise 3, students should observe that there are 5 choices for the first digit. After the first digit is selected, there are 4 choices for the second digit. Thus, applying the counting principle, there are $5 \cdot 4$, or 20, ways of choosing all 2-digit numerals using the 5 digits.

The general formula for the number of permutations, P, of n things taken r at a time is

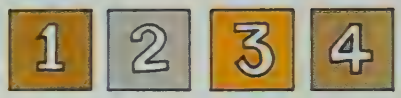
$$P = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

It is not essential that this formula be given to the students for memorizing.

The Counting Principle

Investigating the Ideas

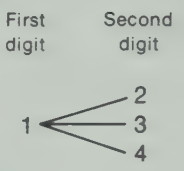
Suppose you had these four cards. You use two of the cards at a time to form a 2-digit numeral.



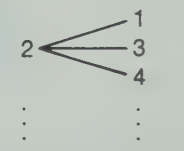
How many different 2-digit numerals can you form using the four cards?

Discussing the Ideas

1. You could use a tree diagram to help you solve the Investigation problem. Complete the tree diagram showing the cases where 3 and 4 are the first digits.



2. **A** How many different choices are there for the first digit?
   B After the first digit has been chosen, how many choices are there for the second digit?



   C How can your answers to parts **A** and **B** be used to find the total number of arrangements?

3. Exercise 2 illustrates an important principle.

THE COUNTING PRINCIPLE

If an operation can be done in any one of ***n*** ways and a second operation can be done in any one of ***m*** ways, then the two operations can be performed in $n \cdot m$ ways.

Explain how this principle could be used to find the number of 2-digit numerals that could be formed from the set $\{1, 2, 3, 4, 5\}$ if no digit is repeated.

4. The previous exercises illustrate problems of a **permutation of *n* objects taken *r* at a time**, when $r \leq n$. For example, using the counting principle repeatedly, we can find the number of arrangements or permutations of the four letters W, X, Y, and Z when taken three at a time to be $4 \cdot 3 \cdot 2 = 24$.

How many permutations are there of five objects taken two at a time?

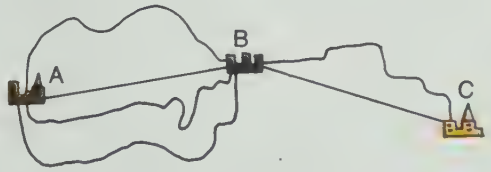
Using the Ideas

1. From the set {A, B, C, D, E}, a 2-letter arrangement is to be selected.
   - **A** How many choices are there for the first letter?
   - **B** How many choices are there for the second letter after the first letter has been selected?
   - **C** How many 2-letter arrangements (permutations) are possible? (Use the counting principle.)

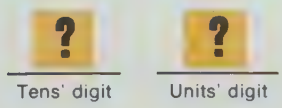
2. In a classroom, there are five seats in the row next to the windows, but only three students sit in these seats. In how many ways can the students arrange themselves in this row? (HINT: The first student can sit in any 1 of 5 seats; the second student can sit in any 1 of 4 seats; the third student can sit in any 1 of 3 remaining seats. Now use the counting principle.)

3. In how many different ways can a president, vice president, secretary, and treasurer be elected from a class of 20 students?

4. There are 4 roads from City A to City B and 2 roads leading from City B to City C. How many different routes are there from City A to City C?



5. Suppose 2-digit numerals are composed from the digits {0, 1, 2, 3 4, 5 6, 7 8, 9}.
   - **A** How many different digits can be used as the tens' digit?
   - **B** How many digits can be used as the units' digit?
   - **C** How many 2-digit numerals in all? Check by direct counting.



6. How many 3-digit numerals are there?

★ 7. How many permutations of 3 different letters of the alphabet are posssible?

UTILIZATION

Assign the exercises as independent work, but reserve time for discussion when students have completed the written work.

Solution, Exercise 6

The hundreds' digit can be any one of 9 digits, 0 excluded. The tens' and units' digits can be any one of 10 digits. Applying the counting principle, we have $9 \cdot 10 \cdot 10 = 900$ 3-digit numerals. As a check, the set of 3-digit numerals consists of 100, 101, . . . 998, 999, or a total of 900 numerals.

EXTENSION

Enrichment: Have students try this NUMBERS problem.

How many 3-letter arrangements are possible using the letters in NUMBERS if each letter can be used only once in each arrangement? (*Answer*: 210)

How many sensible 3-letter words can you find among the 3-letter arrangements of NUMBERS? (*Sample answers*: BUM, URN, RUN, SUM, RUE, ERN (*var.* erne), RUM, RUB, SUB, BUR, EMU, NUB, USE, BUN, NEB, SUE, SUN, BUS, MEN)

Assignments

Minimum 1–4. Average 1–6. Maximum 1–7.

Objective

The student can determine the number of combinations of the elements of a set taken a given number at a time.

PREPARATION

Review the ideas of permutations as ordered arrangements of elements in a set.

INVESTIGATION

Have the students read the Investigation and proceed to try to answer the Investigation questions. The students will soon notice that this is not a problem of permutations. Thus, if problems 2, 4, and 5 are the three selected from the five problems, the arrangement, or order, of the numbers is not important.

There are 10 possible sets of 3 problems each: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,3,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.

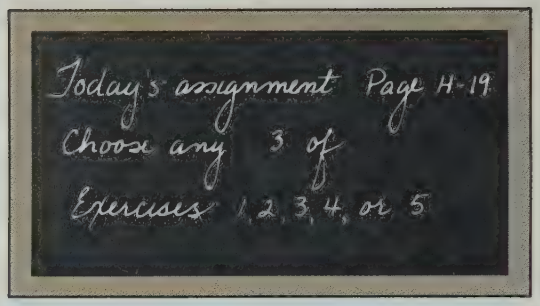
DISCUSSION

Work through the Discussion Exercises carefully. *Combination* is defined in Exercise 3. Exercise 4 relates the problem of finding the number of combinations to the students' previous experience with permutations. If ${}_nP_r$ is the number of permutations of n things taken r at a time, then ${}_nC_r = \frac{{}_nP_r}{r!}$. (It is not necessary to introduce this formula for students to memorize.)

Combinations

Investigating the Ideas

Suppose your teacher made an assignment like this.



How many different choices of problems do you have?

Discussing the Ideas

1. Did you find a systematic way to list the choices of three problems? Describe your method.

2. How is the Investigation problem different from a permutation of 5 things taken 3 at a time?

3. A **combination** is a subset of a given set. The Investigation involved choosing subsets of 3 elements each from a set containing 5 elements. The number of such 3-element subsets is the number of combinations of 5 elements taken 3 at a time. What is the number of combinations?

4. Study the solution below to see how permutations can be used to solve the Investigation problem.

   Number of permutations of 5 elements taken 3 at a time: $5 \cdot 4 \cdot 3 = 60$

   Each 3-element subset has $3! = 6$ permutations.

   Therefore, the number of combinations of

   5 things taken 3 at a time is: $\frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10$

   Show that the number of combinations of 5 elements taken 2 at a time is 10.

5. The symbol ${}_nC_r$ means the number of combinations of n elements taken r at a time. Thus, ${}_6C_2 = \frac{6 \cdot 5}{2!} = \frac{30}{2} = 15$.

   Show how to find the following combinations:

   A ${}_4C_2$ **B** ${}_6C_3$ **C** ${}_5C_1$ **D** ${}_7C_6$

Using the Ideas

1. From a list of 4 books a student can choose to read any 2 of them. How many choices does the student have?

2. If the student may read any 3 books of the 4, how many choices does the student have?

3. **A** If the student may read any one of the 4 books how many choices does the student have?
   B If the student must read all 4 books, how many choices does the student have?

4. A student must choose any 2 of the following activities for Club Day at school:

Dance	Hobbies	Math Club	School Newspaper
Drama	Photography	Science Club	

How many different choices are possible?

5. On a 10-problem quiz, Carrie missed only 2 problems. How many different pairs could she possibly have missed?

6. Compute the number of combinations indicated.
   A ${}_3C_2$ **B** ${}_6C_4$ **C** ${}_5C_4$ **D** ${}_{10}C_3$

7. Six people are waiting for a cab. When the cab arrives, it can take only four of the people. How many different groups of four people could ride together in the cab?

★ 8. From a deck of 52 cards, three cards are chosen. How many different combinations are possible?

think

What whole number is n in this equation?

$$n! = (n+2) \cdot (n+3) \cdot (n+4)$$

More practice, page S-29, Set 56

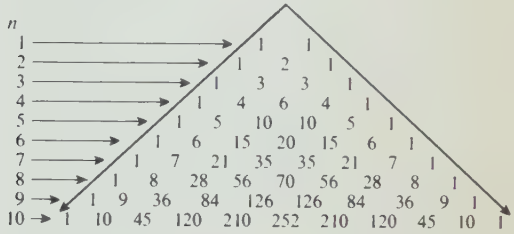
UTILIZATION

Plan to discuss these exercises after students have completed them.

EXTENSION

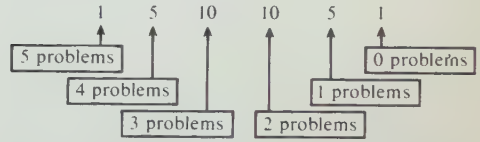
For further practice, see Supplementary Exercise Set 56 on page S–29, *Workbook* page 105, and *Duplicator Masters* page 71.

Enrichment: The triangular array of numbers depicted below is called Pascal's Triangle. It can be extended to as many rows as desired. Using the triangle, we can solve problems involving combinations which would otherwise be quite difficult.



Example: On a 5-problem quiz a student may answer any 3 problems. How many different combinations of problems can the student select? (10)

Solution: Choose the fifth row of Pascal's Triangle.



Thus from the set of 5 problems, there are 10 combinations of 3 that the student can choose.

Make up some other combinations problems like the examples above and ask students to use Pascal's Triangle to help them find the answers. Here are two more examples:

1. How many committees of 3 members can be selected from 8 people? (56)
2. How many committees of 5 members can be formed from 7 people? (21)

Assignments

Minimum 1–5. Average 1–7. Maximum 1–8.

Objective

The student can solve probability problems which involve combinations.

PREPARATION

Materials: Cubes of 2 colors, marbles or beans, paper bags or boxes.

Review the basic ideas of probability by giving some simple examples such as the following:

If two red marbles and three white marbles are in a sock, what is the probability of drawing a red marble? of drawing a white marble?

INVESTIGATION

Students should work in pairs or small groups on the Investigation. Use small cubes, marbles, beans, checker pieces, cardboard squares, or any other available material for the Investigation. The four objects should be placed in a bag or box and two objects withdrawn blindly. Theoretically, the two objects should match in color in $\frac{1}{3}$ of the draws but student predictions will vary from $\frac{1}{4}$ to $\frac{1}{2}$.

DISCUSSION

Exercise 2 presents a way of analyzing the Investigation problem that relies on the students' understanding of combinations. Work through this exercise carefully with students. Since only a small number of objects are involved, you may also demonstrate all the possible combinations by listing them. Let R_1 and R_2 be two objects of one color and W_1 and W_2 be two objects of another color. There are 6 combinations of two: $\{R_1, R_2\}$, $\{W_1, W_2\}$, $\{R_1, W_1\}$, $\{R_1, W_2\}$, $\{R_2, W_1\}$, $\{R_2, W_2\}$.

Of the 6 combinations, only the first two match in color, hence the probability of drawing pairs matching in color is $\frac{1}{3}$. While this problem is easy to analyze in this manner, Exercise 4 presents a more difficult problem because of the larger number of objects:

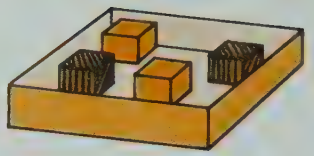
4A. Number of sets of $3 = {}_5C_3 = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$

B. P (3 white) $= \frac{1}{10}$

Combinations in Probability

Investigating the Ideas

Put four objects that are the same size and shape into a box. Two of the objects should be the same color, the other two another color. The experiment consists of drawing two of the objects out of the box and recording their colors. Repeat the experiment 100 times.



Can you predict about how many times two objects will be the same color in the 100 trials?

Discussing the Ideas

1. How close was your prediction to the actual results of the experiment?

2. You can use combinations to find the *theoretical probability* of drawing two of the objects that are the same color.
   - A How many sets of 2 can be selected from a set of 4 objects? That is, what is ${}_4C_2$?
   - B Since there are only two different sets of 2 which will match in color, the probability of drawing two objects that match in color is
     $$P \text{ (same color)} = \frac{2}{{}_4C_2}.$$
     What is this probability?

3. What is the probability of drawing two of the objects that do not match in color?

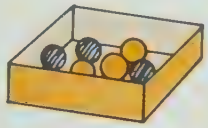
4. Suppose five objects are placed in a box. Three are white and two are black.
   - A How many different sets of three could be drawn?
   - B What is the probability of drawing the three white objects from the box?

Using the Ideas

1. Three students, two girls and a boy, each write their names on a slip of paper and put them into a hat. Two of the names are drawn by a blindfolded student.

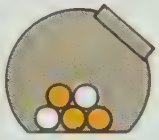
- **A** Suppose the students' names are Vicki, Sue, and Ben. List all the different pairs of names that could be drawn.
- **B** How many pairs consist of a girl's name and a boy's name?
- **C** What is the probability that a girl's name and a boy's name will be drawn?
- **D** What is the probability that the two girls' names will be drawn?

2. Six objects of the same size and shape are placed in a container. Three are one color and the other three are another color. The objects are mixed and three of them are drawn out of the container.



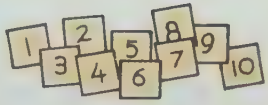
- **A** How many different sets of 3 can be drawn out?
- **B** In how many instances will the set of 3 be the same color?
- **C** What is the probability of drawing three of the objects that have the same color?

3. Three orange marbles and two white marbles are placed in a container and mixed. Then three of the marbles are withdrawn.



- **A** How many different sets of 3 could be drawn?
- **B** What is the probability that the three orange marbles will be drawn?

4. The numerals 1 through 10 are written on cards. The cards are mixed and two of the cards are drawn.



- **A** How many different pairs could be selected? $({}_{10}C_2)$
- **B** How many of the pairs will be pairs of even numbers? $({}_{5}C_2)$
- **C** What is the probability of drawing a pair of even numbers?
- **D** What is the probability of drawing an odd-number pair?
- **E** What is the probability of drawing an odd number and an even number?

UTILIZATION

Assign the problems as independent work. A thorough discussion of the exercises should be beneficial to most students. The exercises may be fairly difficult for some students, because the results are sometimes not in agreement with the student's intuition.

EXTENSION

Enrichment: Probability experiments using the ideas of Exercises 2, 3, and 4 could be set up and carried out by students. At least 100 trials of each experiment should be made so that the experimental results will approximate the theoretical results.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can compute the probability of the union or intersection of two events.

PREPARATION

Review the fact that an event is a subset of a sample space. Use questions concerning tosses of a die: "What is the event of tossing a number less than 3?" (*Answer*: $\{1, 2\}$) "What is the probability of this event?" (*Answer*: $\frac{1}{3}$)

INVESTIGATION

Have the students read the Investigation carefully. Point out that the elements in E **or** F are in the *union* of E and F, that is, in $E \cup F$. The elements in E **and** F are in the *intersection* of E and F; that is, $E \cap F$.

DISCUSSION

In discussing Exercises 2 and 3, it should be made clear that we must subtract $P(E \cap F)$ from the sum of $P(E)$ and $P(F)$ to avoid using the probability of a single outcome more than once in the union of the events and therefore $P(E \cup F)$ is not always the same as $P(E) + P(F)$ unless events E and F are disjoint (that is, $E \cap F = \emptyset$), then $P(E \cap F) = 0$. In this case $P(E \cup F) = P(E) + P(F)$.

Union and Intersection of Events

Investigating the Ideas

An experiment consists of flipping a spinner like the one in the figure. A sample space for the experiment is

$$S = \{1, 2, 3, 4, 5\}.$$

The probability of each outcome in S is $\frac{1}{5}$. Suppose that E is the event $\{1, 2, 3\}$ and F is the event $\{2, 5\}$.

An outcome of the experiment is in event E **or** F if it is in $E \cup F$.

An outcome is in E **and** F if it is in $E \cap F$.

Event	Probability
$E = \{1, 2, 3\}$	$P(E) =$
$F = \{2, 5\}$	$P(F) =$
$E \cap F = \{2\}$	$P(E \cap F) =$
$E \cup F = \{1, 2, 3, 5\}$	$P(E \cup F) =$

Can you find the probability of each event in the table?

Discussing the Ideas

1. What is $P(E) + P(F)$?

2. Explain why $P(E \cup F)$ is not the same as $P(E) + P(F)$.

3. Can you tell how to find $P(E \cup F)$ if you know $P(E)$, $P(F)$, and $P(E \cap F)$?

4. The previous exercise may help you to understand how to find the probability of the union of two events.

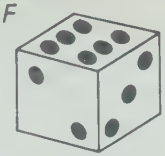
   If S is a sample space and E and F are events in S, then $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

   Can you solve this problem about the spinner experiment?

   What is the probability of getting either an odd number or a number greater than 3?

Using the Ideas

1. Suppose an experiment consists of tossing a die. The sample space is $\{1, 2, 3, 4, 5, 6\}$. E is the event of getting an even number and F is the event of getting 3 or less.

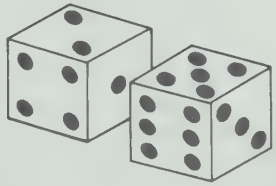


- **A** List the elements in E and F. Then find $P(E)$ and $P(F)$.
- **B** Find $E \cap F$ and give $P(E \cap F)$.
- **C** Find $E \cup F$ and give $P(E \cup F)$.
- **D** Compute $P(E \cup F)$ using the formula $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

2. A spinner is divided into 10 parts of equal size and numbered 1 through 10.

- **A** Let E be the event of getting a multiple of 3. What is $P(E)$?
- **B** Let F be the event of getting a multiple of 2. What is $P(F)$?
- **C** What is the probability of getting a multiple of 3 **and** a multiple of 2?
- **D** What is the probability of getting a multiple of 3 **or** a multiple of 2?

3. An experiment consists of tossing two dice and recording the sum of the number of dots on the two top faces. A sample space and the probability for each outcome are shown.



Sample Space: $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Probability: $\frac{1}{36}\ \frac{1}{18}\ \frac{1}{12}\ \frac{1}{9}\ \frac{5}{36}\ \frac{1}{6}\ \frac{5}{36}\ \frac{1}{9}\ \frac{1}{12}\ \frac{1}{18}\ \frac{1}{36}$

Let E be the event of getting a sum between 3 and 9.
Let F be the event of getting a sum less than 6.

- **A** Give the probability of each outcome in event E.

  $E = \{4, 5, 6, 7, 8\}$

  Probability: ▮ ▮ ▮ ▮ ▮
- **B** What is $P(E)$?
- **C** Give the probability of each outcome in event F.

  $F = \{2, 3, 4, 5\}$

  Probability: ▮ ▮ ▮ ▮
- **D** What is $P(F)$?
- **E** List the elements in $E \cup F$.
- **F** What is $P(E \cup F)$?
- **G** List the elements in $E \cap F$.
- **H** What is $P(E \cap F)$?
- **I** Find the probability of $E \cup F$ using the formula $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

More practice, page S-30, Set 57

UTILIZATION

The exercises provide practice in finding the union and intersection of events and make use of the formula for the probability of the union of two events. A discussion of solutions to the exercises will help to clarify the ideas for many students.

EXTENSION

Remedial: Have students complete a table like the one below.

$P(E)$	$P(F)$	$P(E \cap F)$	$P(E \cup F)$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	▮ ($\frac{7}{12}$)
$\frac{3}{10}$	$\frac{7}{10}$	$\frac{1}{5}$	▮ ($\frac{4}{5}$)
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{5}$	▮ ($\frac{11}{20}$)
$\frac{1}{2}$	$\frac{3}{4}$	▮ ($\frac{3}{8}$)	($\frac{7}{8}$)
$\frac{3}{5}$	▮ ($\frac{1}{5}$)	$\frac{1}{10}$	($\frac{7}{10}$)

Enrichment: Encourage students to try one or both of the Research Projects on page H–25.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–3.

Objective

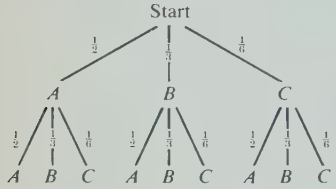
The student can demonstrate the ability to work with the major concepts presented in this module.

UTILIZATION

These exercises review the significant ideas introduced in this module. Provide ample time for students to discuss their work and any topics that caused difficulty, when they have completed the exercises.

Workbook page 106 provides additional exercises that review the concepts of this module.

Answer, Exercise 5



REVIEWING THE IDEAS

1. A coin is tossed 150 times. About how many times would "tails" be expected to come up?

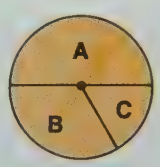
2. A mixture of 100 white beads and 300 red beads is in a jar.
   - **A** What is the probability of drawing a white bead?
   - **B** If you drew out a handful of beads and found that you had 39 beads, about how many of each color would you expect to get?

3. A sample space for flipping the spinner is $\{1, 2, 3, 4, 5, 6\}$. List the elements in the following events.

   - **A** Event A: The event of getting a number greater than 3
   - **B** Event B: The event of getting a number less than 5
   - **C** Event C: The event of getting an even number
   - **D** Event D: The event of getting a prime number

4. Compute each probability, where A, B, C, and D are the events given in Exercise 3.
   A $P(A)$ **B** $P(B)$ **C** $P(C)$ **D** $P(D)$

5. A sample space for an experiment of flipping the spinner is $\{A, B, C\}$, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(C) = \frac{1}{6}$. A symbol such as AC will be used to represent the event of getting A on the first flip and C on the second flip. Make a tree diagram for two flips of the spinner. Label the outcomes and the probabilities along each branch of the tree.



6. Use the tree diagram of Exercise 5 to find the probability of each event below.
   A AA **B** AB **C** BC **D** BB **E** CA **F** CC

7. Four persons run a 1500-metre race. If there are no ties, in how many orders (permutations) can they finish the race?

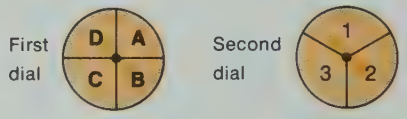
8. A car manufacturer advertises that one make of car is available in 3 body styles, 4 kinds of engine, and 6 colors. How many different models are available?

9. Determine the number of combinations.
   A ${}_4C_2$ **B** ${}_6C_3$ **C** ${}_5C_4$

10. From a group of 5 people, 2 are to be selected. How many different pairs can be selected?

11. Solve each equation.
   A $4 \cdot 3! = n$ **B** $6! \div 5! = n$ **C** $(3!)^2 = n$ **D** $\frac{10 \cdot 9 \cdot 8}{3!} = n$

12.



   - **A** Let E be the event of getting **B** on the first dial. List the elements of E and compute $P(E)$.
   - **B** Let F be the event of getting **2** on the second dial. List the elements in F and compute $P(F)$.
   - **C** What is $P(E \cap F)$?
   - **D** What is $P(E \cup F)$?

TEST YOURSELF

1. **A** What is $P(2)$?
   B What is the probability of getting a number greater than 2?

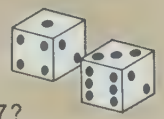
2. **A** What is the sample space for this spinner experiment?
   B Give the probability of each outcome.

3. Two black marbles and three white marbles are placed in a box. One marble is drawn, its color recorded, replaced in the box, and a second marble is drawn.

   A What is the probability of drawing two white marbles in a row?
   B What is the probability of drawing two black marbles in a row?

4. Two coins are tossed. What is the probability that both will come up heads?

5. Two dice are tossed. What is the probability of getting a sum of 7?



6. Four people are leaving a room one at a time through one door. In how many different orders can they leave the room?

7. There are 6! permutations of the letters of the word NOTICE. How many permutations is this?

8. Two people are to be selected from a committee of eight people. How many different pairs are possible?

9. From 5 different cards, 3 are to be selected.

   How many different combinations are possible?

10. E and F are events in which $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{3}$, and $P(E \cap F) = \frac{1}{6}$. Find $P(E \cup F)$.

RESEARCH PROJECTS

A Find out what **random numbers** are and how tables of random numbers are developed and used. (See *Probability: A First Course*, Revised Edition, by Frederick Mosteller, Robert E. K. Rourke, and George B. Thomas, Jr.; Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1970.)

B Create a probability baseball game based on tosses of two dice. For example, Outs: 6, 7, and 8; Singles: 5 or 9; Doubles: 10; Triples: 3; Home runs: 11; Walks: 4; Errors: 2; Sacrifice fly: 12. Draw a baseball diamond on a piece of cardboard and use checkers as base runners. Make up your own rules for playing the game.

TEST YOURSELF

This self-evaluation test should enable students to assess how well they understand the principal concepts presented in the module. Students who perform well on this test should also do well on an achievement test covering this material.

When students have completed the exercises, they should check their work against the answers provided on page H–26. Encourage them to refer back to appropriate lessons within the module to review any topics that proved troublesome for them.

Note that correlations between individual test items and the module objectives are indicated by annotations on the student text page at the left.

RESEARCH PROJECTS

Students who undertake Project A will find that many statistics books have short tables of random digits. Encourage them to explore a book called *A Million Random Digits with 100 000 Normal Deviates*, published by The Free Press of Glencoe, Illinois, copyright 1955 by The Rand Corporation. (Distributed by Collier-Macmillan Canada Ltd., Don Mills, Ontario.) It is possible to simulate many probability experiments such as coin or dice tossing by using a table of random numbers.

Encourage students who are interested in Project B to make a durable model of the game board and playing pieces, and to devise a complete set of rules for playing the game.

MATHEMATICAL RECREATION

To conduct this experiment, students will need a large piece of paper and a toothpick or a needle.

In order to achieve a good approximation for π, the experiment should be repeated several hundred times. If N represents the number of trials and h the number hits in the N trials, then

$$\pi \approx \frac{2N}{h}.$$

It is possible to design the experiment so that the parallel lines are farther apart than the length of the needle provided they are less than twice as far apart as the length of the needle. Then, if d is the distance between the parallel lines and ℓ is the length of the needle,

$$\pi \approx \frac{2N\ell}{hd}.$$

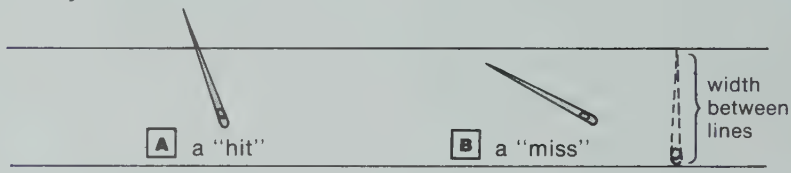
MATHEMATICAL RECREATIONS

Buffon's Needle Problem

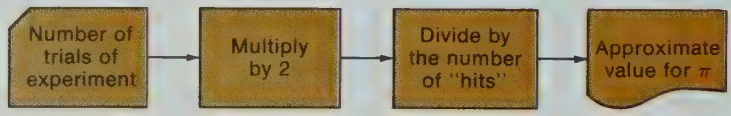
A French naturalist, Comte de Buffon (1707–1788), found an interesting way to find an approximate value for π based upon probability. Buffon's method is described below.

Draw a large number of parallel lines on a paper that are just as far apart as the length of some needle. Then perform the needle experiment as follows:

Drop the needle at random from some height above the paper and record whether the needle hits one of the lines (as in A) or misses (as in B). Repeat the experiment many times.



Follow the flow chart instructions to find an approximate value for π.



What is your approximation for π?

Combine your results with those of several classmates. Do you get a better approximation for π?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $\frac{1}{6}$; B $\frac{5}{6}$ **2.** A $\{1, 2, 3, 4, 5\}$; B $P(1) = \frac{1}{4}$; $P(2) = \frac{1}{4}$; $P(3) = \frac{1}{6}$; $P(4) = \frac{1}{6}$; $P(5) = \frac{1}{6}$ **3.** A $\frac{9}{25}$; B $\frac{4}{25}$ **4.** $\frac{1}{4}$ **5.** $\frac{1}{6}$ **6.** 24 **7.** 720 **8.** 28 **9.** 10 **10.** $\frac{2}{3}$

H–26

UNIT H: SOME SPECIAL TOPICS IN MATHEMATICS

Module 2: Statistics

General Objectives

To familiarize the students with three measures of central tendency–mean, median, and modes.
To show how statistical data can be presented by means of tables or graphs.
To show frequency distributions by means of histograms.
To introduce the student to some simple examples of inferential statistics by using samples.
To relate probability to statistics.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
110	**RED** Given a set of numerical data, the student can find the arithmetic mean of the set.	H-30, 31	**WB** 107 **DM** 72	**ASC** W-13
111	**RED** Given a set of numerical data, the student can find the median of the set.	H-32, 33	**WB** 108 **DM** 73	
112	**RED** Given a set of numerical data, the student can find the mode(s) of the set.	H-34, 35	**WB** 109 **DM** 73	
113	**RED** Given a histogram, the student can interpret the data shown in the graph.	H-36, 37	**WB** 110 **DM** 74	**SWM 3** 328-329
114	**RED** Given a line-segment graph, the student can interpret the data shown in the graph.	H-38, 39	**WB** 111 **DM** 75	
115	**RED** Given a circle graph, the student can interpret the data shown in the graph.	H-40, 41	**WB** 112 **DM** 76	**SWM 3** 338-339
116	**YELLOW** Given a statistical sample of a set (population), the student can draw inferences (conclusions) about the set.	H-42, 43, 46	**WB** 113	**SWM 3** 144-147, 336-337
117	***GREEN*** *Given a probability experiment, the student can carry it out and then compare the experimental results with the theoretical probability of a certain event.*	H-44, 45, 47	**DM** 77	**SWM 3** 132-133
	Reviewing the Ideas	H-48	**WB** 114	

MATHEMATICS

Statistics is often separated into two realms:

A. *Descriptive statistics,* which deals with ways of presenting numerical data so that it can be easily understood.
B. *Inferential statistics,* which deals with drawing probable conclusions about some large population (such as the set of people in a nation) by means of information obtained from a small subset or sample of the population.

There are many aspects of descriptive statistics, all of which attempt to present numerical data in various ways. Graphic devices such as histograms, line-segment graphs, and circle graphs are useful in giving a visual summary of a large collection of measurements. However, it is sometimes more convenient to summarize such information by means of certain numerical measures called *measures of central tendency.* Three such measures are covered in this module.

1. The *arithmetic mean* of a set of n numbers is the sum of the numbers divided by n.
2. The *median* of a set of numbers is the middle number of the set.
3. The *mode* of a set of numbers is the most frequently occurring number in the set, if there is such a number.

In many instances when we desire to know certain facts about a very large population, it would be very difficult, time consuming, and

expensive to gather data about each member of the population. In this case we draw a *random sample* of the population and, from analysis of the sample, draw conclusions concerning the total population. Such a process is called *sampling*. Only very simple cases of sampling are introduced in the module. Selection and analysis of samples requires some expert knowledge and is beyond the objectives of the module.

TEACHING THE MODULE

Materials

Strings, washers, small weights, stop watches or sweep-second watches, metre sticks, newspapers, pins or paper clips, coins or substitute coins, graph paper, protractors, playing cards (one deck per group of students).

Vocabulary

arithmetic mean (average)
bar graph
biased sample
circle graph
data
frequency
histogram (bar graph)
line-segment graph
median
mode
pendulum
random
range
sample

Although much of the material in the module may be a review for your students, there is sufficient new material that almost everyone will benefit from studying it. You will note that there are numerous investigations and exercises that suggest activities that should be carried out by the students in order for them fully to appreciate the ideas presented. Only minimal equipment is needed for these activities.

The first seven lessons deal with measures of central tendency and graphs of data. The lessons on histograms, line-segment graphs, and circle graphs place primary emphasis on reading and interpreting graphs; the construction of graphs is of secondary importance. The remaining lessons introduce the students to problems in sampling through active involvement. The later lessons serve to relate the ideas of probability presented in the previous module to statistics.

Evaluation

The list of specific skills for the module is relatively small. However, the practical importance of the material should not be underestimated. As adults, students will be faced daily with a variety of statistical data, which will need to be understood, analyzed, or interpreted.

Use the review questions and the Test Yourself questions as an aid in designing an achievement test for this module.

Lesson Schedule

You should allow about 10 to 12 days for coverage of the module. The schedule should remain flexible, however, reflecting the amount of coverage and the depth of presentation. If you decide to carry out most of the activities and experiments suggested in the module you may need to allow 3 to 5 more class periods of time.

RESOURCES FOR ACTIVE LEARNING

Activities

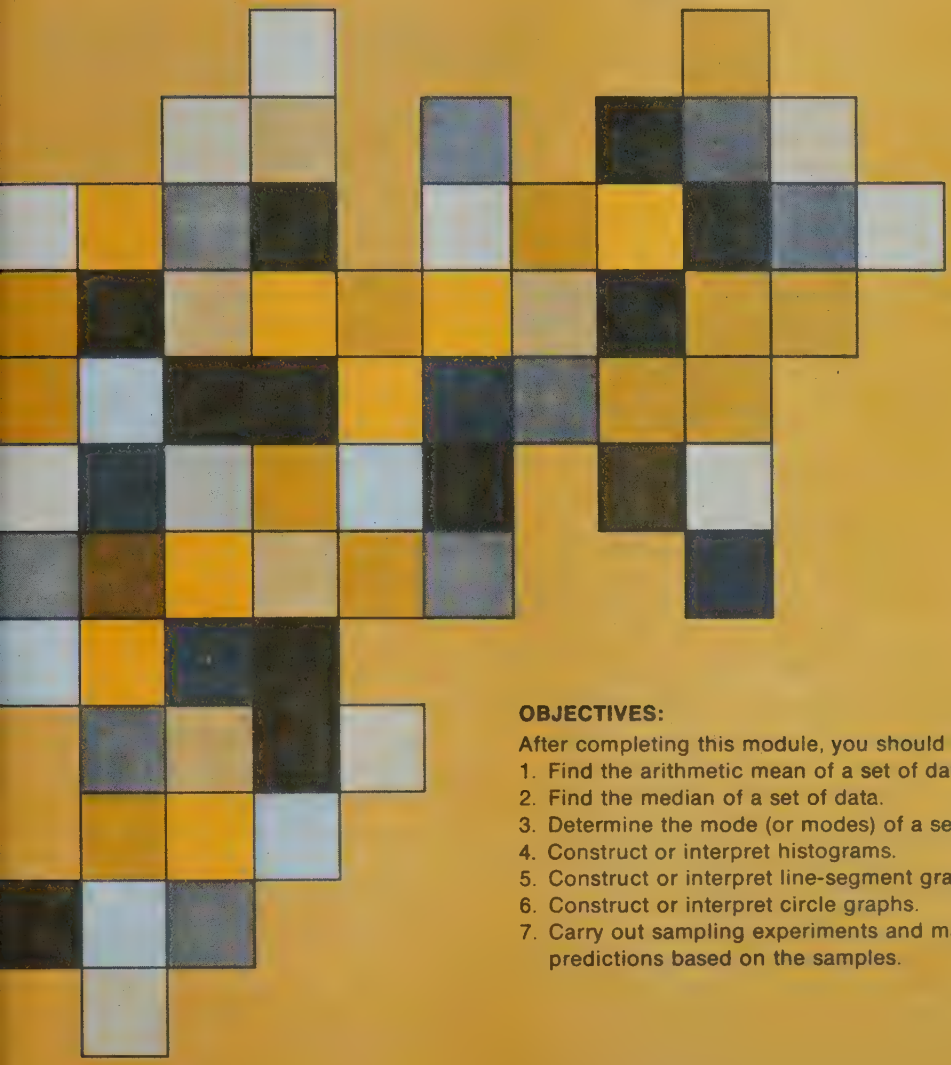
Developmental Math Cards, K1, L7, Addison-Wesley
ESS: Pendulums, Webster, McGraw-Hill
Experiments in Mathematics, Stage 2, pp. 38–39; Stage 3, pp. 14–19, Houghton Mifflin
Experiments in Probability and Statistics, Midwest Publications
Nuffield Project: *Probability and Statistics*, pp. 3–11, 30–44; *Speed and Gradient 1*, Wiley
Sourcebook for Substitutes, p. 100, Addison-Wesley
Statistics, Cuisenaire Co.

Manipulative Devices

Metre Stick (Addison-Wesley)
Probability and Statistics Lab (Gamco; LaPine)
Protractor (Geyer; Selective Educational Equipment; Sigma)
Stopwatch (Educational Teaching Aids)

UNIT H: *Some Special Topics in Mathematics*

MODULE 2: *Statistics*



OBJECTIVES:

After completing this module, you should be able to:

1. Find the arithmetic mean of a set of data.
2. Find the median of a set of data.
3. Determine the mode (or modes) of a set of data.
4. Construct or interpret histograms.
5. Construct or interpret line-segment graphs.
6. Construct or interpret circle graphs.
7. Carry out sampling experiments and make predictions based on the samples.

Objective

The student can obtain data from an experiment and solve problems related to it.

PREPARATION

Materials: String, metal washers or other small weights, stop watch or watches with sweep-second hands, metre sticks.

Have the students begin work on the Investigation immediately.

INVESTIGATION

Students should work in small groups, with at least 3 students in each group. One student should hold the pendulum in place, one or two other students should keep track of the number of swings, and a third student should watch the time. Tie a washer or small weight at the end of the string. If the students use a piece of string a bit longer than 50 cm, they can tie knots at each end of the lengths given in the Investigation. (Each length should be measured to the bottom of the pendulum bob.)

DISCUSSION

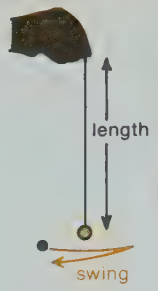
The Discussion should bring out some properties of the pendulum that students may have observed in the Investigation: (1) The shorter the pendulum, the greater the number of swings per minute; (2) The frequency of the swings of the pendulum does not depend on the mass of the pendulum bob. (See Exercise 5.)

The approximate formula given in Exercise 4 is derived from the standard formula for the period T, in seconds, of a pendulum of length ℓ in cm. (The period of a pendulum is the time in seconds for one complete oscillation.) $T = 2\pi\sqrt{\frac{\ell}{g}}$, where g represents the constant, $\frac{980 \text{ cm}}{\text{s}^2}$, the acceleration due to gravity. T is related to the frequency F in swings per minute by $T = \frac{60}{F}$. Substituting for T and solving for ℓ, we obtain $\ell = \frac{60^2 g}{4\pi^2 F^2}$. Substituting for g and π, we obtain the approximate formula $\ell \approx \frac{90\,000}{F^2}$.

Collecting Data from an Experiment

Investigating the Ideas

Make a **pendulum** by tieing a small weight to a piece of string. Hold the string at a measured distance from the weight and count the number of complete swings (back and forth) that the pendulum makes in one minute. Repeat the experiment three times for each length shown in the table.



Length of pendulum	Frequency (Swings per minute)		
10 cm			
20 cm			
30 cm			
40 cm			
50 cm			

Can you estimate how long a pendulum should be in order to make one swing per second?

Discussing the Ideas

1. Did you always get the same number of swings for the same length of the pendulum? Find the average of the number of swings per minute for the three trials.

2. What relationship does your experiment show between the length of a pendulum and its frequency in swings per minute?

3. What was your estimate for the length of a pendulum that should make one swing per second? Compare your estimate with those of your classmates. Check your estimate by actual experiment.

4. If F is the frequency of a pendulum in swings per minute, then the approximate length ℓ of the pendulum in centimetres is given by the formula $\ell \approx \frac{90\,000}{F^2}$. Thus, if $F = 30$ swings per minute, then the length of the pendulum is $\frac{90\,000}{30^2} = \frac{90\,000}{900} = 100$ cm. Show how to use the formula to compute the length of a pendulum that makes 60 swings per minute.

5. Do you think the amount of weight used on the pendulum affects the frequency of the pendulum?

Using the Ideas

1. Which playground swing, **A** or **B**, will make more swings per minute?

2. Use the formula $\ell \approx \frac{90\,000}{F^2}$ to compute the approximate length in centimetres of each pendulum whose frequency F in swings per minute is given below.

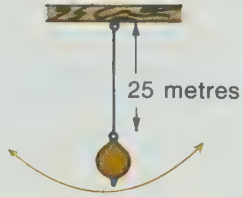
   A $F = 40$ **C** $F = 50$ **E** $F = 12$

   B $F = 15$ **D** $F = 100$ **F** $F = 300$

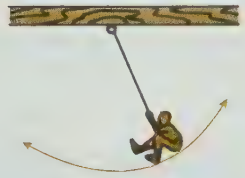
3. A long pendulum on display in a science museum has a frequency of 6 swings per minute.

   A About how long does it take the pendulum to make one swing?

   B What is the approximate length of the pendulum in *metres*?



4. A rope is suspended from a beam in a gymnasium. You could make 10 swings per minute on the rope. How long is the rope?

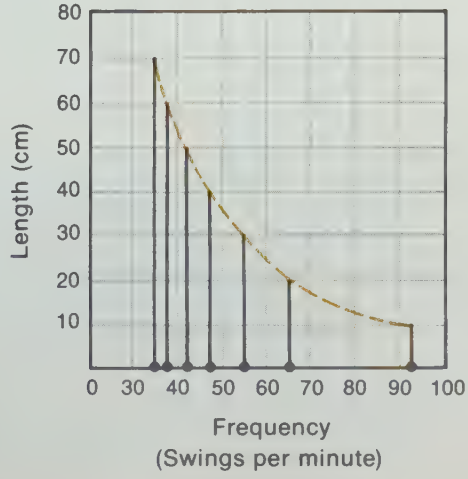


5. A group of students made a graph by taping their pendulums perpendicular to the line of frequency and at the frequency they found for the pendulum. Use the dotted curve connecting the ends to make estimates of the following:

   A The length of a pendulum which will make 60 swings per minute.

   B The frequency of a pendulum 15 centimetres long.

   C The length of a pendulum with a frequency of 50 swings per minute.



UTILIZATION

Remind the students that the approximate formula $\ell \approx \frac{90\,000}{F^2}$ will give the length of a pendulum in centimetres when F is the number of swings per minute of the pendulum. The formula need not be memorized by the students.

EXTENSION

Enrichment: Use the activity described in Exercise 5 as a class project. Place a large posterboard or newsprint grid on a bulletin board. Then have students tape or pin their pendulums at the appropriate frequencies. If several different pendulums are used, a portion of the hyperbolic curve for this relation can be graphically depicted.

Assignments

Minimum 1, 2. Average 1–4. Maximum 1–5.

Objective

Given a set of numbers, the student can compute the arithmetic mean of the set.

PREPARATION

It would be appropriate for students to begin work on the Investigation without any preparatory activity.

INVESTIGATION

The students may need to read the statement that introduces the Investigation more than once before they grasp the nature of the problem involved. They need to understand that they must do two things:

1. Find the total height of the ten students listed (1650 cm)
2. Divide the total height by the number of students to find which student has the required height.

DISCUSSION

The *arithmetic mean*, or *average*, is the sum of all the numbers divided by the number of numbers in the set. Since students will already be familiar with these terms, you need only review the idea with them.

In Exercise 3, the term *range* of a set of numbers is likely to be unfamiliar to most students. You might point out that, though the arithmetic mean of a set of numbers always lies between the smallest and the largest number in the set, it is not necessarily the middle number.

Exercise 4 suggests a class activity that could be carried out now or after the exercises for this lesson have been completed.

Arithmetic Mean

Investigating the Ideas

One student on this list said, "If everyone on the list were as tall as I am, our total height would be just the same as it is now."

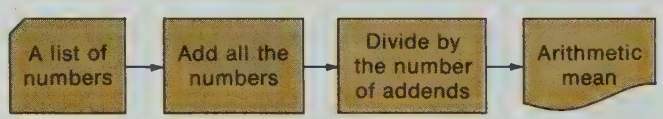
Can you find the name of the student who could have made this remark?

Name	Height (cm)	Weight (kg)
Paula S.	150	43
Jack T.	159	46
Donna M.	160	53
Dale K.	161	56
Kris E.	163	57
Eva B.	165	61
John R.	168	60
Anne W.	169	68
Dick U.	174	73
Tom T.	181	76

Discussing the Ideas

1. Explain how you found the student's name.

2. The student who made the remark above had a height which was the **arithmetic mean** of the heights of all the student heights. The flow chart below may help you to recall how to compute the arithmetic mean of a list of numbers.



A Guess which student's weight is nearest to the arithmetic mean of the group.

B Use the flow chart to compute the arithmetic mean of the weights of the students to the nearest tenth of a kilogram.

3. The **range** of a set of data is the difference between the largest number and the smallest number in the set. Since the heights shown in the table vary from 150 cm to 181 cm, the range in height is 31 cm. What is the range in weight?

4. Guess which student in your class is nearest to the arithmetic mean in height for all the students in the class. Find a way to check your guess.

Using the Ideas

1. The average distance from the sun to each of the nine planets is given in the table.

Planet	Distance (million km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	777
Saturn	1426
Uranus	2869
Neptune	4494
Pluto	5198

- **A** What is the range of the distances from the sun?
- **B** Find the mean distance of the nine planets from the sun.
- **C** Which planet is nearest to the mean distance from the sun?

2. Linda and Mary each bowled three games.

Bowling Score Sheet				
Name	Game 1	Game 2	Game 3	3-game total
Linda	136	148	128	
Mary	158	139	145	

- **A** Find their 3-game totals.
- **B** Find their average (mean) scores. Who had the higher average?
- **C** The "handicap" for each bowler is found by the formula $h = \frac{3}{4} \cdot (200 - A)$ where h represents the handicap and A is the person's average score. Compute each bowler's handicap to the nearest whole number.

3. The scores for a mathematics test given to a class are 82, 88, 72, 94, 90, 69, 74, 81, 89, 85, 72, 92, 80, 66, 75, 85, 96, 72, 83, 87, 84, 82, 77, and 78.

- **A** What is the range of the scores?
- **B** What is the mean score?

4. The weights (in kilograms) of five students are given in the table.

Name	Weight (kg)
Jean	45.9
Arthur	53.1
Jill	43.2
Chip	57.1
Jody	37.8

- **A** Find the range of the weights.
- **B** What is the arithmetic mean of their weights to the nearest tenth of a kilogram?

★ **5.** Melissa's test scores on four tests are 82, 86, 84, and 81. What is the lowest score she can make on the next test to average at least 85 for all five tests?

More practice, page S-30, Set 58

UTILIZATION

Provide time, after students have completed the exercises, for a discussion of them, particularly Exercise 5.

Solution, Exercise 5

If the average of the 5 test scores is to be 85 then the total of the 5 tests must be $5 \cdot 85 = 425$. The total of the 4 previous tests is 333. Therefore the lowest possible score on the 5th test must be $425 - 333 = 92$.

EXTENSION

To provide additional practice, make assignments from Supplementary Exercise Set 58 on page S–30, *Workbook* page 107, *Duplicator Masters* page 72, and *Arithmetic Skill Card* W-13.

Enrichment: The arithmetic mean is one measure of central tendency of a set of numbers. But for certain sets of data the measure may not be appropriate. You might illustrate this by showing that the "average income" of 5 men could be \$200 800 a year when 4 of the men earn \$1000 a year and the fifth man earns \$1 000 000 a year.

Assignments

Minimum 1–3. Average 1–4. Maximum 1–5.

Objective

Given a list of numbers, the student can arrange them in order of size and determine the median of the list.

PREPARATION

Materials: Newspapers, pins or paper clips.

In order to devote as much time as possible to the Investigation, keep the preparation time to a minimum.

INVESTIGATION

This Investigation lends itself to small groups or pairs of students working together. Instead of using a pin, one end of a paper clip can be straightened out and used as the "pointer" to each word.

To increase the likelihood of variation in findings, you might suggest that the groups of students use different sections of the newspaper for their word selections, such as the financial pages, the sports pages, and the editorial pages.

DISCUSSION

Students will be interested in comparing the length of their "middle" words. You may expect some variation in results, but most groups will probably find the "middle word" to be about 5 letters in length.

Exercise 3 illustrates how the median of a list of numbers is found when there is an even number of members in the list. You will want to emphasize this point.

To illustrate the idea of median, present the students with the following 24 test scores:

82, 87, 72, 94, 90, 69, 74, 81, 88, 85, 72, 91, 80, 66, 75, 85, 95, 92, 83, 86, 84, 82, 77, 78

Then ask the students to:

A. arrange the tests scores in order from the highest to the lowest score.
B. find the median score. (Answer: $82\frac{1}{2}$)
C. find the mean score. (Answer: 82)

Finding Medians

Investigating the Ideas

The 15 words in the table were selected at random by dropping a pin on a newspaper and then recording the word that was nearest to the pinpoint. When the words were listed in a table in order from the longest to the shortest, the word large was found to be the word of "middle" length. Just as many words were longer than it as were shorter.

Word	Number of letters
1. possibility	11
2. education	9
3. invested	8
4. programs	8
5. spending	8
6. instead	7
7. united	6
8. large	5
9. here	4
10. said	4
11. the	3
12. are	3
13. and	3
14. new	3
15. is	2

Can you choose 15 words of your own and find the number of letters in the "middle" word?

Discussing the Ideas

1. Does your "middle" word have about the same number of letters as the "middle" word in the table above?

2. The *middle number* in a set of numerical data arranged in order of size is called the **median**. The number "5" in the table above is the middle number of the 15 numbers. The word of median length is "large."
   - **A** What is the length of the word that has the median length in your data?
   - **B** Is your result about the same as your classmates'?

3. On four tests Jody scored 96, 88, 82, and 80. There is no middle number in this set of data. To find the median, find the arithmetic mean of the two numbers nearest the middle.

$$\text{Median} = \frac{88 + 82}{2} = \frac{170}{2} = 85$$

   - **A** If 25 test scores were arranged in order, which score would be the median score?
   - **B** If there were 30 scores, which two scores would be used to find the median score?

Using the Ideas

1. The variation in the price of a certain article in several supermarkets is shown in the table. What is the median price?

Market	Price (¢)
A	83
B	81
C	79
D	77
E	75
F	75
G	73

2. **A** Arrange the numbers in the set in order.
65, 76, 72, 100, 80, 65, 67, 65, 68, 40, 59

B Find the median of the set.

3. On a class test of 25 students, Marjorie found that her score was the median score. No one else made that score.

A How many students had scores higher than Marjorie?

B How many students had scores lower than Marjorie?

4. **A** What is the median amount of rainfall per month?

B Find the arithmetic mean of the rainfall measurements.

C Which is greater, the median or the mean rainfall per month? How much greater is it?

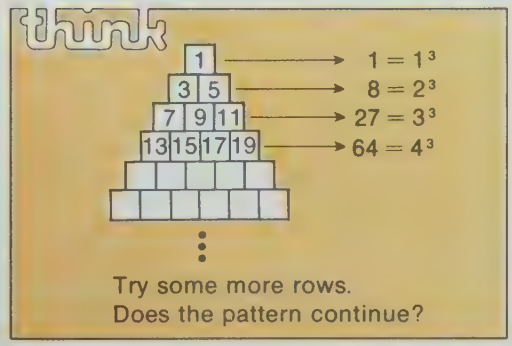
Month	Rainfall (cm)
November	7.3
December	5.9
January	7.5
February	5.5
March	8.8
April	9.2
May	10.6
June	11.0
July	9.1
August	6.6
September	6.9
October	7.1

5. The median income per family in a certain city is $9737. If the set of data is based upon 8461 families, about how many families earned less than $9737?

★ 6. The median of the four numbers 48, 37, **x**, and 19 is 30.

A What number must **x** be?

B What is the mean of the four numbers?



More practice, page S-31, Set 59

UTILIZATION

Assign the exercises as independent work. Discuss the solutions when students have completed them. The discussion should bring out the differences between the median of a set of data and the arithmetic mean of the same set. Students may feel that these two measures of central tendency are very close to each other. To give a counterexample, consider the five numbers 102, 99, 2, 1, 1. The median of the set is 2, but the arithmetic mean is 41.

Solution, Exercise 6

A. When there are an even number of objects, the median is found by averaging the middle two numbers. Since the median is 30, it can be seen that 37 and x are the two middle numbers. Thus,

$$(37 + x) \div 2 = 30$$
$$37 + x = 60$$
$$x = 23$$

B. Mean $= (48 + 37 + 23 + 19) \div 4$
$= 31.75$

EXTENSION

Further practice may be assigned from Supplementary Exercise Set 59 on page S-31 and *Workbook* page 108.

Remedial: Refer students to the height and weight table in the Investigation on page H-30. Ask them to find the median height and median weight of the students listed.

Think Solution

The sums of the numbers in successive rows of the pattern are successive cubes of whole numbers.

$21 + 23 + 25 + 27 + 29 = 125 = 5^3$
$31 + 33 + 35 + 37 + 39 + 41 = 216 = 6^3$
$43 + 45 + 47 + 49 + 51 + 53 + 55 = 343 = 7^3$
$57 + 59 + 61 + 63 + 65 + 67 + 69 + 71 = 512 = 8^3$
$73 + 75 + 77 + 79 + 81 + 83 + 85 + 87 + 89 = 729 = 9^3$
$91 + 93 + 95 + 97 + 99 + 101 + 103 + 105 + 107 + 109 = 1000 = 10^3$

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can find the mode (or modes) for a list of numbers.

PREPARATION

Review the meaning of arithmetic mean and median as measures of central tendency.

INVESTIGATION

The Investigation can be carried out by individual students or by pairs of students. Suggest that they make a tally sheet for each of the ten digits, such as the one shown below.

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	14	12	19	17	15	14	13	10	20	17

The digit 8 occurs the greatest number of times in this decimal approximation of π.

DISCUSSION

The definition of *mode* is given in Exercise 2. The Investigation illustrated a distribution of data that had a single mode. Exercise 3 presents a distribution of last digits of 100 telephone numbers which is *bimodal;* that is, there are two modes, 0 and 3, because both of these digits occur with equally high frequencies. A set of data could be *multimodal;* that is, the set could have many modes.

Modes

Investigating the Ideas

The ratio of the circumference of any circle to its diameter is the number denoted by π. An approximation of π correct to 150 decimal places is given below.

$\pi \approx 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279$
$50288\ 41971\ 69399\ 37510\ 58209\ 74944$
$59230\ 78164\ 06286\ 20899\ 86280\ 34825$
$34211\ 70679\ 82148\ 08651\ 32823\ 06647$
$09384\ 46095\ 50582\ 23172\ 53594\ 08128$

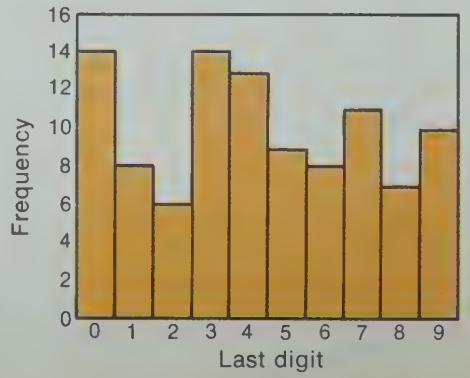
Can you find which digit (or digits) occurred most often in the expression for π?

Discussing the Ideas

1. The number of times each digit of π occurred in the Investigation is called the **frequency** of that digit.
   - A Which digit occurred most often?
   - B Which digit occurred the least number of times?
   - C What was the frequency for the digit 3?

2. The **mode** of a set of numbers is the most frequently occurring number or numbers in a set, if there are such numbers. A number must occur more than once to be a mode.
   - A Does the set of numbers in the table above have a mode?
   - B Does it have more than one mode?

3. The graph shows the frequency of the "last digit" of 100 telephone numbers chosen at random. What mode (or modes) does the graph show for "last digits"?



Using the Ideas

1. The table shows the ages and frequency of the students in a school band. Which age is the mode of all the ages?

Age	11	12	13	14
Frequency	2	17	23	8

2. Rita made a list of the ages of 25 of her classmates' mothers. Which age is the mode of the list?

42, 40, 36, 38, 51, 39, 38, 50, 45
34, 38, 43, 49, 42, 37, 39, 55, 35,
40, 44, 47, 38, 41, 48, 52

3. In 60 tosses of a die, a student found the frequencies for each outcome as given in the table.

Outcome						
Frequency	11	12	7	8	12	10

   A Which outcome (or outcomes) occurred most frequently?
   B What are the modes for the experiment?

4. The results of a student survey concerning the amount of money earned from part-time work per week is shown in the table.

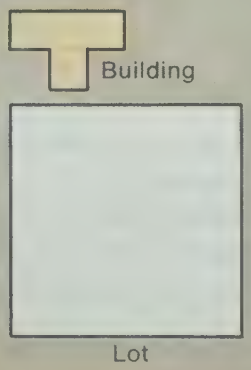
Amount per week	Under $2	$2–4	$4–6	$6–8	$8–10	Over $10
Frequency	10	16	12	9	7	5

   A Which amount per week did the greatest number of students earn?
   ★B Make a graph of the data in the table.

think

1. A man plans to place as many of the T-shaped buildings on the square lot as possible. What is the greatest number of buildings the lot will hold?

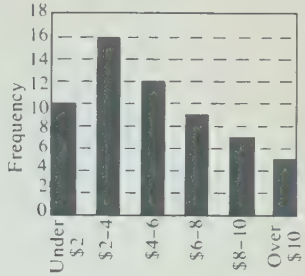
2. An "Open Space Committee" insists that the buildings be placed so that there is room for the fewest possible T-shaped buildings. How many is this?



More practice, page S-31, Set 60

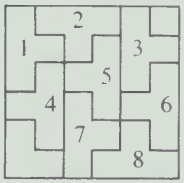
UTILIZATION

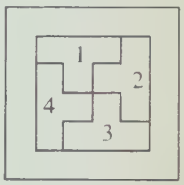
Answer, Exercise 4B



Think Solution

1. Many different arrangements can be used to fit 8 T-shaped buildings into the 6 by 6 square. One of these is below, left:





   Another arrangement would leave the four corner squares empty.

2. Many different arrangements can be used to show a minimum of four buildings. One such arrangement is shown above, right.

EXTENSION

For further practice, see Supplementary Exercise Set 60 on page S-31, *Workbook* page 109, and *Duplicator Masters* page 73.

Enrichment: Some students may be interested in extending the Investigation activity to a large number of decimal places. A table of the first 4000 digits of π is given in *Lore of Large Numbers* by Philip J. Davis, New Mathematical Library, Random House and Yale University, 1961, pages 72, 73. (Available from Random House of Canada, Ltd., Toronto, Ontario.) In connection with this, students might learn about *normal numbers*, infinite decimals in which each of the ten digits occurs $\frac{1}{10}$ of the time. It is not known whether π is a normal number.

Assignments

Minimum 1–3. Average 1–4A. Maximum 1–4.

Objective

The student can interpret a histogram and, given an appropriate set of data, can construct a histogram.

PREPARATION

Materials: Coins or coin substitutes, graph paper (*Duplicator Masters*, page 83).

You will want students to begin work on the Investigation as soon as possible.

INVESTIGATION

Students could work in pairs or small groups to carry out the coin tossing experiment.

Students' estimates of the number of outcomes are likely to vary quite widely. Some students may use a probability tree, since tossing 4 coins at once is the same as tossing 1 coin 4 times in succession. There are 16 possible outcomes, with the following probabilities:

$P(4H, 0T) = \frac{1}{16}$, $P(3H, 1T) = \frac{1}{4}$, $P(2H, 2T) = \frac{3}{8}$, $P(1H, 3T) = \frac{1}{4}$, $P(0H, 4T) = \frac{1}{16}$.

DISCUSSION

Students will be familiar with bar graphs, but use of the term *histogram* to describe a frequency distribution will be new to most of them.

If possible, students should construct a histogram of the results of the experiment. In general, most histograms will be somewhat like the one pictured in Exercise 1, with the outcomes $(4H, 0T)$ and $(0H, 4T)$ the least frequently occurring; however, the frequencies can be expected to vary somewhat.

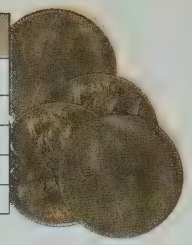
You might show a histogram for the theoretical frequencies given in Exercise 2 and point out the symmetry of the graph. It would also be interesting to combine all student Investigation results and see whether the combined results approach the symmetrical theoretical results.

Histograms

Investigating the Ideas

Suppose you tossed four coins at a time and kept a tally of the number of heads and tails for each toss. About how many times do you think each outcome would occur in 100 trials?

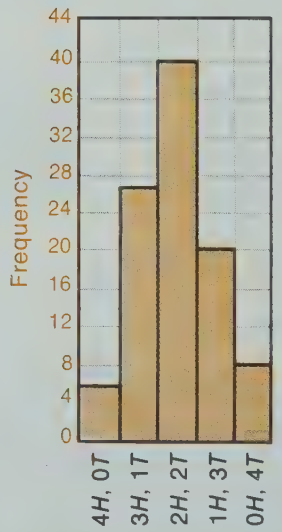
Outcome	Tally
4*H*, 0*T*	
3*H*, 1*T*	
2*H*, 2*T*	
1*H*, 3*T*	
0*H*, 4*T*	



Can you carry out the experiment and check your estimate of the outcomes?

Discussing the Ideas

1. The *bar graph* at the right is a **histogram** of an experiment. Each vertical bar represents the frequency or number of times each outcome occurred.



   A If you made a histogram of your results of the experiment above, in what ways would it be like this one?

   B In what ways would your histogram be different?

2. The theoretical frequency of the outcomes for 64 trials of the experiment is shown in the table. How would a histogram for the experimental data differ from a histogram of the theoretical data?

Outcome	Theoretical Frequency (64 trials)
4*H*, 0*T*	4
3*H*, 1*T*	16
2*H*, 2*T*	24
1*H*, 3*T*	16
0*H*, 4*T*	4

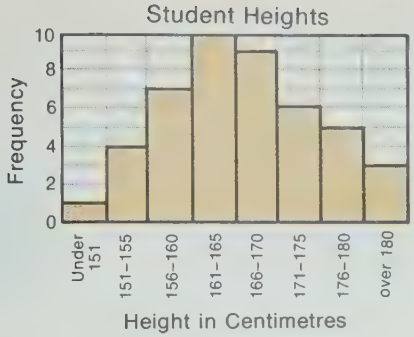
Using the Ideas

1. A How many students are represented by the histogram?

B Give the frequency for each height interval.

C What percent of the students are in the 161–165 centimetre interval?

D What percent are over 180 centimetres tall?

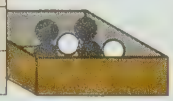


2. Two students tossed a pair of dice 120 times and made a tally of the frequency of the dice totals as shown in the table. Make a histogram of this data.

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	4	6	7	10	19	20	20	17	6	9	2

3. Two white marbles and four black marbles are placed in a box and thoroughly mixed. Two marbles are drawn out at random. The experiment is repeated 30 times. Make a histogram of the outcomes.

Outcome	Tally
2*B*	
1*B*, 1*W*	
2*W*	

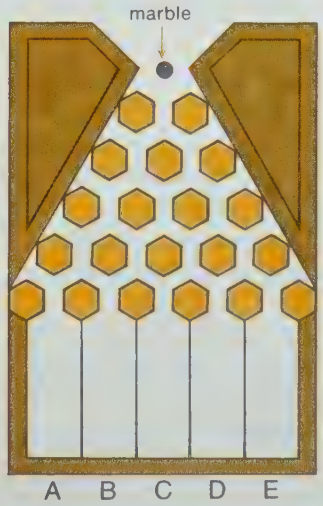


★ **4.** A student made this device for a probability experiment. A marble is dropped in the slot at the top. The probability of falling to the left or to the right of each hexagon is $\frac{1}{2}$.

A Find the probability of a marble reaching each of the slots A, B, C, D, and E.

B Suppose 80 marbles are dropped in the slot. About how many marbles would you expect to fall in each slot?

C Make a histogram of the theoretical frequency of marbles in each slot.

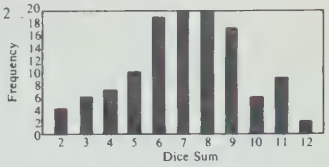


UTILIZATION

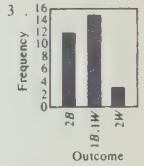
Exercise 1 involves interpretation of a histogram. Exercises 2, 3, and 4 require the construction of histograms.

Solution, Exercises 2, 3, 4

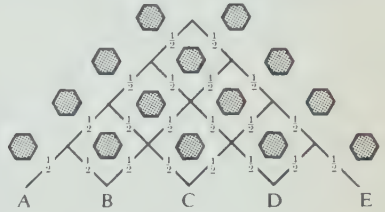
2.



3.



4A. A probability tree can be used to solve this exercise. Each branch has probability $\frac{1}{2}$.



There is 1 path to slot A. $P(\text{A}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = (\frac{1}{2})^4 = \frac{1}{16}$

There are 4 paths to slot B. $P(\text{B}) = 4 \cdot (\frac{1}{2})^4 = \frac{1}{4}$

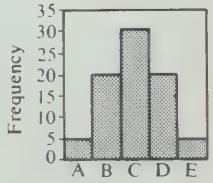
There are 6 paths to slot C. $P(\text{C}) = 6 \cdot (\frac{1}{2})^4 = \frac{3}{8}$

There are 4 paths to slot D. $P(\text{D}) = 4 \cdot (\frac{1}{2})^4 = \frac{1}{4}$

There is 1 path to slot E. $P(\text{E}) = (\frac{1}{2})^4 = \frac{1}{16}$

B. A: $\frac{1}{16} \cdot 80 = 5$ B: $\frac{1}{4} \cdot 80 = 20$
C: $\frac{3}{8} \cdot 80 = 30$ D: $\frac{1}{4} \cdot 80 = 20$
E: $\frac{1}{16} \cdot 80 = 5$

C.



EXTENSION

The exercises on *Workbook* page 110 and *Duplicator Masters* page 74 are appropriate for further practice assignments.

Enrichment: The ideas of Exercises 1, 2, and 3 could be used as the bases for class activities.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objectives

The student can interpret data given as a line-segment graph.

The student can construct a line-segment graph, given an appropriate set of data.

PREPARATION

Materials: Graph paper (*Duplicator Masters*, page 83).

Most students will have had some experience with line-segment graphs. If possible, obtain some graphs of this type from newspapers or magazines and discuss with the students what the graphs show.

INVESTIGATION

The Investigation requires interpretation of a set of data and an experience in making an *extrapolation*—as inference beyond the range of the given data.

Most students will note a gradual slowing down of the plants' growth. Thus a growth of 3 to 5 mm might be a reasonable estimate for the seventh week. Of course, one might also conclude that if growing conditions are altered the plant might grow considerably more or very little growth might occur.

DISCUSSION

Use Exercise 2 to illustrate the construction of a line-segment graph. Such graphs are useful where continuous data such as plant growth, temperature, masses, and so on, are involved.

The estimates for Exercise 3 will vary, but if the graph is projected to 8 weeks it would appear that it would reach about 80 mm.

Showing Data with Line-segment Graphs

Investigating the Ideas

A group of students planted some bean seeds for a science project. They kept a record of the plants' growth. The chart shows the record of one plant's growth by the week.

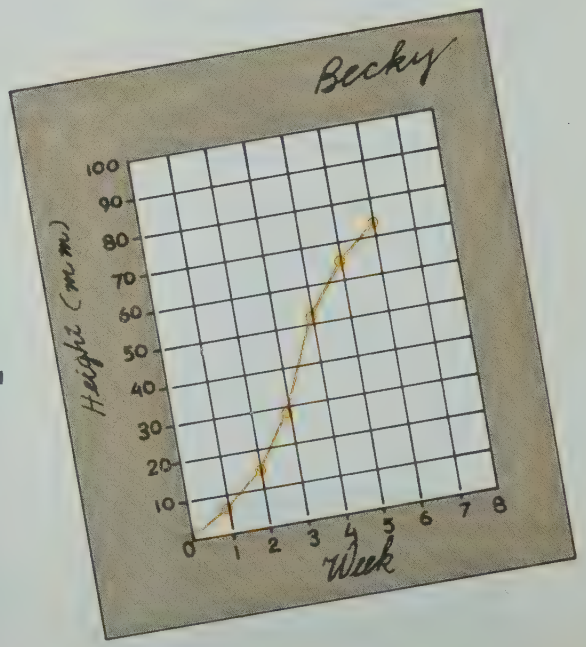
Week	1	2	3	4	5	6
Height (mm)	6	14	29	52	66	72

Can you estimate the height of the plant at the end of seven weeks?

Discussing the Ideas

1. Explain how you made your estimate of the height of the plant after seven weeks of growth.

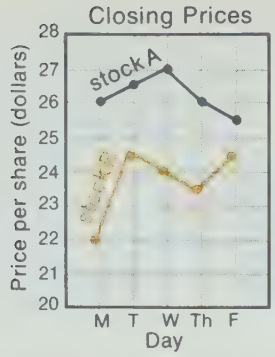
2. Becky made this **line-segment** graph to show the plant's growth for each week.
   - **A** Use the graph to tell in which week the plant grew the most.
   - **B** In which week did the plant grow the least?
   - **C** Assumimg growing conditions remain the same, how can you use the line-segment graph to estimate the plant's height at the end of seven weeks?



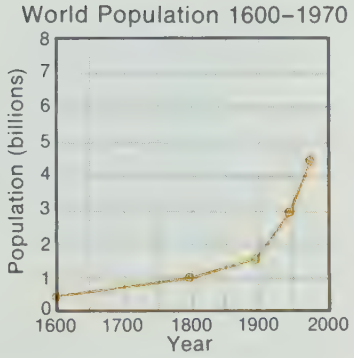
3. **A** What is your estimate of the plant's height at the end of eight weeks?
   - **B** Explain how you made your estimate.

Using the Ideas

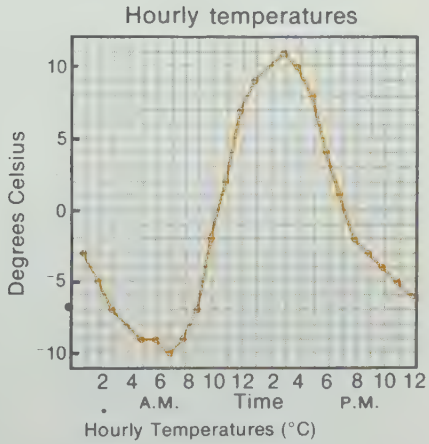
1. The line-segment graph shows the closing prices of two stocks for one week.
   - **A** What were the closing prices for stocks A and B on Monday?
   - **B** What was the highest closing price for each stock during the week?
   - **C** What was the lowest closing price for each stock during the week?
   - **D** Give the range of the closing prices for each stock during the week.



2. The line-segment graph shows the estimated world population from 1600 to the present.
   - **A** Give an estimate of the world populations for 1600, 1700, 1800, 1900, and 1950.
   - **B** The latest estimate shown on the graph is for the year 1970. What is the population for that year?
   - **C** What is your estimate of the world population for the year 2000?



3. The line-segment graph shows the hourly temperatures for a certain day.
   - **A** What was the coldest temperature during the day? At what hour was this temperature recorded?
   - **B** What was the highest temperature?
   - **C** What is the difference between the highest and the lowest temperature?
   - **D** How many degrees did the temperature fall between 3 P.M. and midnight?



4. Make a line-segment graph from the information in the table.

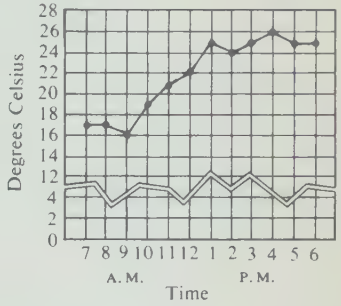
Hourly Temperatures (°C)

7 A.M.	17	10 A.M.	19	1 P.M.	25	4 P.M.	26
8 A.M.	17	11 A.M.	21	2 P.M.	24	5 P.M.	25
9 A.M.	16	Noon	22	3 P.M.	25	6 P.M.	25

UTILIZATION

Exercises 1, 2, and 3 involve interpretation of graphs. Exercise 4 requires construction of a line-segment graph.

Solution, Exercise 4



EXTENSION

Further practice assignments may be made from *Workbook* page 111 and *Duplicator Masters* page 75.

Enrichment: The Investigation theme could be used as a research project for students who are interested in botany. Records of plant growth under various growing conditions could provide some interesting data that could be depicted graphically. A biology or science teacher may be able to provide valuable suggestions and assistance in developing such a project.

Assignments

Minimum 1, 2. Average 1, 2, 4.
Maximum 1–4.

Objectives

The student can interpret data depicted by means of a circle graph.

Given appropriate data, the student can construct a circle graph.

PREPARATION

Materials: Protractors, compasses, straightedges.

If possible, obtain some circle graphs from newspapers or magazines and discuss how they show quantities by means of sectors whose areas are proportional to the quantities represented. You may also want to review some ideas of fraction and percent.

INVESTIGATION

Have the students record their menu selections. These can be used in the Discussion stage. There are several choices that will cost exactly $1.50. Sample Answers:

Hamburger	0.35	Hot dog	0.40
Fishwich	0.50	French fries	0.35
French fries	0.35	Malt	0.35
Milk shake	0.30	2 coffees	0.40
	$1.50		$1.50

DISCUSSION

When discussing circle graphs you will need to emphasize two things:

1. The complete circular region represents a fixed number n.
2. If k represents a part of n, then the central angle of the sector for this part is

$$\frac{k}{n} \cdot 360°.$$

When discussing Exercise 4, you can have students construct a circle graph to represent their choices, or you might use one of the students' selections as an example.

In Exercise 5 the percent for each sector can be found by writing the number of hours for each sector over 24 hours and then giving a percent for this fraction. Each percent multiplied by 360° will give the measure of the central angle for the sector.

Circle Graphs

Investigating the Ideas

Suppose you have $1.50 to spend at the Drive-Inn Diner.

Can you choose several different items and spend exactly $1.50?

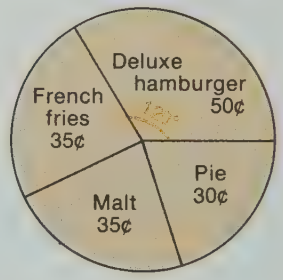
Drive-Inn Menu

Hamburger	35¢	Cola	20¢
Deluxe hamburger	50¢	Root beer	20¢
Fish-wich	50¢	Milk shake	30¢
Ham sandwich	60¢	Malt	35¢
Hot dog	40¢	Pie	30¢
French fries	35¢	Coffee	20¢
Onion rings	45¢	Milk	20¢

Discussing the Ideas

1. What fractional part of the $1.50 did you spend for each item?

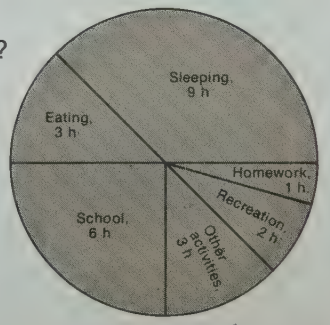
2. The **circle graph** at the right shows one way you could spend $1.50. The measure of the central angle for the sector of the circular region for the deluxe hamburger is 120° because 50¢ is $\frac{1}{3}$ of the total amount spent and $\frac{1}{3} \cdot 360° = 120°$.
   What are the measures of the central angles for each of the other items in the graph?



3. Suppose one of your items was a hot dog. Then the central angle for the item should be $\frac{40}{150} \cdot 360°$. How many degrees is this?

4. What would be the measure of the central angle of a circle graph for each of your choices from the menu? Explain or show how to make a circle graph of your selections from the menu.

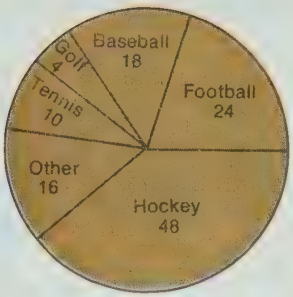
5. The circle graph shows the hours spent in daily activities.
   - A What percent of each day (24 hours) is spent in each activity?
   - B What is the measure of the central angle of the sector for each activity?



Hours spent in daily activities

Using the Ideas

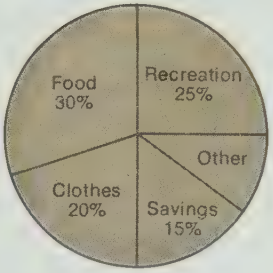
1. A survey of 120 students was made to find out which sports were their favorites. The circle graph at the right shows the number of students who selected a particular sport as their favorite.



   A What fractional part of the circular region is represented by each sport?

   B Express each fractional part as a percent.

2. The circle graph shows how a student accounted for his earnings of $12 for one week.



   A How many dollars were spent on recreation?

   B How many dollars were spent for clothes?

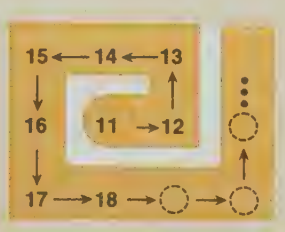
   C What percent is allotted to "other"? How much money is this?

3. In a school of 840 students, 52% come to school by bus, 28% walk, 12% ride bicycles, and the rest of them come by automobile. Find the number of students that come by each mode of transportation.

4. Make a circle graph of the data in Exercise 3.

think

A Begin with 11 and write consecutive whole numbers in a spiral (as in the figure) until you obtain a square with a prime number in each corner. How large is the square?



B Can you find a line of prime numbers in the square?

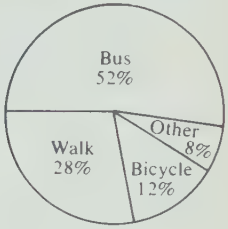
C Make the square larger. What interesting patterns can you notice?

More practice, page S-32, Set 61

UTILIZATION

If Exercise 4 is assigned, students will need to have protractors available. The primary objective for this lesson is interpretation of circle graphs. Construction of circle graphs should be a secondary objective.

Solution, Exercise 4

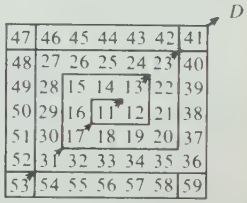


EXTENSION

Appropriate additional practice exercises may be assigned from Supplementary Exercise Set 61 on page S-32, *Workbook* page 112, and *Duplicator Masters* page 76.

Think Solution

A. A 7 by 7 square as shown below.



B. The diagonal labelled D is formed exclusively of prime numbers. Each of these numbers is of the form $n^2 - n + 11$, where $n = 1, 2, 3, \ldots$.

C. If the square is made larger the smallest number on the diagonal that is not a prime is 121 or 11^2.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

Given a sample of a large set of objects, the student can make inferences concerning the large set.

PREPARATION

Materials: Newspapers, paper clips.

Students will probably be familiar with public opinion polls that are published in newspapers. Discuss the fact that the polltakers contact a small portion (sample) of the total population concerning their opinions and then, from this sample, make inferences concerning the population.

INVESTIGATION

Students can work in pairs on the Investigation, making the tosses and completing a tally chart like the one shown. If the paper clip lands on a picture or off the paper, they should disregard the toss. Many students will expect that E or one of the other vowels is likely to get 20 tallies first.

DISCUSSION

You can generate extra interest by combining the results of all the students' tallies for the Investigation. In discussing Exercise 4 inferences might be as follows: high frequencies for letters like E, T, A, O, I, N, in the paper; low frequencies for letters such as J, X, and Z.

A sample from a population is called a *random sample* if it was obtained by a process that enabled it to have the same chance of being drawn from the total population as any other sample of that same size.

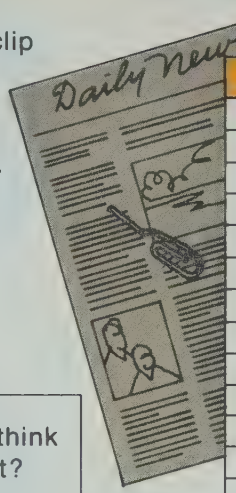
Sampling

Investigating the Ideas

Open one end of a paper clip like this:

Toss the paper clip on the front page of a newspaper. Note the letter nearest to the point of the paper clip and tally the letter. Repeat this experiment until some letter is tallied 20 times.

Which letter do you think will get 20 tallies first?



Letter	Tally	Letter	Tally
A	///	N	/
B		O	
C	/	P	/
D	/	Q	/
E	////	R	//
F	/	S	//
G		T	~~////~~
H	//	U	
I	/	V	
J		W	
K		X	
L	/	Y	
M	/	Z	

Discussing the Ideas

1. Which letter had 20 tallies first? Was it the letter you expected?

2. Which letter got the second most tallies?

3. If you carried out the experiment until some letter got 40 tallies, do you think you would have got a different letter than you did? Try it.

4. The letters that you tallied are a **sample** of all the letters on the newspaper page. By examining a sample of the letters, we may **infer** or conclude some things about all the letters on the page (the **population**). What things do you think you can conclude about the letters on the page, based on your sample?

5. Samples must be chosen at **random** from the population. Otherwise, a **biased sample** may result and inferences or predictions made from such samples may be unreliable. Can you describe some things that might make a sample for your experiment a biased sample?

Using the Ideas

1. In a sample of 362 letters from a newspaper, the ten most frequently occurring letters were found with the frequencies shown in the table.

Letter	Frequency	Letter	Frequency
E	50	N	27
T	34	A	26
R	32	S	25
O	29	D	15
I	28	H	12

- **A** What percent of the letters were E's?
- **B** Suppose the page had about 115 000 letters. About how many letters would be E's?
- **C** About how many of the 115 000 letters would be T's?

2. Suppose you wish to take a survey of the students in your school to see which sport is their favorite. Tell whether each sample below is likely to be biased or unbiased.

- **A** The set of boys in your class.
- **B** Every 5th student passing by a certain corner in the hall.
- **C** The first ten students to leave the school building at the end of the day.
- **D** The set of girls who play violins in the school orchestra.
- **E** Every fourth student who holds a season ticket to hockey games.
- **F** Every 20th student taken from a list of all students arranged alphabetically.

3. Suppose it is expected that 10 000 000 people will vote in a national election. A professional poll samples 5000 voters a week before the election. The table at the right gives the number of people in the sample who planned to vote for each of the three candidates.

Candidate	Number
A	2380
B	1900
C	120
Undecided	600

- **A** On the basis of this information, would you say that candidate *A* will probably win the election?
- **B** Candidate *B* concedes that candidate *A* may get as many votes as predicted by the poll taker, but still claims that he will win the election. Can you find some reasons that might make his claim true?
- **C** Do you think it is very likely that candidate *C* will win the election?
- **D** Assuming that the sample is a fairly accurate indicator of voter preference, give the approximate minimum and maximum number of votes each of the three candidates can expect to receive.

UTILIZATION

The exercises lend themselves to either independent work or class discussion. Part C of Exercise 2 may lead to particularly lively discussion, since student responses will almost certainly vary. Urge a number of students to explain their reasoning in determining whether the given sample would be biased or unbiased.

Answers, Exercise 3

3A. On the basis of the poll, candidate *A* is a slight favorite.

B. Candidate *B* might be assuming that the bulk of the undecided voters will swing to him and thereby give him enough votes to win the election. Note that $1900 + 600 = 2500 > 2380$.

C. Even if all the undecided voters swing to candidate *C*, he will lack the necessary votes to win. His only hope will be that a large number of voters for *A* or *B* can be persuaded to vote for him.

D.

Candidate	Minimum	Maximum
A	33 320 000	41 720 000
B	26 600 000	40 000 000
C	1 680 000	10 080 000

The maximum number of votes is based on the event that all undecided voters should give their votes to the candidate. The minimum number is based on the proportion of the total vote as given by the poll excluding the undecided voters.

EXTENSION

The exercises on page 113 of the *Workbook* would be appropriate for further practice.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–3.

Objective

The student can solve some special problems based on samples of cards.

PREPARATION

Materials: Playing cards (one deck per group of students).

This lesson should be regarded as optional for most students. Since the Investigation will require considerable time to complete, have the students begin work on it without delay.

INVESTIGATION

You may expect that students will have some difficulty making estimates of the number of times they will get at least one ace in 50 4-card trials. For example, since there is 1 chance in 13 of drawing an ace on a single draw, some students may *incorrectly* reason that there are 4 chances in 13 of getting an ace when 4 cards are drawn.

Have the students work in pairs or small groups on this activity.

DISCUSSION

Exercises 2, 3, and 4 are structured so that students can understand how the probability of getting at least 1 ace in a 4-card sample is found. When the probability of getting 0 aces is found, the probability of getting at least 1 ace is 1 minus the probability of getting 0 aces, because the sum of the two probabilities must total 1.

Other Sampling Problems

Investigating the Ideas

Shuffle a deck of playing cards very well. Turn up the first four cards. Make a tally of the number of aces you turn up in the sample of four cards.

Number of Aces	Tally
0 aces	
1 ace	
2 aces	
3 aces	
4 aces	

Can you estimate about how many times you will get at least one ace in the sample of four if you try the experiment 50 times?

Discussing the Ideas

1. How well did your experimental results compare with your estimate?

2. Since there are four aces in a deck of 52 cards, the probability of **not** getting an ace on the first card turned up is $\frac{48}{52}$. The probability of **not** getting an ace on the second card turned up is $\frac{47}{51}$. Why?

3. What is the probability of not getting an ace on either of the first two cards turned up? (Multiply the probabilities.)

4. What is the probability of not getting an ace on the third card that is turned up? of not getting an ace on the fourth card?

5. The product of the four probabilities in Exercises 2 and 4 is the probability of getting no aces in a 4-card sample.

   $P(\text{no aces}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \approx 0.718$

   Therefore, the probability of getting at least one ace in a 4-card sample is

   $1 - P(\text{no aces}) = 1 - 0.718 \approx 0.282.$

   A Did you get at least one ace in 0.282 (about one fourth) of your trials of the experiments?

   B Combine your results with those of your classmates. How does your combined results compare with the theoretical probability of getting at least one ace?

Using the Ideas

1. The theoretical probability of drawing aces in a 4-card sample is shown in the table. Suppose you carried out the experiment 250 times.

OUTCOME	PROBABILITY
No aces	0.718
Exactly 1 ace	0.256
Exactly 2 aces	0.025
Exactly 3 aces	0.0007
Exactly 4 aces	0.000004

- **A** In how many of the 250 trials would you expect to get no aces?
- **B** In how many of the 250 trials would you expect to get one ace?
- **C** About how many times in 250 trials would you expect to get exactly two aces?
- **D** About how many times in 250 trials would you expect to get exactly three aces?
- **E** Are you very likely to draw all four aces in 250 trials? Explain.

2. In a room of 52 people, four have red hair. If four of the people's names are selected at random to win a door prize, what is the probability that at least one of the red-headed persons will win a prize?

3. A sack contained a mixture of 1200 brown and white beans. Two students performed a sampling experiment with the beans. They withdrew one bean from the sack without looking in the sack, tallied its color, and then put it back into the sack. This experiment was repeated 100 times.

BROWN AND WHITE BEANS

OUTCOME	FREQUENCY
Brown	73
White	27

- **A** About what percent of the 1200 beans were brown?
- **B** About how many of all beans were brown?
- **C** About how many of all beans were white?

★ **4.** Carry out an experiment like the one suggested in Exercise 3. Compare the experimental results with the theoretical probability.

UTILIZATION

Some observant students may note that the sum of the probabilities in the table for Exercise 1 is not exactly 1. The small discrepancy is due to errors in rounding for the probabilities given. The exact probabilities are as follows:

$$P(0 \text{ aces}) = \frac{194\,580}{270\,725}$$
$$P(\text{exactly 1 ace}) = \frac{69\,184}{270\,725}$$
$$P(\text{exactly 2 aces}) = \frac{6768}{270\,725}$$
$$P(\text{exactly 3 aces}) = \frac{192}{270\,725}$$
$$P(\text{exactly 4 aces}) = \frac{1}{270\,725}$$

Solution, Exercise 2

This problem can be related to the problem of drawing at least 1 ace in a sample of 4. The probability that at least one red-headed person will win a prize is about 0.282.

Solution, Exercise 3

On the basis of this sample, students would expect that 73% of the beans are brown and 27% are white. Point out that in a practical case, one would obtain other samples to aid in estimating the percent of each color of bean.

EXTENSION

Extra practice exercises are provided on page 77 of the *Duplicator Masters*.

Enrichment: Several experiments of the type suggested in Exercise 3 could be designed, varying the proportion of the two colors of beans. After the students have carried out the experiment, the experimental percentages of colors can be compared to the actual percentages.

Assignments

Minimum 1, 2. Average 1–3. Maximum 1–4.

Objective

The student can carry out a sampling experiment and draw conclusions based on the experiment.

UTILIZATION

Exercises 1 through 6 on this page will require choosing a sample of students and conducting a survey. The size of the samples can be adjusted so that all work can be confined to the classroom if so desired. Broader samples can be used if students can select students outside the immediate class.

Choosing an unbiased sample is a difficult problem and the student should be made aware of the fact that if a sample is poorly chosen no amount of analysis of the sample will produce good inferences from the sample.

The exercises provide highly simplified problems involving sampling. It is not the objective of the lesson to teach refined sampling techniques. Rather, the recognition of the difficulties involved in choosing unbiased samples is more closely related to the aim of the lesson.

This would be a good time for students to work in small groups to obtain the information for the surveys.

EXTENSION

Enrichment: Now would be an opportune time to encourage some students to explore Research Project B on page H–49.

Assignments

Minimum One problem from 1 through 6.
Average Two problems from 1 through 6.
Maximum Three problems from 1 through 7.

SOME SAMPLING EXPERIMENTS

1. Choose a sample of at least 50 students in your school. Find out the favorite kind of television program of each person. What conclusions can you draw from your sample?

Kind of Program	Tally
Musical Variety	𝍸 𝍸
Live Sports	//
Science Fiction	𝍸 ///
Mysteries	𝍸 𝍸 /

2. Choose a sample of 20 girls. Have each girl name her favorite school subject. What conclusions can you draw from your sample?

3. Choose a sample of 20 boys. Have each name the school subject he likes best. Compare the results with those in Exercise 2.

4. Conduct a survey of the students in your class on the question: "How many hours a week do you watch television?"

Make a record, such as the one shown, of their responses.

Hours	Number of Students
0–3	
4–7	
8–11	
12–15	
16–18	
Over 18	

5. Conduct a survey to determine the number and kind of pets that the students in your school have at home.

6. Conduct a survey to find out what kind of job, business, or profession that the students would like to pursue after they become adults.

7. Mark off a square decimetre on your school grounds or lawn. Count, as carefully as possible, the number of blades of grass in the square decimetre.

- **A** About how many blades of grass in one square decimetre?
- **B** Use your sample to estimate the number of blades of grass per square metre.
- **C** Find the area of the lawn. Estimate the number of blades of grass in the lawn.

The BIRTHDAY Problem

The names and birthdates of 25 well-known persons are given in the list at the right.

Osborne, John; London, Eng., 12/12/1929.
O'Toole, Peter; Connemara, Ireland, 8/2/1933.
Owens, Jesse; Decatur, Ala., 9/12/1913.
Paar, Jack; Canton, Ohio, 5/1/1918.
Page, Geraldine; Kirksville, Mo., 11/22/1924.
Page, Patti (Clara Ann Fowler); Claremore, Okla. 11/8/1927.
Palance, Jack; Latimer, Pa., 2/18/1920.
Paley, William S.; Chicago, Ill., 9/28/1901.
Palmer, Arnold; Youngstown, Pa., 9/10/1929.
Palmer, Lilli; Posen, Germany, 5/27/1917.
Parker, Eleanor; Cedarville, Ohio, 6/26/1922.
Parkinson, C. (Cyril) Northcote; Durham, Eng. 7/30/1909.
Pasternak, Joseph; Silagy-Somlyo, Rum., 9/19/1901.
Patterson, Floyd; Waco, N. C., 1/4/1935.
Paul VI (Giovanni Battista Montini); Concesio, It., 9/26/1897.
Pauling, Linus Carl; Portland, Ore., 2/28/1901.
Peale, Norman Vincent; Bowersville, Ohio, 5/31/1898.
Pearson, Drew; Evanston, Ill., 12/13/1897.
Pearson, Lester B.; Toronto, Ont., Can., 4/23/1897.
Peck, Gregory; La Jolla, Calif., 4/5/1916.
Peerce, Jan; New York City, 1904.
Pegler, Westbrook; Minneapolis, Minn., 8/2/1894.
Perelman, S. J. (Sidney J.); Brooklyn, N. Y., 2/1/1904.
Perkins, Millie; New Jersey, 5/12/1940.
Perkins, Tony (Anthony); New York City, 4/14/1932.

SOURCE: Information Please Almanac

1. Do you think it is very likely that any two of these have the same birthday? (Same month and day.)

2. Check through the list of names. Do any two persons have the same birthday?

3. Do any of the persons listed have the same birthday as someone in your class?

4. Do any two students in your class have the same birthday ?

5. Find a list of famous persons' birthdays. Choose 25 names in a row as a sample. See if two of the persons have the same birthday.

6. Choose several different samples of 25 names and repeat the experiment of Exercise 5. In how many of your samples did you find two people with the same birthday?

7. The probability that two people out of a group of people will have the same birthday is given in the table. Note that for 23 people, the probability of two people having the same birthday is a little more than $\frac{1}{2}$.

Number of People	Probability of at least two people with the same birthday
10	0.117
20	0.411
23	0.507
30	0.706
40	0.891
60	0.994

   - **A** Choose a sample of 40 famous people from a list and find their birthdays. How many of them have the same birthday?
   - **B** What is the probability that two of them will have the same birthday?

8. Find a list of presidents, prime ministers, or heads of state and their birthdays. Select some samples of 25 persons. Find which sample contains two persons with the same birthday.

Objective

The student can solve problems relative to two persons having the same birthday.

PREPARATION

Materials: Almanacs or other resource books which list birthdays for prominent persons by day, month, and year.

UTILIZATION

This set of exercises deals with a well-known problem in probability—the probability of two people having the same birthday. To someone who is unfamiliar with the problem, the results are surprising. The problem can be stated as follows:

> Suppose there are n people in a room. What is the probability that at least 2 of the people have the same birthday, that is, their birthdays on the same day and month of the year? If E is the event "No two of the n people have the same birthday" it can be shown that
> $$P(E) = \frac{365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365 - n + 1)}{365^n}$$
> Then P (at least 2 birthdays are the same) $= 1 - P(E)$.

The table in Exercise 7 gives some specific probabilities for various values of n. Note that for 23 people the probability that at least two people have the same birthday is slightly more than $\frac{1}{2}$.

Since most classrooms contain at least 23 students, it is probable that there are two students who have the same birthday in the room.

If Exercises 5, 6, and 8 are assigned, students will need resource material that gives birthdays of famous people.

Assignments

Minimum 1–4. Average 1–7. Maximum 1–8.

Objective

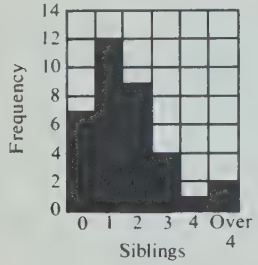
The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

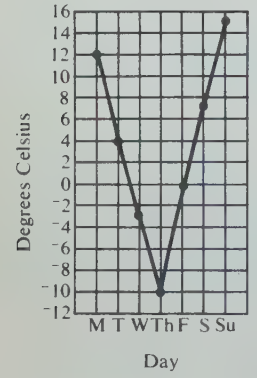
Assign the exercises as independent work. Graph paper will be needed for Exercises 4 and 6. After completing the exercises, a discussion of the exercises would be beneficial to most students.

Additional review exercises are provided on page 114 of the *Workbook*.

Solution, Exercise 4



Solution, Exercise 6



REVIEWING THE IDEAS

1. Georgia earned $10.25, $16.50, and $12.25 in three weeks of part-time work. What was her average (arithmetic mean) earnings per week?

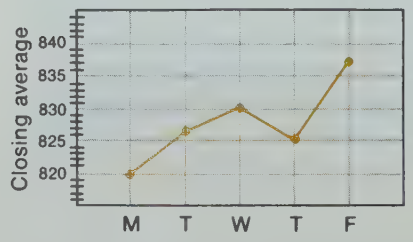
2. Linda had scores on mathematics tests of 89, 83, 94, 81, 78, 85, 90, 96, 88, and 90.
   - A What is the range of the scores?
   - B Find the mean of the scores.
   - C What is the median score?

3. The tally shows the results of a class survey of the number of brothers and/or sisters of each student.
   - A Which number occurred most frequently?
   - B What is the mode of the set of data?

Over 4	𝍸 ||
4	|
3	𝍸
2	𝍸 𝍸
1	𝍸 𝍸 ||
0	𝍸 ||

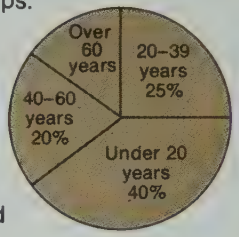
4. Make a histogram of the data given in the table in Exercise 3.

5. The line-segment graph below shows closing averages of the Dow-Jones Industrials. Give the closing average for each day to the nearest whole number.



6. Construct a line-segment graph of the following daily low temperatures: Monday, 12°; Tuesday, 4°; Wednesday, $^{-}3°$; Thursday, $^{-}10°$; Friday, 0°; Saturday, 7°; Sunday, 15°.

7. The circle graph shows the percent of the population of a city in various age groups.
   - A What percent is over 60 years old?
   - B What is the angle of the sector for the 40–60 year old group?



8. Suppose the population of the city in Exercise 7 is 30 000. Determine the approximate number of people in each group.

9. Suppose you ask three of your best friends how they plan to vote in an election for class president. You find that all four of you plan to vote for the same person.
   - A Can you conclude that your candidate will probably be elected?
   - B Is your sample of three friends a random sample of students in your class?

10. In a city of 1 000 000 registered voters, a professional poll of 1000 voters showed that candidate *A* would get 450 votes compared to 325 for candidate *B*. The remaining voters were undecided. If conditions remain as indicated by the poll, what is the maximum number of votes candidate *A* should expect to receive?

11. From the results of several samples, Jeff estimated that about one out of every 15 students was left-handed. If there are 880 students in his school, about how many are left-handed?

TEST YOURSELF

1. In bowling three games, Peggy had scores of 110, 136, and 141. What was her average score?

2. Jack had bowling scores of 108, 124, and 149. What was his average score?

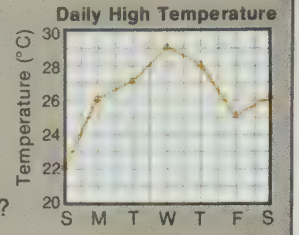
3. Jan's test scores for five tests were 83, 81, 86, 94, and 92. What was the median of these scores?

4. Give the frequency of occurrences of each letter in the word MISSISSAUGA.

5. The price of a certain brand and size of soap flakes was checked in ten different stores. The table shows the results.

Brand	Cost	Brand	Cost
A	72¢	F	78¢
B	69¢	G	72¢
C	66¢	H	69¢
D	72¢	I	68¢
E	69¢	J	69¢

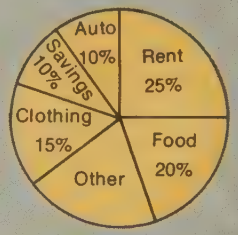
What is the mode (or modes) of the prices?

6. 

   A What is the highest temperature of the week?

   B What is the range of temperatures?

The circle graph shows how Leslie divides her monthly earnings of $800.



7. How much does she spend for food in a month?

8. How much does she spend for clothing?

9. A manufacturer determines from samples that about 3 out of every 800 parts are defective. About how many parts would be expected to be defective out of 100 000?

10. The probability of drawing an ace from a deck of cards is $\frac{1}{13}$. About how many aces would you expect to draw in 500 trials?

RESEARCH PROJECTS

A Carry out a project to determine the "average" person in your class. Use heights, masses, age, waist measurements, foot size, and so on. Determine the mean, median, and mode of each set of data.

B Conduct a survey of the students in your school concerning preferences. Ask questions such as "What is your favorite automobile?" "Who is your favorite TV star?" or "What is your favorite meal?" Select some graphic method of displaying the data. Try to draw conclusions or make predictions from your survey.

TEST YOURSELF

The self-evaluation test should enable students to assess their grasp of the major concepts presented in this module. Encourage students to use the key on page H–50 to check their own answers and then try to correct their mistakes by reviewing the related lessons of the module.

RESEARCH PROJECTS

Some students might use sampling to carry out Research Project A. For example, they could find the average height of the students in the class by selecting a random sample of 5 of their classmates. Several samples of 5 could be taken and the results could be compared. Finally, they could find the actual average height of the group and compare it with the sample averages.

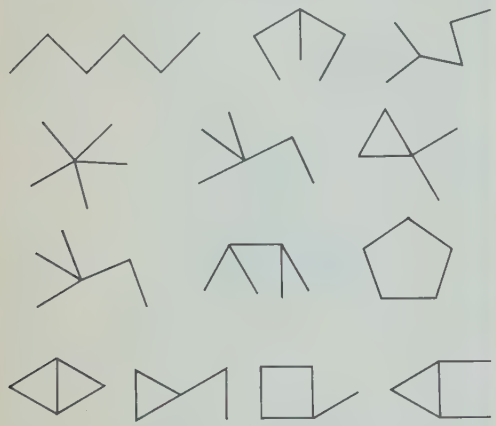
The idea of political polls, television ratings, and various other kinds of surveys are likely to be familiar to many of your students. Try to capitalize on their interest in these things by encouraging them to explore Project B. For example, the popularity rating of a television program is of great concern to all those connected with the program, but how these ratings are derived is not known or understood by many persons. A typical rating might be based upon a survey of, say, 1200 families who represent the nearly 60 million television households. In addition, the survey company might employ supplementary data based upon periodic polls of a greater number of television-viewing families. A committee of interested students could be assigned the project of finding specific information concerning how such ratings are determined.

MATHEMATICAL RECREATION

Students will need toothpicks, matches, or similar objects for this activity.

This mathematical recreation is based upon the topic of *planar graph theory*. Each configuration of toothpicks represents a special planar graph composed of congruent segments in which the only points of intersection are at the endpoints. It is important to note that the main concern is the different ways the toothpicks (segments) are connected rather than the shape of the graph.

The twelve graphs possible with 5 toothpicks are shown below:



Six toothpicks can produce 28 distinct graphs. Seven toothpicks can produce 74 graphs. Eight toothpicks can produce 207 graphs.

Some students might be encouraged to tackle the problem of 6 toothpicks. The toothpicks could be glued or taped to posterboard to form an attractive display.

MATHEMATICAL RECREATIONS

TOOTHPICK GRAPHS

If you use three toothpicks, you can connect them at their ends in only three ways.

A B C

Each figure is called a **simple graph**. Two different-looking figures represent the same simple graph if they are connected in the same way. Thus, graph A above could appear in forms such as those below.

Graph B might appear as these shapes:

There is only one possible shape for graph C.

If you use four toothpicks, you can form only five essentially different simple graphs.

D E F G H

Again different shapes represent the same graph. Thus, graph E might have these shapes:

How many different simple graphs can you make using five toothpicks? Record each graph you find.

RECREATIONS MATHEMATICAL

TEST YOURSELF Answers

1. 129 **2.** 127 **3.** 86 **4.** M: 1; I: 2; S: 4; A: 2; U: 1; G: 1
5. 69 **6.** A 29°C; B 7° **7.** $160 **8.** $120
9. About 375 **10.** About 38

UNIT H: SOME SPECIAL TOPICS IN MATHEMATICS

Module 3: Real Numbers

General Objectives

To introduce irrational numbers.

To define the set of real numbers as the union of the set of rational numbers and the set of irrational numbers.

To introduce square roots of positive numbers.

To provide experiences in finding rational approximations for irrational numbers, particularly square roots of numbers.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
118	**RED** Given a number n, the student can understand the meaning of the square root ($\sqrt{n}$) of the number.	H-52, 53	**WB** 115	
119	**RED** The student can use the Pythagorean Theorem to locate the positive and the negative square root of a number on the real number line.	H-54, 55	**WB** 116	
120	**RED** Given a square root of a number in the form $\sqrt{n}$, the student can find a decimal approximation for it.	H-56 through 61	**WB** 117-9 **DM** 78	**ASC** D-15
121	**RED** Given distance problems involving right triangles, the student can use the Pythagorean Theorem and knowledge of square roots to solve the problems.	H-62, 63	**DM** 79	**DS** f-4
122	**YELLOW** Given a fraction, the student can give a repeating decimal for the fraction and vice-versa.	H-64, 65		
123	**YELLOW** Given appropriate examples, the student can identify decimals representing either rational or irrational numbers.	H-66, 67	**WB** 120	
124	***GREEN*** *The student can identify the set of real numbers.*	H-68, 69	**WB** 121	
125	***GREEN*** *Given two real numbers expressed as square roots, the student can find the product of the numbers:* $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$.	H-68, 69	**WB** 121	
	Reviewing the Ideas	H-70	**WB** 122	

MATHEMATICS

The Pythagorean Theorem plays an important role in this module. It serves to introduce the student to square roots of nonsquare numbers, it aids in locating co-ordinates of points on the real number line, and it is applied in practical problems involving sides of right triangles.

The first two lessons introduce the students to irrational numbers in the form of square roots of numbers. Although no irrational number is rational, each can be closely approximated by rational numbers. Approximations of square roots are found by a flow chart algorithm or by use of a table of square roots. Stress is placed on the fact that each repeating decimal represents some rational number, and vice versa. This leads naturally to the idea that nonrepeating decimals represent irrational numbers. Finally, the set of real numbers is defined as the union of the set of all rational numbers and the set of irrational numbers.

This module can be viewed as a milestone in the development of the *School Mathematics* series. Up to this point, all the work has been with rational numbers. This extension of the number system to include the irrationals is the last major extension the student will meet until

complex numbers are introduced in high school algebra courses.

TEACHING THE MODULE

Vocabulary

cube root	radical sign
decimal approximation	rational number
interpolation	real number
irrational number	real number line
nonrepeating decimal	repeating decimal
Pythagorean Theorem	square root

Although this module's overall objective is to introduce the irrational numbers and define the set of real numbers, other specific objectives include developing concepts of estimation for irrational numbers, working with square roots, and learning to read and use the square root table. Some of the material in this module is quite abstract and will require the best efforts of your most able students. However, all of your students will be able to understand some of the important concepts; and they should be encouraged to participate in the Investigations and Discussions. Knowing that slower students will have difficulty with the more abstract ideas, take special care in making assignments for these students.

At this stage, students are not prepared to do extensive work with irrational numbers, but they can learn how the set of real numbers is defined and the basic characteristics of irrational numbers. Development of the number line is completed in this module by including on it the points for the irrationals. It may surprise students to learn that prior to the introduction of the irrationals, the number line contained gaps, where points on it were not represented by rational numbers. Now these gaps can be filled.

Development of great skill in work with irrational numbers and decimal approximations is not a practical objective for slower students.

Lesson Schedule

At least ten days should be allotted for complete coverage of this module. However, because almost all the material will be new to your students, it may be necessary to spend more than one class period on a given lesson and your schedule should allow for this contingency.

Evaluation

In evaluating your students, you should refer to the general objectives of the module. You should not expect a great amount of skill from students in working with irrational numbers in the form of square roots. More important is students' understanding of concepts such as "What is the meaning of $\sqrt{2}$?" If the student can respond with "$\sqrt{2}$ represents the positive number whose square is 2," you may feel confident that the student has mastered a fairly abstract concept. Some specific skills in estimation of square roots and knowledge of the relation between rational, irrational, and real numbers are also desirable outcomes for the student who has studied this module.

RESOURCES FOR ACTIVE LEARNING

Activities

Experiments in Mathematics, Stage 2, "Square Roots" and "The Golden Triangle," pp. 17–21, Houghton Mifflin

Let's Play Games in Mathematics, Vol. 7, Activities 89–93, National Textbook Co.

Madison Project: *Explorations*, "Matrices," pp. 325–409 (TT), Addison-Wesley

Mathematics: Man's Key to Progress, Book A, pp. 41–52, Rand McNally

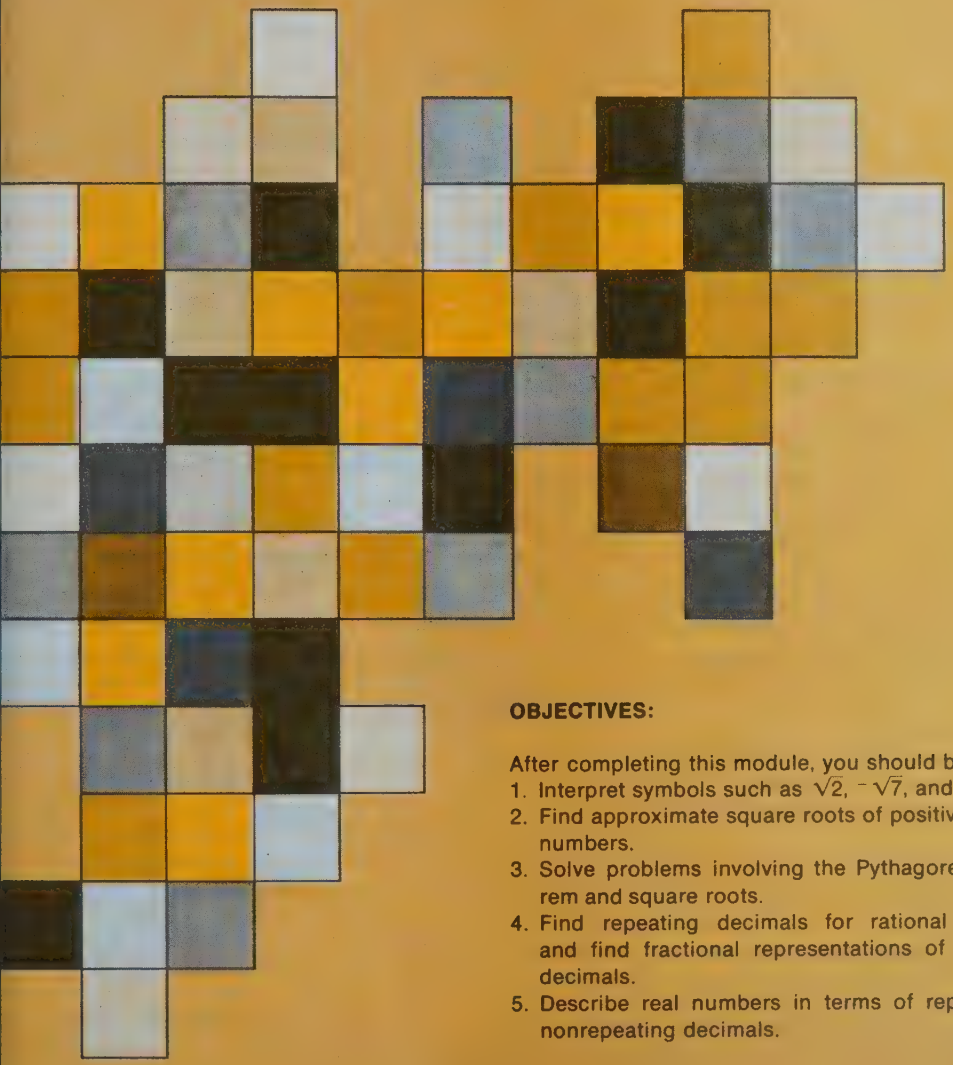
Powers, Roots, and Logarithms, Set 5E, Cards 18–30, Herder and Herder

Manipulative Devices

Square Root Charts (Geyer)

UNIT H: *Some Special Topics in Mathematics*

MODULE 3: *Real Numbers*



OBJECTIVES:

After completing this module, you should be able to:

1. Interpret symbols such as $\sqrt{2}$, $-\sqrt{7}$, and $\sqrt{11}$.
2. Find approximate square roots of positive rational numbers.
3. Solve problems involving the Pythagorean Theorem and square roots.
4. Find repeating decimals for rational numbers and find fractional representations of repeating decimals.
5. Describe real numbers in terms of repeating or nonrepeating decimals.

Objective

The student can demonstrate an understanding of the symbol $\sqrt{n}$ as representing the positive number whose square is n.

PREPARATION

Before beginning the lesson, play a guessing game with students by asking oral questions such as the following:

"I am thinking of a positive number which when squared is 16. What is the number?" (4)

"I am thinking of a negative number whose square is 100. What is the number?" ($^{-}10$)

"I am thinking of a number whose square is 36. What is the number?" (6 or $^{-}6$)

INVESTIGATION

Many students will never have thought about finding a number whose square is exactly 2 and will be surprised that they cannot readily find such a number. The number line picture suggests that the number is between 1.2 and 1.5. By a succession of trials, students will find that $(1.41)^2 = 1.9881$ and $(1.42)^2 = 2.0164$. Either number is within 0.02 of 2. However, this will enable students to realize that there is a rational number expressed in thousandths whose square may be even closer to 2.

DISCUSSION

Exercises 1, 2, and 3 are designed to help students see how increasingly close estimates can be made for a number whose square is 2.

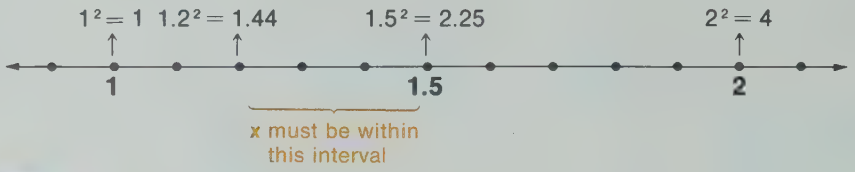
The Pythagorean Theorem is used in Exercise 4 to give the students an appreciation of the need for a number whose square is 2. Otherwise, there would be a "hole" in the number line at the distance x from the origin where $x^2 = 2$.

Emphasize the definition given in Exercise 5: $\sqrt{2}$ represents the *positive* number whose square is 2. Since the product of two negative numbers is a positive number, we can also expect to find a negative number whose square is 2. This negative number is denoted by the symbol $^{-}\sqrt{2}$, and is read "negative square root of 2."

The Square Root of Two

Investigating the Ideas

Is there a number whose square is 2? That is, is there a number **x** such that $x^2 = 2$? Study the number line and the examples below.



Can you find a decimal whose square is a number within 0.02 of 2?

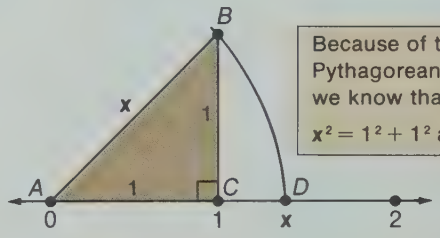
Discussing the Ideas

1. Is $(1.4)^2$ greater than 2 or less than 2?

2. Is $(1.45)^2$ greater than 2 or less than 2?

3. If $x^2 = 2$, is $1.4 < x < 1.45$?

4. The Pythagorean Theorem can help you mark a point on the number line for the number whose square is 2. Make a drawing like this on your paper. Make $\overline{AC}$ and $\overline{BC}$ each 1 decimetre long. Then measure the length of $\overline{AB}$ (distance **x**).



Because of the Pythagorean Theorem, we know that $x^2 = 1^2 + 1^2$ and $x^2 = 2$.

   A Is the length a number whose square is very nearly 2?

   B What is your estimate of the co-ordinate **x**?

5. We shall assume that there is a number that multiplies by itself to give 2. We make the following definition:

> The symbol represents the positive number which when multiplied by itself gives 2. We call this number **the positive square root of 2**.

   A Is there a negative number whose square is 2?

   B What symbol would you use for the negative number whose square is 2?

Using the Ideas

These exercises will help you become familiar with the use of the symbol $\sqrt{\ }$ (called a radical sign) used in writing numerals for square roots. Study the examples.

A $\sqrt{2}$ is the **positive** number which when multiplied by itself gives 2.

B $^{-}\sqrt{2}$ is the **negative** number which when multiplied by itself gives 2.

1. A ▒ is the positive number which when multiplied by itself gives **3**.
B ▒ is the negative number which when multiplied by itself gives **3**.
C ▒ is the positive number which when multiplied by itself gives **17**.
D ▒ is the negative number which when multiplied by itself gives **17**.
E $\sqrt{6}$ is the ? number which when multiplied by itself gives ▒.
F $^{-}\sqrt{6}$ is the ? number which when multiplied by itself gives ▒.
G $\sqrt{7}$ is the ? number which when multiplied by itself gives ▒.
H ▒ is the negative number which when multiplied by itself gives **47**.
★ I For a positive integer n, $\sqrt{n}$ is the positive number which when multiplied by itself gives ▒.

2. Using the definitions that you completed in Exercise 1, give an integer for n in each exercise.

A $\sqrt{2} \cdot \sqrt{2} = n$
B $\sqrt{5} \cdot \sqrt{5} = n$
C $(^{-}\sqrt{6}) \cdot (^{-}\sqrt{6}) = n$
D $(^{-}\sqrt{15}) \cdot (^{-}\sqrt{15}) = n$
E $\sqrt{100} \cdot \sqrt{100} = n$
F $(^{-}\sqrt{225}) \cdot (^{-}\sqrt{225}) = n$

3. Give a positive number that makes each equation true.

A $n \cdot n = 9$
B $n \cdot n = 7$
C $n \cdot n = 11$
D $n \cdot n = 16$
E $n^2 = 17$
F $n^2 = 19$
G $n^2 = 63$
H $n^2 = 64$

4. In the following sequence of square roots, those representing whole numbers are indicated. Use a radical sign to write symbols that represent the whole numbers 6, 7, 8, 9, and 10. EXAMPLE: $6 = \sqrt{36}$

1 ↓			2 ↓					3 ↓	
$\sqrt{1}$	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{4}$	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	$\sqrt{9}$	$\sqrt{10}$
					4 ↓				
$\sqrt{11}$	$\sqrt{12}$	$\sqrt{13}$	$\sqrt{14}$	$\sqrt{15}$	$\sqrt{16}$	$\sqrt{17}$	$\sqrt{18}$	$\sqrt{19}$	$\sqrt{20}$
				5 ↓					
$\sqrt{21}$	$\sqrt{22}$	$\sqrt{23}$	$\sqrt{24}$	$\sqrt{25}$	$\sqrt{26}$	$\sqrt{27}$	$\sqrt{28}$	$\sqrt{29}$	$\sqrt{30}$

UTILIZATION

If you assign the exercises as independent work, be certain to allow time for discussing them afterward. Note that other square roots are introduced in the exercises.

When discussing Exercise 4, point out that each symbol like $\sqrt{4}$ represents only one number, the positive square root. You should avoid writing statements like $\sqrt{4} = \pm 2$. According to our definition, $\sqrt{4} = 2$ and $^{-}\sqrt{4} = ^{-}2$. That is, $\sqrt{4}$ is the positive number whose square is 4, and $^{-}\sqrt{4}$ is the negative number whose square is 4.

EXTENSION

Additional practice exercises are provided on page 115 of the *Workbook*.

Enrichment: Challenge the students to compute as decimals to the nearest thousandth the following square roots. (Answers are given in parentheses.)

A. $\sqrt{2}$ (1.414)
B. $\sqrt{3}$ (1.732)
C. $\sqrt{10}$ (3.162)

Students should find these decimal approximations by squaring various decimals until the best approximation is found.

If hand-held computers or desk computers are available, you might increase the number of decimal places required.

Assignments

Minimum 1, 2, oral; 3, 4.
Average 1, oral; 2–4. Maximum 1–4.

Objective

The student can find irrational numbers for co-ordinates of points on the number line by using the Pythagorean Theorem.

PREPARATION

Review the meaning of the square root of a positive number, which was presented in the previous lesson.

INVESTIGATION

The purpose of the Investigation is for students to explore the possibility that $\sqrt{2}$ is a rational number. Since $(\frac{3}{2})^2 = \frac{9}{4} > 2$, $(\frac{7}{5})^2 = \frac{49}{25} < 2$ and $\frac{141}{100} = \frac{19\,881}{10\,000} < 2$, none of these rational numbers can be equal to $\sqrt{2}$. Of course, this does not prove that there is no rational number whose square is 2, but it suggests this fact.

DISCUSSION

When discussing Exercise 2, students are likely to be divided in their opinions of finding a rational number whose square is 2. You might suggest they try $\frac{99}{70}$ or $\frac{577}{408}$. $(\frac{99}{70})^2 = \frac{9801}{4900}$ and $(\frac{577}{408})^2 = \frac{332\,929}{166\,464}$. Note that the squares are very nearly, but not exactly, 2.

The proof that $\sqrt{2}$ is not rational is not given. If you wish to present a proof for your students, a simple proof based on the Fundamental Theorem of Arithmetic (Unique Factorization) is given in the Extension section of this lesson.

It would be well to put a diagram like the one in Exercise 3 on the chalkboard. Work through the exercise carefully. Use a compass to show that there are two numbers, one negative and one positive, whose square equals 5. Thus, if $x^2 = 5$ then $\sqrt{x^2} = \sqrt{5}$, hence $x = \sqrt{5}$ or $x = {}^-\sqrt{5}$. Note that this is in agreement with the formal definition that is sometimes given for the absolute value of a number: $|n| = \sqrt{n^2}$.

Irrational Numbers

Investigating the Ideas

In the previous lesson you learned that certain square roots ($\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, $\sqrt{25}$, $\sqrt{36}$, and so on) represent whole numbers. Do you think other square roots, such as $\sqrt{2}$, might represent *rational numbers*?

$\sqrt{2}$ is the positive number which when squared gives 2. If there were a positive *rational* number which when squared gave 2, it would be equal to $\sqrt{2}$.

Do any of these rational numbers equal $\sqrt{2}$?

A $\frac{3}{2}$ B $\frac{7}{5}$ C $\frac{141}{100}$

Discussing the Ideas

1. Which of the rational numbers above is closest to $\sqrt{2}$?

2. Do you think it is possible to find a rational number which when squared gives 2?

3. It can be proved that there is no rational number $\frac{a}{b}$ such that $(\frac{a}{b})^2 = 2$ and that $\sqrt{2}$ is not a rational number. We shall work with other numbers which, like $\sqrt{2}$, are not rational numbers. We shall call such numbers **irrational numbers** and, as with $\sqrt{2}$, agree that there are points on the number line for these numbers. The Pythagorean Theorem can help us locate such points.

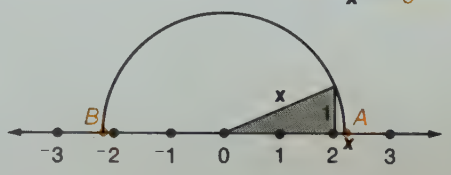
   A Explain why the calculations shown help us to determine the distance **x**.

   $x^2 = 2^2 + 1^2$

   $x^2 = 5$

   B How far is the distance **x** from zero to point *A*?

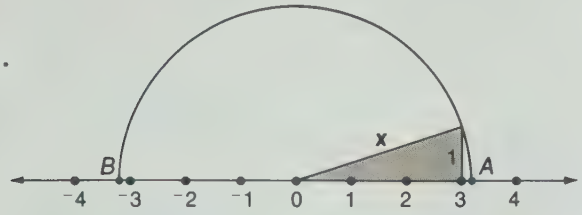
   C What number goes with point *A* on the number line? with point *B*?



Using the Ideas

For each figure, the Pythagorean Theorem can help you find numbers for points *A* and *B* on the number line. Give the missing numbers for each ▒. Then give the numbers for points *A* and *B*.

1.



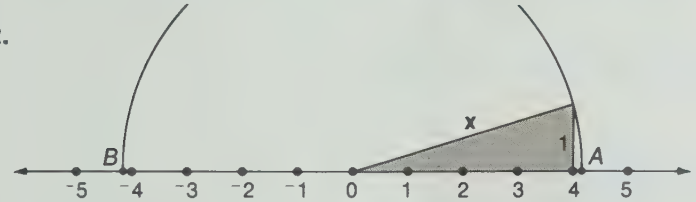
$x^2 = 3^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

2.



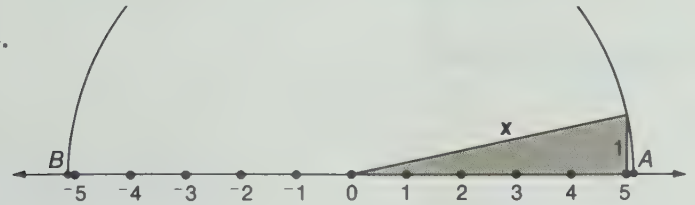
$x^2 = 4^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

3.



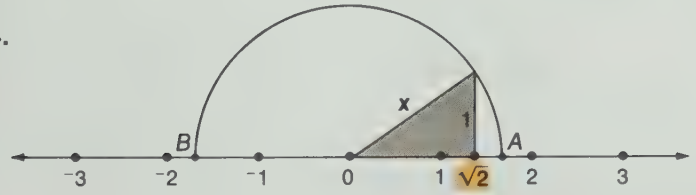
$x^2 = 5^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

4.



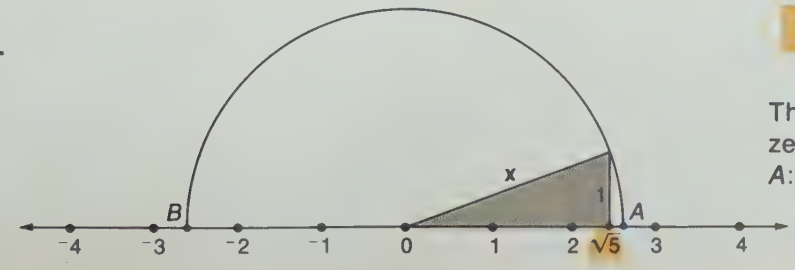
$x^2 = (\sqrt{2})^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

5.



$x^2 = (\sqrt{5})^2 + 1^2$

$x^2 =$ ▒

The distance **x** from zero to *A* is ▒.

A: ▒ *B*: ▒

UTILIZATION

There are relatively few exercises on the page, but they are important in helping students develop a good understanding of the nature of irrational numbers and how they can be related to points on the number line.

EXTENSION

The exercises on page 116 of the *Workbook* may be used to provide further practice.

Enrichment: Some students will be able to understand a proof that $\sqrt{2}$ is not a rational number. A simple proof can be presented as follows:

In order for $\sqrt{2}$ to be a rational number, represented by a fraction $\frac{a}{b}$ (where a and b are nonzero whole numbers), we must be able to find whole numbers a and b so that the numerator, $a \cdot a$, is twice the denominator, $b \cdot b$.

$$\sqrt{2} \cdot \sqrt{2} = 2$$
$$\frac{a}{b} \cdot \frac{a}{b} = 2$$
$$\frac{a \cdot a}{b \cdot b} = 2$$
$$a \cdot a = 2 \cdot b \cdot b$$

Any square has an *even* number of 2's in its prime factorization.

$\underbrace{a \cdot a}$

This number must have an *even* number of 2's in its prime factorization

This number must have an *even* number of 2's in its factorization,

$\stackrel{?}{=}$ $2 \cdot \overbrace{b \cdot b}$

This number must have an *odd* number of 2's in its prime factorization.

Therefore, there are no nonzero whole numbers a and b so that the number $a \cdot a$ is the same number as $2 \cdot b \cdot b$ because they have different prime factorizations.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–5.

Objective

The student can give rational approximations for square roots of selected whole numbers.

PREPARATION

Review the method of locating the point on the number line for a given irrational number by using the method of the previous lesson.

INVESTIGATION

Have the students read through the Investigation material. It reviews the fact that rational approximations for $\sqrt{2}$ in decimal form can be found by successive approximations. Students can use trial-and-error methods to find a decimal approximation for $\sqrt{3}$. $(1.73)^2 = 2.9929$ and $(1.74)^2 = 3.0276$, hence $1.73 < \sqrt{3} < 1.74$.

DISCUSSION

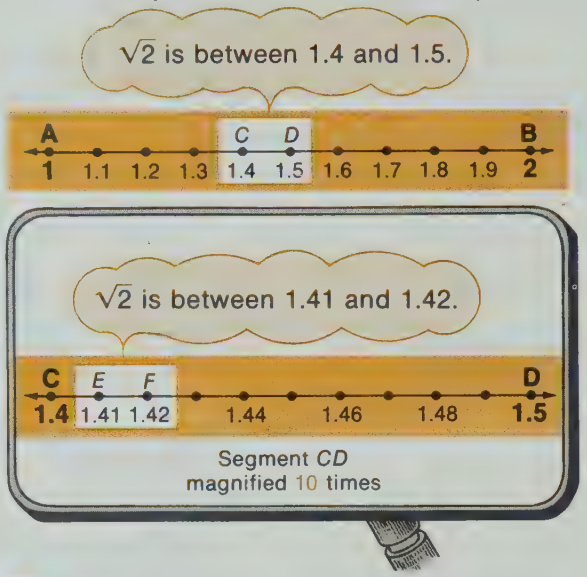
The Discussion Exercises emphasize that to find a rational approximation of an irrational number, we first locate the number between successive tenths, then with further trials locate the number between successive hundredths, and so on. When discussing Exercises 1 and 2, you might refer back to Exercise 4 on page H–55 for geometric construction of the distance $\sqrt{3}$. If a careful construction of the problem is made on graph paper with each square representing 0.1 unit, an approximation of $\sqrt{3}$ to the nearest hundredth can be found.

In Exercise 4, since 10.24 is closer to 10 than 9.61, this means that 3.2 is nearer to $\sqrt{10}$ than 3.1. You might have students show that $3.16 < \sqrt{10} < 3.17$ and 3.16 is nearer to $\sqrt{10}$ than 3.17.

Rational Approximations for Irrational Numbers

Investigating the Ideas

Although $\sqrt{2}$ is not a rational number, you have seen that there are rational numbers that are "very close" to $\sqrt{2}$. Study the number lines and equations below.



$(1.4)^2 = 1.96$ (less than 2)

$\sqrt{2}^2 = 2$

$(1.5)^2 = 2.25$ (greater than 2)

We write: $1.4 < \sqrt{2} < 1.5$

$(1.41)^2 = 1.9881$ (less than 2)

$\sqrt{2}^2 = 2$

$(1.42)^2 = 2.0164$ (greater than 2)

We write: $1.41 < \sqrt{2} < 1.42$

? Can you locate $\sqrt{3}$ between successive hundredths using the method above?

Discussing the Ideas

1. Which set of inequalities is correct? How can you prove it?

   A $1.6 < \sqrt{3} < 1.7$ $\quad (1.6)^2 < 3 < (1.7)^2$

   B $1.7 < \sqrt{3} < 1.8$ $\quad (1.7)^2 < 3 < (1.8)^2$

2. Which inequality is correct? Prove it.

   A $1.71 < \sqrt{3} < 1.72$ B $1.72 < \sqrt{3} < 1.73$ C $1.73 < \sqrt{3} < 1.74$

3. Is $\sqrt{10}$ closer to 3 or to 4? Explain your reasoning.

4. Use the equations to help you complete the sentence.

   To the nearest tenth, $\sqrt{10} \approx$ ▮.

$(3.1)^2 = 9.61$
$\sqrt{10}^2 = 10$
$(3.2)^2 = 10.24$

Using the Ideas

1. Give the largest possible integer for a. Give the smallest possible integer for b. Study the examples.

A $\underbrace{2 \cdot 2}_{a \cdot a} < \sqrt{7} \cdot \sqrt{7} < \underbrace{3 \cdot 3}_{b \cdot b} \rightarrow \begin{cases} a = 2 \\ b = 3 \end{cases}$

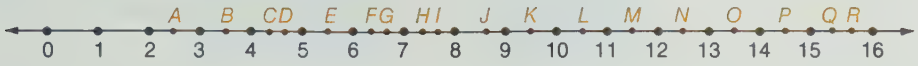
B $\underbrace{4 \cdot 4}_{a \cdot a} < \sqrt{19} \cdot \sqrt{19} < \underbrace{5 \cdot 5}_{b \cdot b} \rightarrow \begin{cases} a = 4 \\ b = 5 \end{cases}$

A $\sqrt{13} \cdot \sqrt{13}$ → $a \cdot a < 13 < b \cdot b$

B $\sqrt{27} \cdot \sqrt{27}$ → $a \cdot a < 27 < b \cdot b$

C $\sqrt{153} \cdot \sqrt{153}$ → $a \cdot a < 153 < b \cdot b$

2. Write the letter for the point on the number line that goes with each number in the exercises.



A $\sqrt{39}$	**D** $\sqrt{6}$	**G** $\sqrt{51}$	**J** $\sqrt{24}$	**M** $\sqrt{129}$	**P** $\sqrt{156}$
B $\sqrt{79}$	**E** $\sqrt{115}$	**H** $\sqrt{12}$	**K** $\sqrt{231}$	**N** $\sqrt{222}$	**Q** $\sqrt{250}$
C $\sqrt{18}$	**F** $\sqrt{33}$	**I** $\sqrt{91}$	**L** $\sqrt{48}$	**O** $\sqrt{174}$	**R** $\sqrt{61}$

3. Give the largest possible one-place decimal for a. Give the smallest possible one-place decimal for b.

A $\sqrt{8} \cdot \sqrt{8}$ → $a \cdot a < 8 < b \cdot b$

B $\sqrt{52} \cdot \sqrt{52}$ → $a \cdot a < 52 < b \cdot b$

C $\sqrt{113} \cdot \sqrt{113}$ → $a \cdot a < 113 < b \cdot b$

4. Use Exercise 3 to give the decimal to the nearest tenth that correctly completes each sentence.

A $\sqrt{8} \approx$ ▮▮▮ **B** $\sqrt{52} \approx$ ▮▮▮ **C** $\sqrt{113} \approx$ ▮▮▮

In Exercises 5 through 8, we are "closing in on" an irrational number. Give the largest possible number for ℓ (lower bound) and the smallest possible number for u (upper bound) in each exercise.

	Between integers	Between tenths	Between hundredths	Between thousandths
5.	$\ell < \sqrt{3} < u$	$\ell < \sqrt{3} < u$	$\ell < \sqrt{3} < u$	$\ell < \sqrt{3} < u$
6.	$\ell < \sqrt{5} < u$	$\ell < \sqrt{5} < u$	$\ell < \sqrt{5} < u$	$\ell < \sqrt{5} < u$
★ **7.**	$\ell < \sqrt{29} < u$	$\ell < \sqrt{29} < u$	$\ell < \sqrt{29} < u$	$\ell < \sqrt{29} < u$
★ **8.**	$\ell < \sqrt{87} < u$	$\ell < \sqrt{87} < u$	$\ell < \sqrt{87} < u$	$\ell < \sqrt{87} < u$

UTILIZATION

These exercises require considerable computation with decimals. Therefore you should adjust the assignment according to the needs of the students and in such a way that it will not be unduly burdensome. If calculators are available, a larger number of exercises could be assigned.

EXTENSION

Additional practice exercises are provided on page 117 of the *Workbook*.

Enrichment: This would be an appropriate time to encourage some students to explore Research Project A.

Many different proofs of the irrationality of $\sqrt{2}$ have been made. A variety of such proofs are found in the January 1971 issue of *The Mathematics Teacher*. See pages 19–21, Vol. LXIV, Number 1, "On Proofs of the Irrationality of $\sqrt{2}$," by V. C. Harris. A terminal digit proof in this article can be understood by junior high school students.

Assignments

Minimum 1, 2A–I, 3.
Average 1, 2A–I, 3–5.
Maximum 1–8.

Objective

The student can use an algorithm involving estimation and division to find rational approximations for square roots of numbers.

PREPARATION

Discuss the fact that, thus far in the study of irrational numbers, square roots of small integers have been the main concern. Rational approximations of these numbers have been somewhat difficult to find. Students will be pleased to know that this lesson presents a systematic way of finding such approximations

INVESTIGATION

Have the students study the flow chart and discuss any questions they may have. If you wish, try a sample problem in which a perfect square, such as 1444, is the input for the flow chart. Students might estimate that $\sqrt{1444}$ is a number between 30 and 40, say 32.

(1) Carry out the division: $32\overline{)1444}$ (45 r 4)

(2) Find the average of 32 and 45:

$$\frac{32+45}{2} = \frac{77}{2} = 38\frac{1}{2}$$

(3) Try 38 as a divisor: $38\overline{)1444}$ (38 r 0)

(4) Since $38 \cdot 38 = 1444$, $\sqrt{1444} = 38$

Then have students try $\sqrt{18.49}$ on their own.

DISCUSSION

The first trial for $\sqrt{18.49}$ should be a number between 4 and 5 since $4^2 < 18$ and $5^2 > 18$. However, students' initial choices will vary.

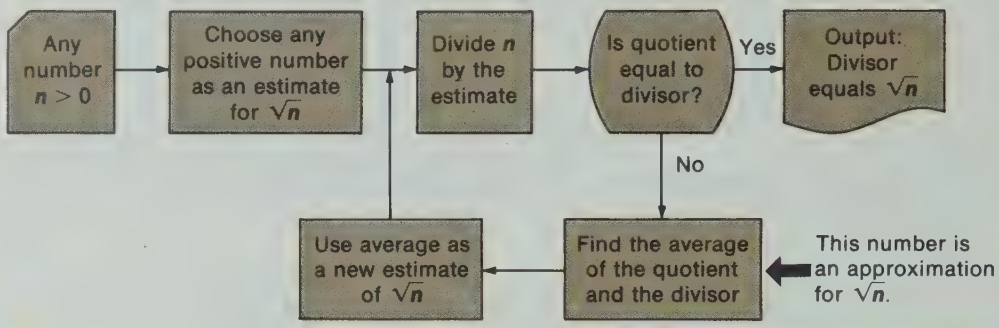
In discussing Exercise 3, point out that if the $\sqrt{n}$ is irrational, the flow chart, when used repeatedly, will yield closer and closer approximations as new averages of quotients and divisors are found. Thus, rational approximations to any number of significant digits are possible by this method.

If calculators are available, you might have students use them in conjunction with the flow chart.

Finding Square Roots

Investigating the Ideas

The flow chart shows instructions for finding the positive square root of a number to any desired accuracy.


Any number $n > 0$
Choose any positive number as an estimate for $\sqrt{n}$
Divide n by the estimate
Is quotient equal to divisor?
Yes
Output: Divisor equals $\sqrt{n}$
No
Find the average of the quotient and the divisor
Use average as a new estimate of $\sqrt{n}$
This number is an approximation for $\sqrt{n}$.

? Can you use the flow chart to find $\sqrt{18.49}$?

Discussing the Ideas

1. What number did you first choose for $\sqrt{18.49}$? How did you make your choice?

2. Paul chose 4.5 on his first estimate for $\sqrt{18.49}$.
   - A What should be his next estimate for $\sqrt{18.49}$?
   - B What is $\sqrt{18.49}$ exactly?

Paul

$$4.5\overline{)18.49}\ \ 4.1$$
$$180$$
$$49$$
$$45$$
$$4$$

3. Not every number has an exact square root. Study the example below.

EXAMPLE: Find $\sqrt{537}$.

A Estimate of $\sqrt{537}$:
$20^2 = 400$
$25^2 = 625$
Try 22.5

B
$$22.5\overline{)537.000}\ \ 23.86$$
$$450$$
$$870$$
$$675$$
$$1950$$
$$1800$$
$$1500$$
$$1350$$
$$150$$

C The average of 22.5 and 23.86 is 23.18. Therefore, $\sqrt{537} \approx 23.2$ to the nearest tenth.

D By carrying out the division to thousandths, we can get better estimates of $\sqrt{537}$. Thus $\sqrt{537} \approx 23.17$

Use this method to find $\sqrt{747}$ to the nearest hundredth.

Using the Ideas

1. Using the estimation and division method shown in the flow chart, find the positive square root of each of the following numbers. Each can be found exactly.

A 676	C 12.25	E 44.89	G 3249
B 6241	D 0.0324	F 28.09	H 139.24

2. Find an approximation of each number to the nearest tenth.

A $\sqrt{17}$	C $\sqrt{163}$	E $\sqrt{172.89}$	G $\sqrt{69.48}$	I $\sqrt{0.18}$
B $\sqrt{88}$	D $\sqrt{21.4}$	F $\sqrt{435}$	H $\sqrt{200}$	J $\sqrt{139.5}$

3. Find an approximation of each number to the nearest hundredth.

A $\sqrt{86}$	C $\sqrt{13}$	E $\sqrt{31.89}$	G $\sqrt{7.91}$	I $\sqrt{0.394}$
B $\sqrt{623}$	D $\sqrt{18.99}$	F $\sqrt{52.89}$	H $\sqrt{3000}$	J $\sqrt{1.05}$

★**4.** Which number is the best approximation of $\sqrt{2}$ to the nearest millionth?

[A] 1.414213 [B] 1.414214

★**5.** The **cube root** of 27 is 3 because $3^3 = 27$.
The symbol for cube root is $\sqrt[3]{\ }$. Thus, $\sqrt[3]{27} = 3$.

What number is represented by each of the symbols?

A $\sqrt[3]{8}$	B $\sqrt[3]{1000}$	C $\sqrt[3]{64}$	D $\sqrt[3]{125}$	E $\sqrt[3]{-1}$

★**6.** Each number in the sequence below is a whole number. Can you give the first four whole numbers of the sequence?

$\sqrt{4489}$, $\sqrt{444\,889}$, $\sqrt{44\,448\,889}$, . . .

think

A toy train is made up of an engine, a dining car, and a caboose. The ratio of the length of the dining car to the length of the engine is 3 to 2. The ratio of the caboose to the dining car is 1 to 2.

1. Compare the length of the caboose with the length of the engine.
2. Compare the sum of the lengtns of the engine and the caboose with the dining car.
3. If the whole train is 26 centimetres long, how long is each part?

More practice, page S-32, Set 62

UTILIZATION

Since the exercises entail considerable computation, you will probably want to limit the number of problems assigned. It would be desirable for some students to present their solutions on the chalkboard.

Solution, Exercise 5E

Note that the cube root of $^-1$ is $^-1$ since $(^-1)^3 = {}^-1$. Thus, while it is possible to find the cube roots of negative numbers, it is not possible to find real square roots of negative numbers, because there is no real number whose square is a negative number.

EXTENSION

To provide further practice, make selective assignments from Supplementary Exercise Set 62 on page S-32, *Workbook* page 118, *Duplicator Masters* page 78, and *Arithmetic Skill Card* D-15.

Think Solution

A drawing may help in analyzing parts 1 and 2 of the problem.

1. Dining Car Engine — Ratio 3:2

Caboose Dining Car — Ratio $1\frac{1}{2}$:3 or 1:2

Caboose Engine — Ratio $1\frac{1}{2}$:2 or 3:4

2. Engine + Caboose Dining Car — Ratio $3\frac{1}{2}$:3 or 7:6

3. If x is the unit of length the dining car is $3x$ units, the engine $2x$ units, and the caboose $1\frac{1}{2}x$. Hence $6\frac{1}{2}x = 26$ cm.

ℓ(caboose) = 6 cm
ℓ(engine) = 8 cm
ℓ(dining car) = 12 cm

Assignments

Minimum 1A–D, 2A–D, 3A–D
Average 1A–F, 2A–F, 3A–F, 4.
Maximum 1–6.

Objective

Given a table of square roots of whole numbers, the student can use the table to find approximate square roots of selected numbers.

PREPARATION

The previous lesson should have provided sufficient preparation for this lesson.

INVESTIGATION

The Investigation provides an opportunity for the student to study the table of squares and square roots, and discover patterns that will enable him to find the three missing numbers:

A. Clearly this number is 14^2, or 196.

B. This number must be $\sqrt{17}$. A decimal approximation can be found for $\sqrt{17}$ using the square root flow-chart method of the previous lesson. $\sqrt{17} \approx 4.123$. (Note that a good approximation for $\sqrt{17}$ is the average of $\sqrt{16}$ and $\sqrt{18}$, values which are given in the table.)

C. This number is $\sqrt{10 \times 3}$ or $\sqrt{30}$. Since 30 is not in the table, this number must be approximated. The average of ($\sqrt{20} \approx 4.472$) and ($\sqrt{40} \approx 6.325$) is about 5.4, which is a fair approximation for $\sqrt{30}$.

DISCUSSION

Work through the Discussion Exercises carefully, making certain that students can read the table efficiently.

The method for checking the estimate of $\sqrt{30}$ in Exercise 1C can be justified:

$$(\sqrt{3} \cdot \sqrt{10})^2 = \sqrt{3} \cdot \sqrt{3} \cdot \sqrt{10} \cdot \sqrt{10}$$
$$= 3 \cdot 10 = 30$$
$$(\sqrt{30})^2 = 30$$

Hence $\sqrt{3} \cdot \sqrt{10} = \sqrt{30}$.

Solution, Discussion Exercise 5

$\sqrt{18} \approx 4.243$ and $\sqrt{17} \approx 4.123$

Difference ≈ 0.120

$0.5 \times 0.120 = 0.060$

Therefore, $\sqrt{17.5} \approx 4.123 + 0.060$, or 4.183.

Check: $(4.183)^2 = 17.497489$, which is very close to 17.5.

Approximations for Square Roots

Investigating the Ideas

Certain entries are missing in this table of squares and square roots.

Can you give the missing entries to complete the table?

Table of Squares and Square Roots

Number n	Square n^2	Square root $\sqrt{n}$	Square root of 10 times n $\sqrt{10 \times n}$
1	1	1.000	3.162
2	4	1.414	4.472
3	9	1.732	C
4	16	2.000	6.325
5	25	2.236	7.071
6	36	2.449	7.746
7	49	2.646	8.367
8	64	2.828	8.944
9	81	3.000	9.487
10	100	3.162	10.000
11	121	3.317	10.488
12	144	3.464	10.954
13	169	3.606	11.402
14	A	3.742	11.832
15	225	3.875	12.247
16	256	4.000	12.649
17	289	B	13.038
18	324	4.243	13.416
19	361	4.359	13.784
20	400	4.472	14.142

Discussing the Ideas

1. **A** How did you find the number for **A** in the table?
   B Explain your method for finding the number for **B**.
   C The number for **C** is $\sqrt{30}$. Check your estimate by multiplying $\sqrt{3} \cdot \sqrt{10}$.

2. Explain how to use the table to find a decimal approximation of each of the following.

   A $\sqrt{5}$ **B** $\sqrt{8}$ **C** $\sqrt{13}$ **D** $\sqrt{20}$

3. Explain how you can use the table to find a decimal approximation of each of these.

   A $\sqrt{30}$ **B** $\sqrt{90}$ **C** $\sqrt{120}$ **D** $\sqrt{190}$

4. How can you use the table to find each of these?

   A $\sqrt{324}$ **B** $\sqrt{169}$ **C** $\sqrt{256}$ **D** $\sqrt{361}$

5. Close approximations of square roots of numbers not found in the table can be found by a process of **interpolation**. To find $\sqrt{7.5}$, we observe that this number should be between $\sqrt{7}$ and $\sqrt{8}$. The work can be arranged as follows:

   $\sqrt{7} \approx 2.646$ and $\sqrt{8} \approx 2.828$
   difference ≈ 0.182
   $0.5 \times$ difference ≈ 0.091

   $\sqrt{7.5}$ is approximately $2.646 + 0.091$, or 2.737
   $\sqrt{7.5} \approx 2.737$

   Use interpolation to find $\sqrt{17.5}$.

Using the Ideas

1. Use the table of squares and square roots in the Investigation to find the following.

A $\sqrt{6}$	C $\sqrt{13}$	E $\sqrt{18}$	G $\sqrt{11}$	I $\sqrt{196}$
B $\sqrt{60}$	D $\sqrt{130}$	F $\sqrt{180}$	H $\sqrt{110}$	J $\sqrt{169}$

2. Using the table, give a decimal approximation for a and b.

A $\sqrt{8} = a$	B $\sqrt{20} = a$	C $\sqrt{12} = a$
$\sqrt{4} \cdot \sqrt{2} = b$	$\sqrt{4} \cdot \sqrt{5} = b$	$\sqrt{4} \cdot \sqrt{3} = b$

3. Use the table to find the number for n.

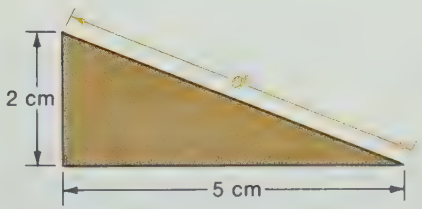
A $\sqrt{n} = 2.646$	C $\sqrt{n} = 12.247$	F $\sqrt{225} = n$
SOLUTION: $n = 7$	D $\sqrt{n} = 9.487$	G $\sqrt{10} = n$
B $\sqrt{n} = 3.317$	E $\sqrt{n} = 3.875$	H $\sqrt{121} = n$

4. Using the table and interpolation, find decimal approximations for each of these.

A $\sqrt{4.5}$	C $\sqrt{10.5}$	E $\sqrt{8.5}$	G $\sqrt{19.4}$	I $\sqrt{13.6}$
B $\sqrt{1.5}$	D $\sqrt{17.5}$	F $\sqrt{18.3}$	H $\sqrt{16.2}$	J $\sqrt{14.8}$

Complete the solutions in Exercises 5 and 6 by giving the answers as decimal approximations to the nearest hundredth.

5. Find the hypotenuse of a triangle whose legs are 2 centimetres and 5 centimetres respectively.



$d^2 = 2^2 + 5^2$
$d^2 = 4 + 25$
$d^2 = 29$
$d =$ ▮

6. It is 4 km along the road from Dock A to Dock C. It is 2 km from Dock C to Dock B. How far is it by boat straight across the lake from Dock A to Dock B?

$= 4^2 + 2^2$,
$=$ ▮

UTILIZATION

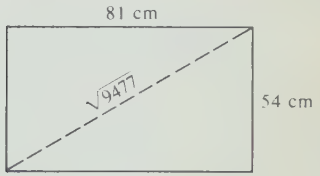
A larger table of powers and roots of whole numbers is given on page S-34. You may refer students to this page, if you wish, although the table in the Investigation is sufficient for this lesson.

EXTENSION

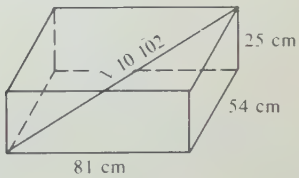
Additional appropriate practice exercises are provided on page 119 of the *Workbook*.

Enrichment: Here is a two-part challenge problem for very able students:

A. Can a metre stick lie flat in a box that is 81 cm by 54 cm by 25 cm? (*Answer:* No. The diagram of the 81 by 54 rectangular base of the box is $\sqrt{81^2 + 54^2} = \sqrt{9477}$ and $\sqrt{9477} \approx 97.349884$, which is less than 100 cm.)



B. Can the metre stick be placed completely inside the box? (*Answer:* Yes. The diagonal of the box is $\sqrt{(\sqrt{9477})^2 + 25^2} = \sqrt{9477 + 625} = \sqrt{10\,102}$, and $\sqrt{10\,102} \approx 100.5087$, which is greater than 100 cm.)



Assignments

Minimum 1A–F, 2, 3.
Average 1–3, 4A–F. Maximum 1–6.

Objective

The student can solve problems by applying the Pythagorean Theorem and using a table of square roots.

PREPARATION

Allow a few minutes for a review of the use of the Table of Powers and Roots on page S–34.

INVESTIGATION

The Pythagorean Theorem should be familiar to all students by this time. The Investigation provides a review of its use. When students find $c = \sqrt{13}$ cm, they should use the table of square roots to find that the approximation of $\sqrt{13}$ cm is 3.6 cm to the nearest tenth.

DISCUSSION

A careful drawing of the right triangle in Exercise 1 will reinforce the work of the Investigation. The hypotenuse of the triangle will be about 3.6 cm long.

Exercise 2 will provide additional practice using the Pythagorean Theorem. The diagonal is $\sqrt{80}$ cm. Using the table of square roots, $\sqrt{80}$ cm ≈ 8.94 cm.

Exercise 3 illustrates a common student error in the use of the Pythagorean Theorem. Since the hypotenuse and one leg are given, the correct equation is

$$13^2 = 5^2 + p^2$$

Hence, $p^2 = 13^2 - 5^2$

$$p^2 = 169 - 25$$

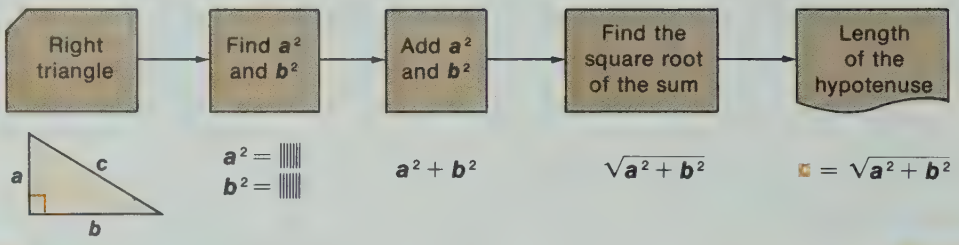
$$p^2 = 144$$

$$p = 12$$

Problem Solving with Square Roots

Investigating the Ideas

You can use the Pythagorean Theorem to find the length of the hypotenuse of a right triangle when the lengths of the two legs are known. Study this flow chart.



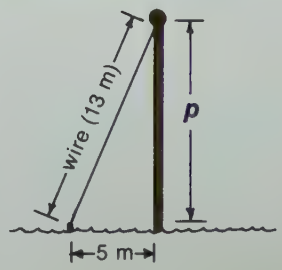
Can you find the length of the hypotenuse (to the nearest tenth) of a right triangle with legs 2 centimetres and 3 centimetres?

Discussing the Ideas

1. Draw a right triangle with legs 2 centimetres and 3 centimetres in length. Measure the length of the hypotenuse. Is this length about the same as the length you found using the Pythagorean Theorem?

2. A draftsman wanted to give the diagonal distance (to the nearest hundredth) across this rectangle. How can he find this distance?

3. A 13-metre wire supported a pole. It was fastened 5 metres from the base of the pole. How long was the pole?



A A student started to solve the problem this way: $p^2 = 5^2 + 13^2$

$$p^2 = 25 + 169$$

$$p^2 = 194$$

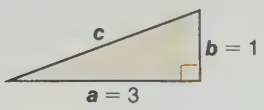
What is wrong with this solution?

B How would you solve this problem?

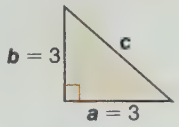
Using the Ideas

1. Find the length of the hypotenuse of each right triangle.

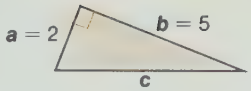
A



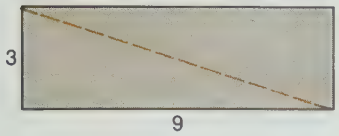
B



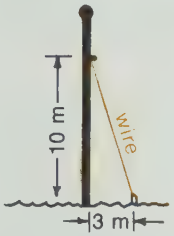
C



2. A rectangle is 9 units long and 3 units wide. How long is the diagonal of the rectangle, to the nearest tenth of a unit?

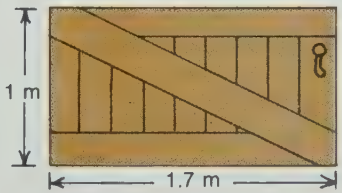


3. A guy wire is attached to an upright light pole 10 metres above the ground. The wire is anchored to the ground 3 metres from the base of the pole. How long is the wire to the nearest tenth of a metre?



4. Can an umbrella 60 centimetres long be packed to lie flat in a box 52 centimetres long and 30 centimetres wide?

5. What is the length of the diagonal brace for the gate in the figure?



6. It is 10 kilometres from Fisherman's Point to the Lodge and 8 kilometres from Fisherman's Point to Townsville. How far is it from the Lodge across the lake to Townsville?

UTILIZATION

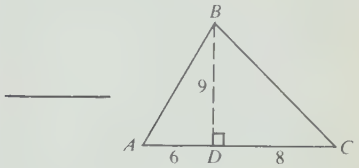
In Exercise 1, you may direct students to use the table of square roots and express the length of the hypotenuse as a decimal. Have selected students place their solutions on the chalkboard and explain them to the class.

EXTENSION

Duplicator Masters page 79 may be used for additional practice assignments.

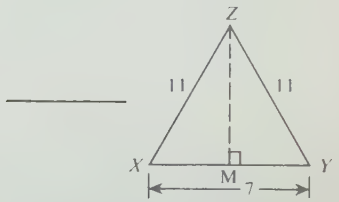
Enrichment: Challenge the more able students with problems like the ones that follow.

1. What is the perimeter of $\triangle ABC$ to the nearest tenth of a unit?



(*Answer:*
$\ell(\overline{AB}) = \sqrt{6^2 + 9^2} = \sqrt{117} \approx 10.82$
$\ell(\overline{BC}) = \sqrt{8^2 + 9^2} = \sqrt{145} \approx 12.04$
$\ell(\overline{AC}) = 14$
$P(\triangle ABC) \approx 36.9$ units)

2. Find the area of this isosceles triangle.



(*Solution:* The height of the triangle is the perpendicular line from point Z to $\overline{XY}$. This segment bisects the base, $\overline{XY}$, at point M. Then $\ell(\overline{MY}) = \ell(\overline{MX}) = 3.5$
and $\ell(\overline{MZ}) = \sqrt{11^2 - (3.5)^2} =$
$\sqrt{121 - 12.25} = \sqrt{108.75}$
$\ell(\overline{MZ}) \approx 10.43$
$A(\triangle XYZ) = \frac{1}{2} \cdot 7 \cdot (10.43) = 36.505 \approx 36.5$ sq units)

Assignments

Minimum 1–3. Average 1–5. Maximum 1–6.

Objective

The student can find a repeating decimal for a fraction or find a fraction for a repeating decimal.

PREPARATION

Repeating decimals were introduced in Unit F, Module 2. You might review the method of finding a decimal for a given fraction. Also, distinguish between terminating decimals (the repeating part consists of zeros) and nonrepeating decimals.

INVESTIGATION

Have the students carry out the division for $\frac{3}{17}$. This will take some students several minutes as there will be 16 different remainders.

$$17\overline{)3.0000000000000000}\quad 0.\overline{1764705882352941}$$

The 16 remainders, in succession, are 13, 11, 8, 12, 1, 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3. Since all the different possible remainders have occurred, the decimal must start repeating at this point.

DISCUSSION

A convincing argument that every rational number has a repeating decimal representation can be made by using the fraction $\frac{3}{17}$ as an example. If a zero is the remainder at any step in the division process, the decimal would repeat in zeros. If no zero remainders occur, there cannot be more than $(d-1)$ remainders where d is the divisor. Note that there are 16 different remainders where the divisor is 17. Whenever one of the remainders occurs twice the decimal starts repeating.

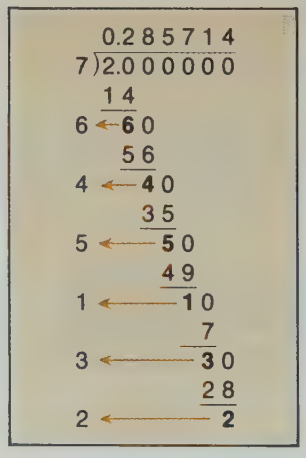
Work through Exercise 3 carefully with the class, making certain that they understand the steps.

Repeating Decimals Again

Investigating the Ideas

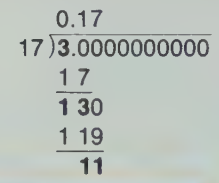
In computing the decimal for the fraction $\frac{2}{7}$, a student found that six different remainders occurred before the decimal started to repeat.

Can you find all the different remainders when the decimal for $\frac{3}{17}$ is computed?



Discussing the Ideas

1. Are you sure that $\frac{3}{17}$ can be represented by a repeating decimal? Why?



2. It can be proved that:

Every rational number has a repeating decimal representation.

How would you convince someone that this is true?

3. It can be proved that:

Every repeating decimal represents a rational number.

The examples below illustrate a way to prove the statement above for some particular cases.

1 To find a fraction q for $0.\overline{7}$

$q = 0.777\ldots \rightarrow 10q = 7.777\ldots$

$10q = 7.777\ldots$

$1q = 0.777\ldots$

$9q = 7$

$q = \frac{7}{9}$

2 To find a fraction q for $0.\overline{45}$

$q = 0.4545\ldots \rightarrow 100q = 45.4545\ldots$

$100q = 45.4545\ldots$

$1q = 0.4545\ldots$

$99q = 45$

$q = \frac{45}{99} = \frac{5}{11}$

Use this method to find a fraction for these repeating decimals.

A $0.\overline{5}$ **B** $0.\overline{73}$ **C** $0.\overline{129}$

Using the Ideas

1. The **period** of a repeating decimal is the number of digits in the entire group of digits that repeat over and over again. For example, the period of the repeating decimal $.\overline{54}$ is 2. Write a repeating decimal for each rational number and give its period.

A $\frac{2}{3}$ **B** $\frac{2}{7}$ **C** $\frac{5}{8}$ **D** $\frac{5}{13}$ **E** $\frac{5}{6}$ **F** $\frac{1}{27}$ **G** $\frac{9}{11}$

2. Find the products.

A $10 \cdot 0.\overline{6}$
Answer: $6.\overline{6}$

B $10 \cdot 0.\overline{4}$

C $10 \cdot 0.\overline{38}$

D $100 \cdot 0.\overline{07}$

E $1000 \cdot 0.\overline{842}$

In Exercises 3, 4, and 5, copy the exercises, giving the correct decimal for ***a***, the number for ***b***, and the fraction for ***c***.

3. $q = 0.4444\ldots$

$10q = a$

$1q = 0.4444\ldots$

$9q = b$

$q = c$

4. $q = 0.\overline{27}$

$100q = a$

$1q = 0.272727\ldots$

$99q = b$

$q = c$

5. $q = 0.\overline{074}$

$1000q = a$

$1q = 0.074074\ldots$

$999q = b$

$q = c$

6. Using the method illustrated in Exercises 3, 4, and 5, find a fraction for each repeating decimal.

A $0.555\ldots$ **B** $0.\overline{37}$ **C** $2.\overline{27}$ **D** $0.\overline{015}$ **E** $0.\overline{123}$ **F** $0.0111\ldots$ **G** $0.0\overline{5}$ **H** $4.0\overline{27}$ **I** $3.\overline{2}$ **J** $0.00\overline{1001}$

7. Find the repeating decimal for each fraction.

A $\frac{1}{9}$ **B** $\frac{2}{9}$ **C** $\frac{1}{99}$ **D** $\frac{2}{99}$ **E** $\frac{1}{999}$ **F** $\frac{2}{999}$

8. These exercises suggest another way to find a fraction for a repeating decimal. Give the number for ***a***. Then give the number for ***b***.

A $0.11\overline{1} = a \rightarrow 0.22\overline{2} = b$
B $0.11\overline{1} = a \rightarrow 0.33\overline{3} = b$
C $0.11\overline{1} = a \rightarrow 0.44\overline{4} = b$
D $0.0\overline{101} = a \rightarrow 0.0\overline{202} = b$
E $0.0\overline{101} = a \rightarrow 0.0\overline{909} = b$
F $0.0\overline{101} = a \rightarrow 0.1\overline{010} = b$
G $0.0\overline{101} = a \rightarrow 0.11\overline{11} = b$
H $0.0\overline{101} = a \rightarrow 0.12\overline{12} = b$
I $0.0\overline{101} = a \rightarrow 0.23\overline{23} = b$
J $0.0\overline{101} = a \rightarrow 0.68\overline{68} = b$
K $0.00\overline{1001} = a \rightarrow 0.002\overline{002} = b$
L $0.00\overline{1001} = a \rightarrow 0.009\overline{009} = b$
M $0.00\overline{1001} = a \rightarrow 0.023\overline{023} = b$
N $0.00\overline{1001} = a \rightarrow 0.46\overline{7467} = b$

UTILIZATION

Exercises 7 and 8 illustrate an interesting property of the repeating decimals for fractions of the form $\frac{1}{10^n - 1}$ where n is a positive integer. Since $\frac{1}{9} = 0.\overline{1}$, $\frac{2}{9} = 0.\overline{2}$, $\frac{3}{9} = 0.\overline{3}$, and so on. $\frac{1}{99} = 0.\overline{01}$, $\frac{2}{99} = 0.\overline{02}$, . . . $\frac{1}{999} = 0.\overline{001}$, $\frac{2}{999} = 0.\overline{002}$, ... Thus, given a repeating decimal such as $0.\overline{73}$, we can immediately write $0.\overline{73} = \frac{73}{99}$.

EXTENSION

Enrichment: Have students find repeating decimals for $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, and $\frac{6}{7}$. Then ask: "What pattern in the digits for these decimals can be found?" (*Answer:* The digits are cyclic and always occur in the order 1, 4, 2, 8, 5, 7.)

This property of the repeating digits is the basis of a clever mathematical trick described by Martin Gardner in "Mathematical Games" in *Scientific American*, Volume 222, Number 3, March 1970. Some students might be interested in finding this article in the library and demonstrating the trick for the class.

Assignments

Minimum 1–5. Average 1–6. Maximum 1–8.

Objective

Given an appropriate decimal, the student can identify it as representing either a rational number or an irrational number.

PREPARATION

Review the fact that each repeating decimal represents a rational number and, conversely, every rational number can be represented by a repeating decimal.

INVESTIGATION

Most of the patterns for these decimals are easy to discover.

A. Repeating decimal: $0.\overline{233}$
B. The only digits are 5 and 0 with the 5's separated by 1 zero, then by 2 zeros, then by 3 zeros, and so on.
C. The numeral is formed by writing the natural numbers in succession.
D. The repeating decimal for $\frac{1}{7} = 0.\overline{142857}$
E. The numeral is formed by writing the even numbers in succession.
F. Repeating decimal: $6.\overline{26}$
G. Starting with the ten thousands' place, each successive natural number is separated by 2 zeros.
H. The pattern indicates the repeating decimal $0.4\overline{14253}$. Note that the repeating part does not begin immediately.

DISCUSSION

As you discuss Exercise 1 with the students, you may have them describe the patterns they found in the decimals in the Investigation.

The term *irrational number* is defined in Exercise 2. Emphasize this point by showing that the decimals for parts B, C, E, and G in the Investigation were nonrepeating decimals; hence they represent irrational numbers.

Be sure the students understand that if the square root of a given number is not a rational number, then it does not have a repeating decimal representation. If it did, it would be rational.

Repeating and Nonrepeating Decimals

Investigating the Ideas

The digits in each decimal below form a pattern.

Can you discover the pattern and extend each decimal five more places?

A 0.233233 . . .
B 0.050050005 . . .
C 0.1234567891011 . . .
D 0.1428571428571 . . .
E 0.2468101214 . . .
F 6.262626 . . .
G 0.1001002003 . . .
H 1.41425314253142 . . .

Discussing the Ideas

1. All decimals can be considered to be "unending." Certain decimals, in which the same sequence of digits repeats again and again, are called **repeating decimals**. Every repeating decimal represents a rational number and every rational number can be represented by a repeating decimal. Which decimals in the Investigation represent rational numbers?

2. A **nonrepeating decimal** is "unending," but the same sequence of digits is not repeated over and over again. An **irrational number** is any number that is represented by a nonrepeating decimal. Which decimals in the Investigation represent irrational numbers?

3. Can you create your own nonrepeating decimal? What kind of a number does it represent?

4. The number $\sqrt{10}$ is irrational. What kind of decimal would represent $\sqrt{10}$?

5. Many irrational numbers do not have patterns in their decimal representations. The ratio of the circumference to the diameter of any circle is an irrational number denoted by the symbol π. $\pi \approx 3.141592653589793\ldots$

   What must be true of the decimal for π?

6. A Does $0.\overline{72}$ represent a rational number? Explain.
   B Does 0.9217 represent a rational number? Explain.

Using the Ideas

1. Tell whether each decimal represents a rational number or an irrational number.

A 0.127127 . . .
B 0.4
C 3.17117111711117 . . .
D 0.142857142857 . . .
E 0.0369121518 . . .
F 3.249999999 . . .
G 189.647764776 . . .
H 0.01020304050 . . .

2. Find a decimal approximation to the nearest hundredth for each of the following expressions. Use $\sqrt{2} \approx 1.414$.

A $3 + \sqrt{2}$ **B** $2 \cdot \sqrt{2}$ **C** $\sqrt{2} - 1$ **D** $\sqrt{2} + \sqrt{2}$

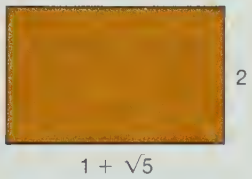
E $\frac{\sqrt{2}}{5}$ **F** $\frac{4 + \sqrt{2}}{6}$ **G** $\frac{1}{\sqrt{2}}$ **H** $\frac{\sqrt{2}}{2}$

3. Use $\pi \approx 3.14$ to find a decimal approximation to the nearest hundredth for each of the following.

A $10 \cdot \pi$ **B** $4 \cdot \pi$ **C** $\pi \cdot 3^2$ **D** $\frac{1}{\pi}$

E $\frac{\pi}{180}$ **F** π^2 **G** $5.6 \cdot \pi$ **H** $\frac{25\pi}{4}$

★ **4.** The ratio $\frac{1 + \sqrt{5}}{2}$ is called the **Golden Ratio**. A rectangle whose length and width are in this ratio is said to be the most pleasing rectangle to the artistic eye. Find a decimal approximation of this ratio to the nearest hundredth. Use $\sqrt{5} \approx 2.236$.



Think

π is an interesting irrational number. The decimal approximation of π has been calculated to more than 100 000 decimal places. An approximation to π is given below to 45 places.

3.14159265358979323846264338327950288419716939 9

A Which of the following rational numbers is closest to π?

$\frac{22}{7}$, $\frac{179}{57}$, $\frac{355}{113}$

B Give the decimal for a rational number between $\frac{355}{113}$ and π.

UTILIZATION

Exercises 2 and 3 involve computations with decimal approximations for irrational numbers. You may want to have some of the exercises put on the chalkboard and discussed.

EXTENSION

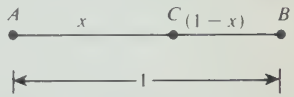
Workbook page 120 provides further exercises appropriate for use as an extra practice assignment.

Enrichment: The Golden Ratio can be described in the following terms.

Let $\overline{AB}$ be a segment of unit length, and let C be a point between A and B such that $\ell(\overline{AC}) = x$, where $\overline{AC} > \overline{CB}$. If C divides $\overline{AB}$ so that

$$\frac{\ell(\overline{AB})}{\ell(\overline{AC})} = \frac{\ell(\overline{AC})}{\ell(\overline{BC})},$$

then either ratio is called the Golden Ratio.



In terms of the lengths in the drawing above, we have

$$\frac{1}{x} = \frac{x}{1 - x}.$$

Solving for x, we obtain $x^2 + x - 1 = 0$, whence

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

The positive number $\frac{1 + \sqrt{5}}{2}$ is the Golden Ratio.

Think Solution

A. Using division, $\frac{355}{113}$ is the best approximation for π. $\frac{355}{113} = 3.1415929 \ldots$, since it differs with the given value of π only in the ten millionths' place ($\pi = 3.1415926 \ldots$). $\frac{22}{7} \approx 3.1428571 \ldots$ and $\frac{179}{57} \approx 3.1403508 \ldots$

B. Many rational numbers could be given. For example, $\pi < 3.1415927 < \frac{355}{113}$.

Assignments

Minimum 1, 2A–D, 3A–D.
Average 1–3. Maximum 1–4.

Objectives

The student can identify real numbers as being represented by repeating or nonrepeating decimals.

The student can find products of square roots of numbers.

PREPARATION

You should begin work on the Discussion Exercises without delay.

DISCUSSION

Up to this point, the number line has had "gaps" in it. By combining the rationals and irrationals to form the set of real numbers, these gaps are filled. There is now a number for each point on the number line.

Although the objective of this lesson includes computing with radicals, students are not expected to become proficient at this type of computation. This lesson does provide sufficient experience with real numbers so that students will be comfortable with these new ideas.

Exercise 3 shows that the product of two irrational numbers can be a rational number. However, students should not be lead to think that this is always the case. For example, $\sqrt{2} \cdot \sqrt{3}$ is the irrational number $\sqrt{6}$. To prove this we observe that

$$(\sqrt{2} \cdot \sqrt{3})^2 = (\sqrt{2} \cdot \sqrt{2}) \cdot (\sqrt{3} \cdot \sqrt{3})$$
$$= 2 \cdot 3 = 6$$

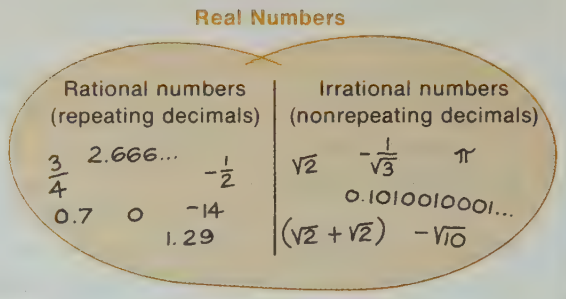
But $(\sqrt{6})^2 = 6$

Hence $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$

You may find it helpful to review the complete statements of the basic principles for real numbers as you discuss Exercise 3.

The Set of Real Numbers

Discussing the Ideas



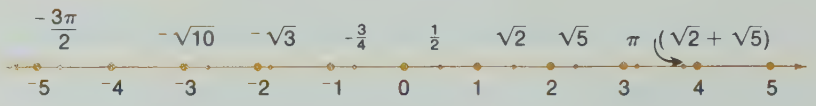
1. The union of the set of rational numbers and the set of irrational numbers is a set of numbers called the **real numbers**.

   A Is an integer a real number? Is an integer rational or irrational?

   B A certain real number is not a rational number. What kind of number must it be?

2. Since every decimal (repeating or nonrepeating) determines a certain point on the number line and every point on the number lines goes with a certain decimal, we say that the number line is **complete**, and call it the **real number line**.

   Some real numbers matched with their points on the real number line are shown below.



   A Name a rational number between $\sqrt{2}$ and $\sqrt{5}$.

   B Can you name an irrational number between $\sqrt{2}$ and $\sqrt{5}$?

3. You are familiar with the basic principles for addition and multiplication of rational numbers. Because the set of real numbers contains the set of rational numbers, we accept the same basic principles for addition and multiplication of real numbers.

Commutative Principle	Zero Principle
Associative Principle	One Principle
Distributive Principle	

   Use the commutative and associative principles to find the products. Study the examples.

   A Multiply these factors first.

   $(3 \cdot \sqrt{2}) \cdot \sqrt{2} = 3 \cdot 2 =$ ▮

   B Multiply these factors first.

   $(\sqrt{5} \cdot 4) \cdot \sqrt{5} = 4 \cdot 5 =$ ▮

   C $(\sqrt{3} \cdot \sqrt{5}) \cdot (\sqrt{3} \cdot \sqrt{5}) =$ ▮

   D $(8 \cdot \sqrt{7}) \cdot \sqrt{7} =$ ▮

Using the Ideas

1. Give the word (**rational** or **irrational**) that best describes the number n.

- **A** n is a real number represented by a repeating decimal.
- **B** n is a real number that is not represented by a repeating decimal.
- **C** n is a real number that is not a rational number.
- **D** n can be represented by a fraction.

2. Find the products.

- **A** $(2 \cdot \sqrt{3}) \cdot \sqrt{3}$
- **B** $\sqrt{2} \cdot (\sqrt{2} \cdot 9)$
- **C** $\sqrt{7} \cdot (3 \cdot \sqrt{7})$
- **D** $\sqrt{6} \cdot (\sqrt{6} \cdot 4)$
- **E** $(2 \cdot \sqrt{2}) \cdot (3 \cdot \sqrt{2})$
- **F** $(5 \cdot \sqrt{3}) \cdot (7 \cdot \sqrt{3})$

3. Give the integers for a and b. Then give the final product for c.

- **A** $(\sqrt{2} \cdot \sqrt{3})^2 = (\sqrt{2} \cdot \sqrt{3}) \cdot (\sqrt{2} \cdot \sqrt{3}) = a \cdot b = c$ (Multiply)
- **B** $(\sqrt{3} \cdot \sqrt{5})^2 = (\sqrt{3} \cdot \sqrt{5}) \cdot (\sqrt{3} \cdot \sqrt{5}) = a \cdot b = c$
- **C** $(\sqrt{2} \cdot \sqrt{7})^2 = (\sqrt{2} \cdot \sqrt{7}) \cdot (\sqrt{2} \cdot \sqrt{7}) = a \cdot b = c$
- **D** $(\sqrt{7} \cdot \sqrt{5})^2 = (\sqrt{7} \cdot \sqrt{5}) \cdot (\sqrt{7} \cdot \sqrt{5}) = a \cdot b = c$

4. In Exercise **3A**, $\sqrt{2} \cdot \sqrt{3}$ gave 6 when squared. What other symbol represents the number that gives 6 when squared?

5. **A** Using the facts that $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$, find a decimal approximation for $\sqrt{2} \cdot \sqrt{3}$.

B What is a decimal approximation for $\sqrt{6}$?

6. Give a simple square root for the product.

- **A** $\sqrt{3} \cdot \sqrt{5}$ Answer: $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$
- **B** $\sqrt{2} \cdot \sqrt{7}$
- **C** $\sqrt{7} \cdot \sqrt{5}$
- **D** $\sqrt{3} \cdot \sqrt{7}$
- **E** $\sqrt{5} \cdot \sqrt{11}$
- **F** $\sqrt{13} \cdot \sqrt{11}$
- **G** $\sqrt{11} \cdot \sqrt{17}$
- **H** $\sqrt{3} \cdot \sqrt{17}$
- **I** $\sqrt{13} \cdot \sqrt{13}$

7. Use the distributive principle to give a rational number for the product $\sqrt{2} \cdot (\sqrt{2} + \sqrt{8})$.

UTILIZATION

Assign the exercises as independent work. Careful discussion of these exercises should be beneficial to all students.

EXTENSION

Additional practice exercises are provided on page 121 of the *Workbook*.

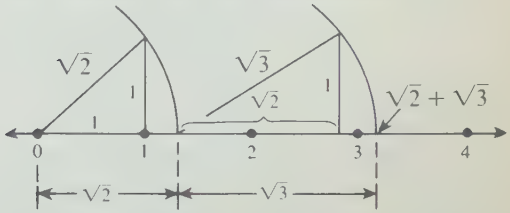
Remedial: Exercise 5 can be used to extend the use of the table of square roots. For example, $\sqrt{115}$ is not found in the table on S–34, but since $115 = 5 \cdot 23$, we can find a decimal approximation as follows:

$\sqrt{115} = \sqrt{5} \cdot \sqrt{23} = 2.236 \cdot 4.796 \approx 10.724$.

Ask students to use the table and this method to find these square roots:

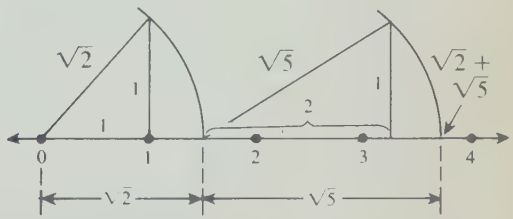
A. $\sqrt{200} = \sqrt{2} \cdot \sqrt{100}$ (14.142)
B. $\sqrt{120} = \sqrt{10} \cdot \sqrt{12}$ (10.954)
C. $\sqrt{136} = \sqrt{8} \cdot \sqrt{17}$ (11.660)
D. $\sqrt{232} = \sqrt{29} \cdot \sqrt{8}$ (15.229)
E. $\sqrt{555} = \sqrt{15} \cdot \sqrt{37}$ (23.559)
F. $\sqrt{1001} = \sqrt{7} \cdot \sqrt{11} \cdot \sqrt{13}$ (31.649)

Enrichment: A sum such as $\sqrt{3} + \sqrt{2}$ cannot be represented by a simpler real number symbol. However we can locate the point for the sum as shown by the drawing below.



Show this method; then challenge students to show how to locate $\sqrt{2} + \sqrt{5}$ by a similar drawing.

Solution:



Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The following statements summarize the main concepts developed in this module. These points need to be stressed as you work through the review material.

1. Using the Pythagorean Theorem, we can establish points on the number line for numbers such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$.
2. It can be proved that these numbers, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, . . . , are not rational numbers. We call them *irrational numbers*.
3. We can find decimals that represent rational numbers close to the square roots. We call these *decimal approximations*.
4. We can build a useful table of decimal approximations to square roots and use it to solve practical problems.
5. Every rational number can be represented by a repeating decimal and every repeating decimal represents some rational number. Irrational numbers, such as $\sqrt{2}$, cannot be represented by repeating decimals, but do have nonending decimal representations. We say that irrational numbers are represented by nonrepeating decimals.
6. The set of *real numbers* includes both the rational numbers (repeating decimals) and the irrational numbers (nonrepeating decimals).

Further exercises which review the major concepts of this module are provided on page 122 of the *Workbook*.

REVIEWING THE IDEAS

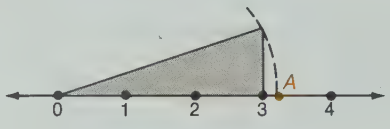
1. A ▒ is the **positive** number that multiplied by itself gives 5.

B ▒ is the **negative** number that multiplied by itself gives 5.

C $\sqrt{13}$ is the ? number that multiplied by itself gives ▒.

D $^{-}\sqrt{13}$ is the ? number that multiplied by itself gives ▒.

2. Give the number for point *A* on the number line.



3. Between what two integers is the point for $^{-}\sqrt{37}$?

4. Give the largest possible integer for ***a***. Give the smallest possible integer for ***b***.

A $\sqrt{12} \cdot \sqrt{12}$

$a \cdot a < 12 < b \cdot b$

B $\sqrt{29} \cdot \sqrt{29}$

$a \cdot a < 29 < b \cdot b$

C $\sqrt{333} \cdot \sqrt{333}$

$a \cdot a < 333 < b \cdot b$

5. Give each square root to the nearest integer.

A $\sqrt{12}$ C $\sqrt{333}$ E $\sqrt{250}$

B $\sqrt{29}$ D $\sqrt{19}$ F $\sqrt{163}$

6. A Which number, 3.6 or 3.7, is closer to $\sqrt{13}$?

B Which number, 7.28 or 7.29, is closer to $\sqrt{53}$?

7. Find a decimal approximation of $\sqrt{18.44}$ to the nearest tenth.

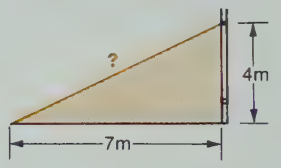
8. Find a decimal approximation of $\sqrt{163}$ to the nearest hundredth.

9. These numbers are approximations of the square roots of certain whole numbers. Find these numbers without using a table of square roots.

A 1.7 C 7.21

B 3.16 D 2.646

10. The wire braces the post. Find the length of the wire to the nearest tenth of a metre.



11. Write a repeating decimal for $\frac{6}{7}$.

12. Find a fraction for the number $.23\overline{23}$.

13. A Write a nonrepeating decimal.

B Does it represent a rational number or an irrational number?

14. Find the product $(2 \cdot \sqrt{7}) \cdot \sqrt{7}$.

15. Give a decimal approximation for $3 \cdot \sqrt{5}$.

16. The product $\sqrt{3} \cdot \sqrt{11}$ is the square root of what number?

17. Write the decimal for a real number between 4.8629 . . . and 4.8643 . . . .

★ **18.** Find the length of the diagonal of a square whose sides are 12 centimetres long.

TEST YOURSELF

1. What positive number squared is equal to 5?

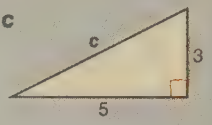
2. The symbol $-\sqrt{2}$ represents a number whose square is ▨.

3. Which number is the best approximation of the positive square root of 5?
   [A] 2.1 [B] 2.2 [C] 2.3

4. Find $\sqrt{34.76}$ to the nearest tenth.

5. Give the co-ordinate of point *P* on the number line.

6. Find the length of the hypotenuse **c** of the right triangle.



7. What is the repeating decimal for $\frac{5}{6}$?

8. What rational number in fractional form is represented by $0.\overline{4}$?

9. The union of the set of rational numbers and the set of irrational numbers is the set of ? numbers.

10. The number **x** has a nonrepeating decimal representation. Is **x** a rational or an irrational number?

RESEARCH PROJECTS

A Measure the length and width of your mathematics classroom. Then, by using the Pythagorean Theorem, determine the length of the diagonal of the room. Check your computation by measuring the diagonal length of your classroom. Find the height of your classroom. Use the height and the diagonal length of the floor to compute the diagonal length from one corner of the floor of your classroom to the diagonally opposite corner of the ceiling.

B Do you think there are other kinds of numbers that are not real numbers? You may be surprised to find that there is no real number **x** such that $x^2 = -1$. Mathematicians have invented complex numbers or imaginary numbers so that such equations can be solved. Find out some things about imaginary numbers and give a report to the class or write a paper about them. (See *Numbers: Fun and Facts* by John Newton Friend; New York: Charles Scribner's Sons, 1954, pp. 34-41; John Wiley.)

TEST YOURSELF

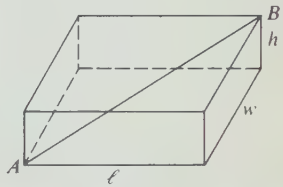
This self-evaluation test should enable students to assess how well they understand the principal concepts presented in the module. Students who perform well on this test should also do well on a module achievement test.

When students have completed the exercises, they should check their work against the answers provided on page H–72. Encourage them to refer back to appropriate lessons within the module to review any topics that proved troublesome for them.

Note that correlations between individual test items and the module objectives are indicated by annotations on the student text page.

RESEARCH PROJECTS

In Research Project A, if l, w, and h are the length, width, and height, respectively, of a rectangular prism, then the distance d, to diagonally opposite corners such as A and B is $d = \sqrt{l^2 + w^2 + h^2}$.



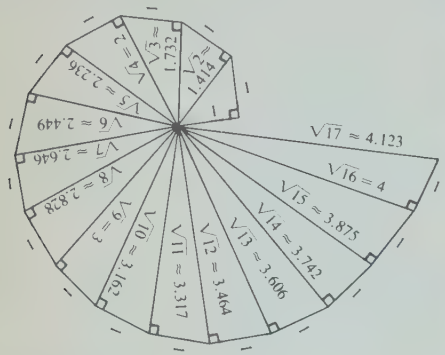
In discussing Project B, note that the term *real numbers*, as applied to the set of rationals and irrationals, is in many respects unfortunate. It gives the uninitiated the impression that only the rationals and irrationals can be considered true numbers and that other sorts of numbers are somehow "artificial" or "synthetic."

No one type is any more "real" or "imaginary" than the others, but they are different. While the origin of the real numbers goes back into antiquity, imaginary numbers are a relatively recent addition to the number family, having come into use only in the last three centuries.

See *The Giant Golden Book of Mathematics* by Irving Adler (Golden Book Press, 1960, 1958, pp. 44–45) for a brief but readable account of imaginary numbers. (Available from Fitzhenry and Whiteside Ltd., Don Mills, Ontario.)

MATHEMATICAL RECREATION

There are 16 triangles that can be constructed without overlapping. The total angular measure about the central point is a little more than 352°. If a careful construction of this figure is made on a piece of posterboard and the triangular regions are colored in various colors, a striking spiral design will result.



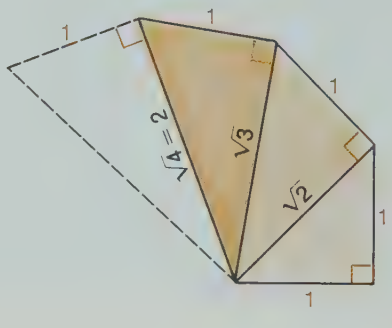
Possible answers:

1. See figure above.
2. For a figure of 16 triangles (as shown above), the perimeter is
$$17 + \sqrt{17} \approx 17 + 4.123 \approx 21.123$$
3. The area is $\frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot \sqrt{2} + \frac{1}{2} \cdot 1 \cdot \sqrt{3} + \ldots + \frac{1}{2} \cdot 1 \cdot \sqrt{17}$
$= \frac{1}{2} \cdot 1 \,(1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{17})$
$= \frac{1}{2} \,(1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{17})$
$= \frac{1}{2} \,(48.594)$
$= 24.297$

MATHEMATICAL RECREATIONS

A SQUARE ROOT SPIRAL

How many right triangles like these can you draw before they start to overlap? The outer leg is always 1 unit long and the other leg is the hypotenuse of the preceding right triangle.



1. Try to find the length of the hypotenuse of each triangle you drew.
2. Can you find the total perimeter of the polygon you constructed?
3. Can you find the total area of the polygon?

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. $\sqrt{5}$ 2. 2 3. B 4. 5.9 5. $\sqrt{5} \approx 2.236$
6. $\sqrt{34} \approx 5.8$ 7. $0.8\overline{3}$ 8. $\frac{4}{9}$ 9. real
10. irrational

UNIT H: SOME SPECIAL TOPICS IN MATHEMATICS

Module 4: Other Topics in Mathematics

General Objectives

To introduce the student to some finite mathematical systems.
To compare basic principles for real numbers with similar principles in other mathematical systems.
To investigate the properties of congruent integers, modulo 4.
To introduce the system of vectors in a plane and the operation of addition of vectors.
To examine the logic of electrical circuits.
To stimulate interest in mathematics.

	Performance Objectives	Pupil Text	Reteach-Reinforce	Related Activities
126	**YELLOW** Given two integers in modulo 4, the student can determine whether they are congruent.	H-74, 75	**WB** 123	
127	**YELLOW** The student can find sums and products in modulo 4.	H-76 through 79	**WB** 124, 125 **DM** 80, 81	**DS** e-1 through 4
128	**YELLOW** Given problems involving an octic mathematical system (rotations and flips of a square), the student can solve them.	H-80, 81 H-82, 83	**WB** 126	
129	**YELLOW** Given a vector, the student can find the magnitude (length) of the vector and can find the sum of two vectors.	H-84, 85 H-86, 87	**WB** 127 **DM** 82	
130	***GREEN*** *Given problems involving series and parallel electrical circuits, the student can solve them.*	H-88, 89 H-90, 91		
	Reviewing the Ideas	H-92	**WB** 128	

MATHEMATICS

There are four separate mathematical topics introduced in the module.

There are many modular arithmetic systems; the one introduced here is arithmetic of 4 elements, or a modulo 4 arithmetic.

If n is a whole number greater than 1, then the relation *is congruent to* ($\equiv$) is defined on the set of integers as follows:

> If a and b are any pair of integers, then $a \equiv b \pmod{n}$ if and only if $a - b$ is a multiple of n.

The relation denoted by "$\equiv$" partitions the set of integers into n disjoint sets or equivalence classes. For $n = 4$, we have the equivalence classes $\mathbf{0} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$, $\mathbf{1} = \{\ldots -7, -3, 1, 5, 9, \ldots\}$, $\mathbf{2} = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$, $\mathbf{3} = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$. Using the four elements 0, 1, 2, 3, we define operations of "addition" and "multiplication" on these elements and observe that these new operations on these objects have many of the properties of the arithmetic of real numbers. Thus in congruence arithmetic, modulo 4, we observe that there are identity elements for both addition and multiplication; the commutative, associative, and distributive principles hold; and each element has an "additive" inverse, but not all elements have multiplicative inverses.

A second mathematical system studied in this module is an 8-element system based on the rotations of a square around its center and the reflections of the square (flips) about its four lines of symmetry. This "octic system," obtained by combining rotations and flips, has an identity element and is associative, and each element has an inverse element with respect to the operation "followed by" ($\bigstar$). This finite mathematical system is particularly interesting because it can be shown that the commutative principle does not hold for the operation "followed by." Technically, this system is an example of a noncommutative group.

Vectors are introduced as ordered pairs of real numbers which are represented by arrows emanating from the origin. The magnitude of a vector (x, y) is the length of the vector $\sqrt{x^2 + y^2}$. Addition of vectors is defined and graphically illustrated by the "parallelogram law" of addition. The set of vectors is closed under the operation addition, the commutative and associative principles hold for addition, there is an identity element for addition, and each vector has an additive inverse. Mathematically, the set of vectors in the plane with the operation of addition constitute a mathe-

matical system called a *group*. This term is not introduced to the students.

The final special topic in the module deals with the switching networks of electrical circuits. This topic introduces the students to some elementary aspects of Boolean algebra and is closely related to logic. Series and parallel circuits are considered, and mathematical models are constructed to analyze these circuits.

TEACHING THE MODULE

Materials

Cardboard for a trundle wheel, adding machine tape, paper strips, small cardboard squares (one per student), graph paper, 1.5 volt dry cell, model door bell, two knife switches, connecting wire.

Vocabulary

arrow
associative
commutative
congruent integers
flip
identity
inverse
magnitude
modulo
null vector
octic mathematical system
parallel
rotation
series
vector
zero vector

Since a central objective of this module is to stimulate interest, you should take care not to be overly formal in your approach. The special topics in this module are closely connected to number theory, abstract algebra, and logic, but they should be approached intuitively, in a manner that will whet the mathematical appetities of your students. However, there are other than purely motivational purposes that justify the introduction of other mathematical systems in this module. First, these systems often contain structures quite similar to those of the more familiar number systems, so that by examining the structure of finite systems the structure of the infinite systems will be better understood and appreciated. Second, the computational "facts" are quite limited; therefore the student can concentrate more on the structure of the system, rather than spend an inordinate amount of time learning to compute in the system. Third, the introduction of such material will provide an introduction to some work in simple abstractions of mathematics that may be useful to some students in later work. Nonetheless, probably the most important justification is that the material will stimulate in many students a deeper curiosity about the many diverse mathematical paths that can be explored in future studies.

Each of the topics introduced in this module can be taught independent of the others, if you prefer. Students should not be expected to master a great many facts or become skillful with the operations introduced in the module. If students gain insight into the structure of mathematics through their work in the module, if they find these new systems intriguing and motivating, the main objective of the module will have been accomplished.

Lesson Schedule

Coverage of the complete module will probably require a minimum of 8 days. Since the material will be new, you may find that progress is slower than normal and extra time will be needed.

Evaluation

You should not be greatly concerned about development of specific computation skills with the operations of each of the mathematical systems of this module. You might even consider an "open book" examination in which students have access to the tables of operations in the module. You can use the review questions and the "Test Yourself" page as models for any teacher-made evaluation device, or you may choose to administer the achievement test that is available as part of the evaluation program for this series.

RESOURCES FOR ACTIVE LEARNING

Activities

Aftermath 3, pp. 32, 40; *Aftermath 4*, pp. 13, 39–40, 56–58, 88–89, Creative Publications

Boxes, Squares and Other Things (motions), pp. 23–40, NCTM

Cloudburst, Vol. 5 (topology), Nos. 8500 series, Midwest Publications

Creative Geometry Experiments (topology), Nos. 18–20, 25, Midwest Publications

Developmental Math Cards, L14, L15, L18, Addison-Wesley

Experiments in Mathematics, Stage 3, pp. 22–23, Houghton Mifflin

Geoboard Activity Cards, "Vectors," Geocards 141–142, Cuisenaire Co.

Mathematics Around the Clock, Rand McNally

Mathematics: Man's Key to Progress, Book B (topology), pp. 73–83, Rand McNally

Measure and Find Out, Book 3, "Measuring Electricity," pp. 37–78, Scott, Foresman

Nuffield Project: *Computation and Structure 4* "Modular Arithmetic," pp. 13–21; *Problems*—Red Set, Nos. 3, 9B, 10–12, 20–A, 24, 26; *Shape and Size 4* (vectors and topology), pp. 11–23, 45–47, Wiley

Puzzles and Graphs, NCTM

Sourcebook for Substitutes, pp. 81, 92–93, Addison-Wesley

Manipulative Devices

Geoboards (Addison-Wesley)

Orbit Materials for Mathematics (Webster, McGraw-Hill)

Trundle Wheels (Addison-Wesley.)

Games and Puzzles

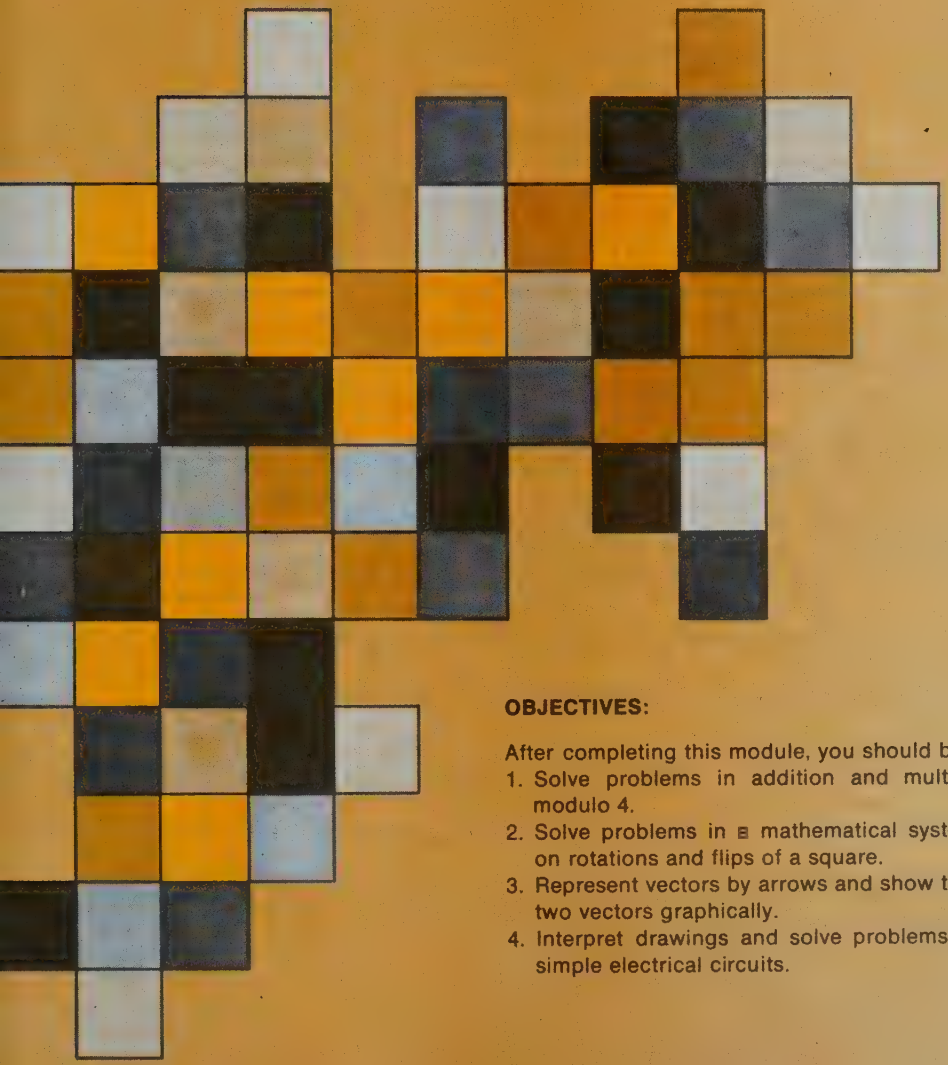
Operational Systems Games—clock arithmetic and modular games (Webster, McGraw-Hill)

Psyche-Paths (Cuisenaire Co.)

Vector (Cuisenaire Co.)

UNIT H: *Some Special Topics in Mathematics*

MODULE 4: *Other Topics in Mathematics*



OBJECTIVES:

After completing this module, you should be able to:

1. Solve problems in addition and multiplication, modulo 4.
2. Solve problems in a mathematical system based on rotations and flips of a square.
3. Represent vectors by arrows and show the sum of two vectors graphically.
4. Interpret drawings and solve problems involving simple electrical circuits.

Objective

Given two integers, the student can decide whether their difference is a multiple of 4.

PREPARATION

Materials: Cardboard trundle wheel, adding machine tape or strips of paper.

Successful work in the lesson will depend upon students' ability to work with integers, particularly subtraction of integers. Therefore, it would be helpful to review the integers and operations with integers.

The Investigation will be enhanced if you or some students would make a simple cardboard trundle wheel. A pencil pushed through the centre of the wheel can be used to help roll the wheel. Make an integer number line on a piece of adding machine tape by rolling the wheel along the line and marking points where 0, 1, 2, and 3 touch the line.

INVESTIGATION

If students actually make the trundle wheel, the pairing of 0, 1, 2, and 3 with the various integers will be easily seen (Discussion Exercise 2).

DISCUSSION

As you discuss the questions, you will want to emphasize these two points.

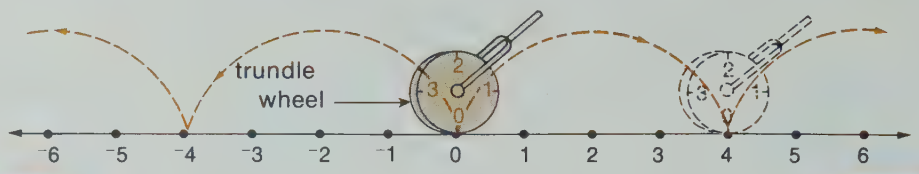
1. Each integer falls into exactly one of the four sets listed in Exercise 2.
2. If two integers are in the same set, their difference is a multiple of 4, and conversely.

The notation for congruent integers, introduced in Exercise 4, will be new to most students. The symbol $\equiv$ is used to distinguish it from the usual equality symbol or geometric symbol for congruence. The *modulus* of two integers is an integer which divides the difference of two numbers exactly. The modulus in this lesson is 4.

Congruent Integers, Modulo 4

Investigating the Ideas

The trundle wheel starts on the number line with the zeros aligned and rolls back and forth. The wheel is designed so that 0 on the trundle wheel will touch the points for each integer in the set $\{\ldots, -16, -12, -8, -4, 0, 4, 8, 12, 16, \ldots\}$.



Can you list several positive and negative integers in the sets that match with 1, 2, and 3 on the trundle wheel?

Discussing the Ideas

1. The numbers in the set that match with 0 on the trundle wheel are the integers that are multiples of 4. Describe the pattern for the numbers in the sets for 1, 2, and 3.

2. Suppose we designate the four sets above by the four trundle wheel numbers.
   - A Is every integer in exactly one of the four sets?
   - B Is the difference between any two numbers in the same set a multiple of 4?

   $0 \leftrightarrow \{\ldots, -8, -4, 0, 4, 8, \ldots\}$
   $1 \leftrightarrow \{\ldots, -7, -3, 1, 5, 9, \ldots\}$
   $2 \leftrightarrow \{\ldots, -6, -2, 2, 6, 10, \ldots\}$
   $3 \leftrightarrow \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

3. The integers -15 and 5 are in the same set, the 1 set, because $5 - (-15) = 20$ and 20 is a multiple of 4. Which of these pairs are in the same set?
   A 21, 1 B 3, -14 C -2, -34 D 105, -7

4. We say that 18 **is congruent to** -6, **modulo 4** because their difference is a multiple of 4: $18 - (-6) = 24$.
   We write: $18 \equiv -6 \pmod 4$
   Similarly, $-11 \equiv -3 \pmod 4$ because $-11 - (-3) = -8$ and -8 is a multiple of 4.

   Which of these statements are true and which are false?
   A $1 \equiv -7 \pmod 4$ C $16 \equiv -8 \pmod 4$ E $-47 \equiv 1 \pmod 4$
   B $15 \equiv -9 \pmod 4$ D $0 \equiv 100 \pmod 4$ F $23 \equiv -51 \pmod 4$

Using the Ideas

The exercises on this page refer to these four sets.

0 ↔ {. . ., $^-12, ^-8, ^-4, 0, 4, 8, 12$, . . .}

1 ↔ {. . ., $^-11, ^-7, ^-3, 1, 5, 9, 13$, . . .}

2 ↔ {. . ., $^-10, ^-6, ^-2, 2, 6, 10, 14$, . . .}

3 ↔ {. . ., $^-9, ^-5, ^-1, 3, 7, 11, 15$, . . .}

1. In which set is the given integer?

A 19	**C** 26	**E** $^-101$	**G** $^-16$	**I** $^-1002$
B $^-14$	**D** 44	**F** 43	**H** 201	**J** 77

2. Find the differences.

A $6 - {}^-2$	**C** $17 - 8$	**E** $13 - 53$	**G** $^-42 - 2$
B $^-3 - 21$	**D** $^-1 - 27$	**F** $10 - {}^-20$	**H** $^-53 - 7$

3. Tell whether each statement is true or false. Use Exercise 2 to help you decide.

A $6 \equiv {}^-2 \pmod 4$	**C** $17 \equiv 8 \pmod 4$	**E** $13 \equiv 53 \pmod 4$
B $^-3 \equiv 21 \pmod 4$	**D** $^-1 \equiv 27 \pmod 4$	**F** $10 \equiv {}^-20 \pmod 4$

4. Copy and complete each statement with one of the integers 0, 1, 2, or 3.

A $9 \equiv$ ▮ (mod 4)	**D** $27 \equiv$ ▮ (mod 4)	**G** $^-39 \equiv$ ▮ (mod 4)
B $^-6 \equiv$ ▮ (mod 4)	**E** $^-19 \equiv$ ▮ (mod 4)	**H** $^-40 \equiv$ ▮ (mod 4)
C $^-16 \equiv$ ▮ (mod 4)	**F** $76 \equiv$ ▮ (mod 4)	**I** $^-41 \equiv$ ▮ (mod 4)

5. Complete each sentence.

A Every number in set ? is congruent to 0, modulo 4.
B Every number in set ? is congruent to 1, modulo 4.
C Every number in set ? is congruent to 2, modulo 4.
D Every number in set ? is congruent to 3, modulo 4.

★**6.** Find the sum of the two integers. Then tell whether the sum is congruent to **0**, **1**, **2**, or **3**.

EXAMPLE 1: $7 + {}^-16 = {}^-9$, $^-9 \equiv$ **3** (mod 4)

EXAMPLE 2: $^-14 + {}^-8 = {}^-22$, $^-22 \equiv$ **2** (mod 4)

A $12 + {}^-5$	**C** $8 + 7$	**E** $23 + {}^-47$	**G** $38 + 52$
B $^-10 + {}^-6$	**D** $^-5 + 15$	**F** $^-19 + {}^-2$	**H** $64 + {}^-32$

UTILIZATION

The exercises can be assigned as independent work and then discussed when students have completed them. The problems offer an excellent opportunity to review operations with integers. You should refer to the four sets at the top of the page as "the zero set," "the one set," "the two set," and "the three set."

EXTENSION

Further appropriate practice exercises are provided on *Workbook* page 123.

Enrichment: Although this lesson deals only with modulo 4, some students may be interested in other moduli. If so, refer them to Research Project A on page H-93, which suggests a modulo 7 arithmetic. This system is often called "calendar arithmetic" because the difference between any given day of one week and the same day of another week is a multiple of 7. For example, the 5th day of each month and the 19th day of each month fall on the same day of the week because the difference $19 - 5$ is a multiple of 7; that is, $19 \equiv 5 \pmod 7$.

Assignments

Minimum 1, 2A–D, 3A–D, 4.
Average 1–5. Maximum 1–6.

Objectives

The student can find sums and solve addition equations in the modulo 4 system.

The student can compare properties of modulo 4 arithmetic with the arithmetic of real numbers.

PREPARATION

Spend a few minutes reviewing congruent integers, modulo 4, which was presented in the previous lesson. You may also want to review addition of positive and negative integers, which play an important role in this lesson.

INVESTIGATION

Suggest that students copy and complete each equation; it will be helpful to have these equations available during the Discussion stage. Each problem requires two steps. First, the sum of the two integers must be found. Then the sum must be shown to be congruent to either 0, 1, 2, or 3. Thus, for 1 the student might show the following:

$$^{-}3 + 9 = 6,\ 6 \equiv 2 \pmod 4$$

DISCUSSION

The main purpose of the Discussion is to show that, instead of working with the complete set of integers, we may simply restrict ourselves to the four modulo 4 integers—0, 1, 2, 3.

Exercise 4 defines addition in modulo 4. The symbol $\oplus_4$ is used to denote this operation. Although this is not a standard symbol, we use it here to distinguish this kind of addition from the ordinary addition operation with real numbers.

When discussing Exercise 5, students may suggest such things as commutativity and the property of zero as being shared by modulo 4 addition and addition of real numbers. However, the equation $2 \oplus_4 2 = 0$ suggests some properties of modulo 4 addition which are quite different from real number addition. Equations such $3 \oplus_4 2 = 1$ may cause students to wonder if 1 (mod 4) is greater than 3 (mod 4). It is not possible to order modulo 4 integers in a meaningful way.

Addition, Modulo 4

Investigating the Ideas

$0 \leftrightarrow \{\ldots, ^{-}8, ^{-}4, 0, 4, 8, \ldots\}$ $\quad$ $2 \leftrightarrow \{\ldots, ^{-}6, ^{-}2, 2, 6, 10, \ldots\}$

$1 \leftrightarrow \{\ldots, ^{-}7, ^{-}3, 1, 5, 9, \ldots\}$ $\quad$ $3 \leftrightarrow \{\ldots, ^{-}5, ^{-}1, 3, 7, 11, \ldots\}$

Can you complete each equation below by using only the four integers 0, 1, 2, or 3?

1. $(^{-}3 + 9) \equiv \underline{?} \pmod 4$
2. $(5 + ^{-}2) \equiv \underline{?} \pmod 4$
3. $(32 + ^{-}20) \equiv \underline{?} \pmod 4$
4. $(7 + 10) \equiv \underline{?} \pmod 4$
5. $(11 + ^{-}5) \equiv \underline{?} \pmod 4$
6. $(^{-}7 + ^{-}1) \equiv \underline{?} \pmod 4$
7. $(14 + ^{-}2) \equiv \underline{?} \pmod 4$
8. $(^{-}4 + 14) \equiv \underline{?} \pmod 4$
9. $(13 + ^{-}4) \equiv \underline{?} \pmod 4$
10. $(^{-}5 + ^{-}8) \equiv \underline{?} \pmod 4$

Discussing the Ideas

1. A In Equation 1 above, in which set was $^{-}3$?
   B In which set was 9?
   C In which set was $(^{-}3 + 9)$?
   D If any two numbers from set 1 are chosen, in which set is their sum?

2. A In Equation 2, in which sets were the two addends 5 and $^{-}2$?
   B In which set is their sum?

3. If you know one integer is in 1 and a second integer is in 2, in which of the four sets is the sum of the integers?

4. Suppose that an operation of **addition**, **modulo 4**, is defined on the set $\{0, 1, 2, 3\}$ and that $\oplus_4$ designates the operation.

   We write $2 \oplus_4 1 = 3$ because the sum of any number in set 2 and any number in set 1 is a number in set 3.

   A What does this equation mean? $1 \oplus_4 3 = 0$
   B What is $3 \oplus_4 2$?
   C What is $2 \oplus_4 0$?

5. How is modulo 4 addition like ordinary addition? How is it different?

Using the Ideas

1. Copy and complete the addition table for modulo 4.

$\oplus_4$	0	1	2	3
0			2	
1				0
2		3		
3			1	

2. Use the table to solve each equation in modulo 4.
   - A $1 \oplus_4 2 = x$
   - B $2 \oplus_4 2 = y$
   - C $3 \oplus_4 z = 0$
   - D $r \oplus_4 3 = 1$
   - E $2 \oplus_4 n = 2$
   - F $3 \oplus_4 t = 2$

3. Is the operation $\oplus_4$ commutative?

4. Does 0 in modulo 4 have the same property as 0 in ordinary addition? That is, is 0 the identity element for addition?

5. Find the modulo 4 solution for each equation.
   - A $0 \oplus_4 p = 0$
   - B $1 \oplus_4 q = 0$
   - C $2 \oplus_4 r = 0$
   - D $3 \oplus_4 s = 0$

6. In modulo 4, 1 and 3 are additive inverses because $1 \oplus_4 3 = 0$.
   - A Give the additive inverse of 2.
   - B What is the inverse of 0?

7. One interpretation of modulo 4 addition can be made by using "clock arithmetic." Using a clock with the four numbers as shown, you can write $3 \oplus_4 2 = 1$ because 2 hours after 3 o'clock would be 1 o'clock.

   Complete each statement using the modulo clock.
   - A 1 hour after 2 o'clock will be ▒ o'clock.
   - B 2 hours after 1 o'clock will be ▒ o'clock.
   - C 3 hours after 3 o'clock will be ▒ o'clock.
   - D 2 hours after 2 o'clock will be ▒ o'clock.
   - E 3 hours after 0 o'clock will be ▒ o'clock.

8. Write equations using modulo 4 addition for each statement in Exercise 7.

UTILIZATION

The addition table accompanying Exercise 1 should be referred to freely as students do the exercises. It is not necessary that students memorize the table, although many students will be able to do so.

One of the objectives of this lesson is to compare modulo 4 arithmetic with ordinary arithmetic of real numbers. For addition, the properties of commutativity, associativity, the zero principle and additive inverse principle hold for both systems.

Exercises 7 and 8 relate modulo 4 to a kind of "clock arithmetic."

EXTENSION

Additional practice exercises are provided on *Workbook* page 124 and *Duplicator Masters* page 80.

Enrichment: Military forces use a 24 hour clock, with noon "12 o'clock" (1200 hours) and midnight "24 o'clock" (2400 hours). Encourage some students to investigate this system and to develop a "24 clock" addition table.

Because each integer, modulo 4, has an additive inverse, subtraction can be defined in the modulo 4 system. If we use $\ominus_4$ to denote the operation, we can write

$3 \ominus_4 1 = 2$ because $2 \oplus_4 1 = 3$
$2 \ominus_4 3 = 3$ because $3 \oplus_4 3 = 2$
$0 \ominus_4 1 = 3$ because $3 \oplus_4 1 = 0$

Create some subtraction problems and have students solve the problems.

Assignments

Minimum 1–5. Average 1–6. Maximum 1–8.

Objectives

The student can find products of integers, modulo 4.

The student can compare properties of multiplication, modulo 4, with multiplication of real numbers.

PREPARATION

A quick review of multiplication of integers would be helpful for students before they begin the Investigation.

INVESTIGATION

Students should copy and complete each equation. This is a two-stage process. For example, for part 1, $2 \times 3 = 6$ and $6 \equiv 2 \pmod 4$. Hence, $2 \times 3 \equiv 2 \pmod 4$. The equations will be used during the Discussion stage.

DISCUSSION

The purpose of the Discussion is to show that, rather than working with all the integers in multiplication, we can define a kind of multiplication on the set of integers in modulo 4. Since the product of any two integers in set 2 is found in set 0, in modulo 4 we define $2 \otimes_4 2 = 0$, where $\otimes_4$ refers to modulo 4 multiplication.

You may prefer that students complete the multiplication table for Exercise 1 on pages H–79 at this time.

Answers, Discussion Exercise 2

Part 2. $^-2$ is in set 2; 2 is in set 2; $^-2 \times 2 \equiv 0 \pmod 4$.
Part 3. $^-3$ is in set 1; 3 is in set 3; $^-3 \times 3 \equiv 3 \pmod 4$.
Part 4. 4 is in set 0; 3 is in set 3; $4 \times 3 \equiv 0 \pmod 4$.
Part 5. 6 is in set 2; $^-4$ is in set 0; $6 \times ^-4 \equiv 0 \pmod 4$.
Part 6. 5 is in set 1; $^-3$ is in set 1; $5 \times ^-3 \equiv 1 \pmod 4$.
Part 7. 8 is in set 0; 12 is in set 0; $8 \times 12 \equiv 0 \pmod 4$.
Part 8. 7 is in set 3; $^-1$ is in set 3; $7 \times ^-1 \equiv 1 \pmod 4$.

Multiplication, Modulo 4

Investigating the Ideas

How many of these equations can you complete using only the integers 0, 1, 2, or 3?

1. $2 \times 3 \equiv \underline{?} \pmod 4$
2. $^-2 \times 2 \equiv \underline{?} \pmod 4$
3. $^-3 \times 3 \equiv \underline{?} \pmod 4$
4. $4 \times 3 \equiv \underline{?} \pmod 4$
5. $6 \times ^-4 \equiv \underline{?} \pmod 4$
6. $5 \times ^-3 \equiv \underline{?} \pmod 4$
7. $8 \times 12 \equiv \underline{?} \pmod 4$
8. $7 \times ^-1 \equiv \underline{?} \pmod 4$

Discussing the Ideas

1. **A** In Equation 1 above, in which of the four sets 0, 1, 2, or 3 is 2?
   B In which set is 3?
   C In which set is 2×3?

   $0 \leftrightarrow \{\ldots, ^-8, ^-4, 0, 4, 8, \ldots\}$
   $1 \leftrightarrow \{\ldots, ^-7, ^-3, 1, 5, 9, \ldots\}$
   $2 \leftrightarrow \{\ldots, ^-6, ^-2, 2, 6, 10, \ldots\}$
   $3 \leftrightarrow \{\ldots, ^-5, ^-1, 3, 7, 11, \ldots\}$

2. Repeat question 1**A–C** using the factors in the other equations above.

3. If an integer is from set 2 and another integer is from set 1, in which set is the product?

4. If two integers are both from set 2, in which set is the product?

5. The examples above suggest a kind of "multiplication" for integers, modulo 4. The operation is designated by $\otimes_4$.

Any integer in 3		Any integer in 2		An integer in 2
3	$\otimes_4$	2	=	2

   Explain these products.
   A $2 \otimes_4 2 = 0$ **B** $3 \otimes_4 1 = 3$ **C** $0 \otimes_4 3 = 0$

Using the Ideas

1. Copy and complete the multiplication table for modulo 4.

$\otimes_4$	0	1	2	3
0				0
1				
2			0	
3			2	

2. Use the table to solve each equation in modulo 4.
   - A $1 \otimes_4 2 = x$
   - B $2 \otimes_4 2 = y$
   - C $1 \otimes_4 z = 3$
   - D $3 \otimes_4 t = 1$
   - E $s \otimes_4 1 = 0$
   - F $2 \otimes_4 3 = n$

3. Is the operation $\otimes_4$ a commutative operation?

4. The equation $2 \otimes_4 k = 0$ has **two** solutions in modulo 4: $k = 2$ or $k = 0$. What are the solutions for $2 \otimes_4 m = 2$?

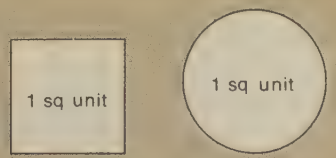
5. In ordinary arithmetic, the identity element for multiplication is 1 because for any real number n, $n \cdot 1 = n$. What is the identity element for modulo 4 multiplication?

6. The rational numbers $\frac{3}{2}$ and $\frac{2}{3}$ are multiplicative inverses because $\frac{3}{2} \cdot \frac{2}{3} = 1$.
   - A Does each modulo 4 integer 0, 1, 2, and 3, have a multiplicative inverse?
   - B Give the multiplicative inverses of the modulo 4 integers which have inverses.

★7. Perform the indicated operations.
   - A $2 \otimes_4 (3 \oplus_4 2)$
   - B $(2 \otimes_4 3) \oplus_4 (2 \otimes_4 2)$
   - C $3 \otimes_4 (2 \oplus_4 1)$
   - D $(2 \otimes_4 3) \oplus_4 (1 \otimes_4 3)$
   - E $1 \otimes_4 (0 \oplus_4 2)$
   - F $(1 \otimes_4 0) \oplus_4 (1 \otimes_4 2)$

think

Which is larger, the ratio of the circumference of the circle to the perimeter of the square or the ratio of the area of the circle to the area of the square?



UTILIZATION

The completed table for Exercise 1 should be used freely by students as they do the other exercises. Memorization of the modulo 4 multiplication is not an objective for this lesson.

Exercises 2 through 7 pertain to basic principles of modulo 4 multiplication. The operation $\otimes_4$ is commutative, associative, and distributive over addition, modulo 4, and 1 is the multiplicative identity. However, there is one major difference for modulo 4 multiplication as compared to multiplication of real numbers; this is shown in Exercise 6. The nonzero number 2 (mod 4) does not have a multiplicative inverse. This creates serious problems if we attempt to define division in modulo 4. For example, $2 \ominus_4 2 = 1$, and $2 \ominus_4 2 = 3$ would both be true statements.

EXTENSION

For further practice exercises, see *Workbook* page 125 and *Duplicator Masters* page 81.

Enrichment: Have students investigate a modulo 5 arithmetic system and compare it to modulo 4. In particular, show that each nonzero integer in modulo 5 has a unique multiplicative inverse so that division modulo 5 can be defined and will give unique quotients.

Think Solution

$$\frac{\text{area (circle)}}{\text{area (square)}} = \frac{1}{1}$$

$$\frac{\text{circumference of circle}}{\text{perimeter of square}} = \frac{2\sqrt{\pi}}{4} \approx \frac{2 \times 1.77}{4}$$

$$= \frac{3.54}{4} = 0.885 \text{ and } 0.885 < 1$$

This shows that a circle enclosing the same area as a square will have a shorter boundary than the square.

Assignments

Minimum 1–5. Average 1–6. Maximum 1–7.

Objective

The student can solve problems in a mathematical system based on four rotations around the centre of a square.

PREPARATION

Materials: Small cardboard squares (one per student).

Rotations were studied earlier, in Unit G, Module 2. Review the idea of rotations, centre of rotation, and direction of rotation before referring students to the Investigation.

INVESTIGATION

The Investigation introduces four specific rotations of a square. The outline of the square and the arrow are useful in helping students think about the relation between the square and its rotation image. You might suggest to the students that in performing a combination of rotations, for convenience, they should start the first rotation with the arrow pointing upward. This will enable them to identify the combined effect of the two rotations.

You might have designated students demonstrate some of the combined motions for the other students.

DISCUSSION

Rotation *I* can be thought of as either a 360° rotation or a 0° rotation. Rotation *U* might be remembered as turning the square *U*pside down. *R* and *L* refer to *R*ight (clockwise 90°) and *L*eft (counterclockwise 90°) turns.

The elements *I*, *R*, *U*, and *L* with the operation "followed by" (★), which denotes a combination of the elements, can form a simple mathematical system.

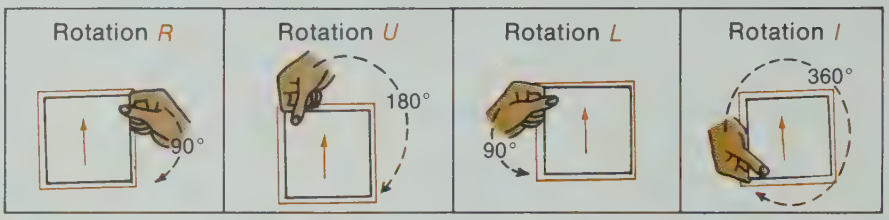
Each student should construct the table in Exercise 3 and use it freely throughout this lesson. Students should not be asked to memorize this table; instead, they should be allowed to use the cardboard square to determine the result of a combination of rotations.

Work through the examples in Exercise 4, making certain that the students interpret the diagram correctly.

The Rotations of a Square

Investigating the Ideas

Cut out a small cardboard square. Draw an outline of the square on a piece of paper. The display below shows four rotations in which the square is turned so that it fits back into the outline again. The arrow on the square shows its starting position.



Can you show that a combination of any two of these rotations will give the same result as one of the four rotations?

Discussing the Ideas

1. Rotation *I* is so named because it is the **identity** rotation. It leaves the square in its original position. How can the other letters *U*, *R*, and *L* help you to remember these rotations?

2. What single rotation produces the same result as a combination of these two rotations?

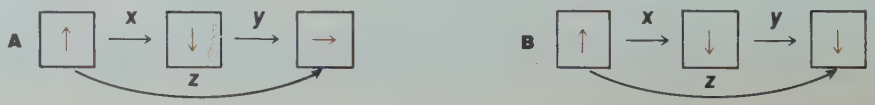
   A *R* followed by *U* **C** *R* followed by *R* **E** *L* followed by *L*

   B *R* followed by *L* **D** *U* followed by *U* **F** *I* followed by *U*

3. Suppose the operation "followed by" is denoted by a star (★). Copy and complete the "followed by" table shown at the right. It has been completed for $U \star R = L$

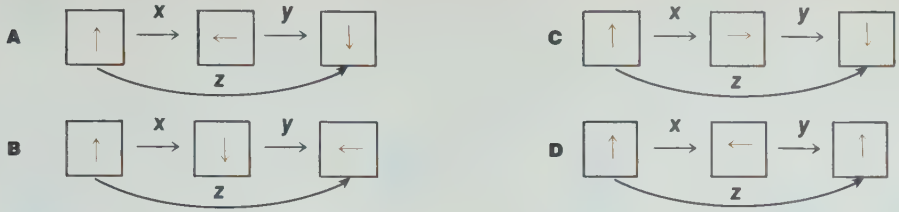
★	*I*	*R*	*U*	*L*
I				
R				
U		*L*		
L				

4. For each part, give the rotations for **x** and **y**. Then give the single rotation for **z** that is the same as the combined rotations.



Using the Ideas

1. Give the rotations for **x** and **y**. Then give the combined rotation **z**.



2. Using elements of the set $\{I, R, U, L\}$ and the operation ★, write an equation for each part of Exercise 1. The equation for part **A** of Exercise 1 is $L \star L = U$

3. Solve each equation. The replacement set is $\{I, R, U, L\}$. Use the table completed in the Discussion or actually use a square to help you find the solution.

A $L \star x = R$ **B** $R \star x = U$ **C** $x \star R = I$ **D** $x \star U = I$ **E** $U \star x = U$ **F** $I \star x = R$ **G** $L \star L = x$ **H** $L \star x = I$ **I** $x \star L = R$

4. Solve each equation for **x** in the *IRUL* system. Use the table of Discussion Exercise 3 to help you.

A $(R \star L) \star U = x$ **B** $R \star (L \star U) = x$ **C** $(I \star R) \star R = x$ **D** $I \star (R \star R) = x$ **E** $(U \star R) \star U = x$ **F** $U \star (R \star U) = x$ **G** $(L \star R) \star L = x$ **H** $L \star (R \star L) = x$

5. What basic principle for the operation ★ is suggested by the equations of Exercise 4?

6. In what way is the identity element I for the operation ★ like an identity element for arithmetic operations?

7. For each equation, give the rotation for **y**.

A $I \star y = I$ **B** $R \star y = I$ **C** $U \star y = I$ **D** $L \star y = I$

8. In the set of integers, each integer has an **additive inverse**. That is, if **x** is any integer, $x + (-x) = 0$.

Does each rotation I, R, U, and L have an **inverse** with respect to the operation ★? Give the inverse of each rotation. (See Exercise 7.)

UTILIZATION

Mathematicians call the finite mathematical system introduced in this section the "group of rotations of a square." Students may simply call it the *IRUL* system.

Exercises 5 through 8 deal with the mathematical properties of the *IRUL* system. Mathematically, the system is an example of a group. That is, the elements I, R, U, L, and the operation ★ are such that commutative and associative principles hold; there is a unique identity element (I), and each element has a unique inverse with respect to the operation ★.

EXTENSION

Enrichment: An additional problem to challenge your students is described below.

A toy soldier is constructed to make four moves:

A. At *A*ttention (Looks straight ahead). This is move *A*.
B. *L*eft face (Turns 90° counterclockwise). This is move *L*.
C. *R*ight face (Turns 90° clockwise). This is move *R*.
D. About face (Turns to the *O*pposite direction). This is move *O*.

1. Make a table for combinations of these moves.
2. Compare this table with the table for the *IRUL* system. (This table has the same form as the *IRUL* system.)

Solution (table)

Let ★ be the operation "followed by."

★	*A*	*L*	*R*	*O*
A	*A*	*L*	*R*	*O*
L	*L*	*O*	*A*	*R*
R	*R*	*A*	*O*	*L*
O	*O*	*R*	*L*	*A*

Assignments

Minimum 1–3. Average 1–6. Maximum 1–8.

Objective

The student can show rotations and flips (reflections) of a square and solve problems related to these motions.

PREPARATION

Materials: Cardboard squares (one per student).

Review the basic ideas of reflections—reflection image and lines of symmetry. (See Unit G, Module 2, pages G–8 and G–9.)

INVESTIGATION

Have the students read the Investigation and practice the four flips illustrated. Point out that the "flips" are simply the reflection images of the square in its four lines of symmetry. Then have them try to answer the Investigation question. It may surprise the students to discover that a combination of any two of the flips is not the same as a single flip, but is equivalent to one of the four rotations *I*, *R*, *U*, or *L*.

DISCUSSION

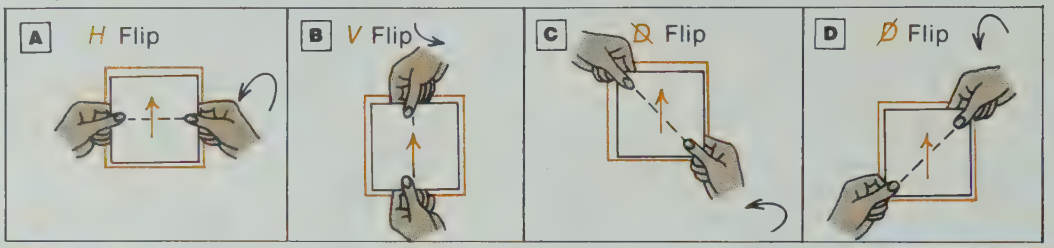
The names of the diagonal flips are arbitrarily designed to aid students in remembering the flips. In order to visualize the results of rotations and flips, students should draw a dotted arrow on the reverse side of their cards that points in the same direction as the solid arrow.

Discuss the exercises carefully, keeping in mind that, since all the material will be new to your students, you need to allow additional time for experimentation and assimilation of the ideas.

Rotations and Flips of a Square

Investigating the Ideas

You can get some new moves of a square by turning, or flipping, a square around its four lines of symmetry.



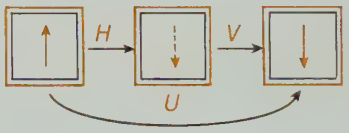
Can you find a combination of two of the flips that will give the same result as one of the four rotations (*I*, *R*, *U*, or *L*)?

Discussing the Ideas

1. The flip is called the "left diagonal" flip and is the "right diagonal" flip. How would you name the flips *H* and *V*?

2. Will the combination of any two flips give the same result as any single flip?

3. The figure shows that an *H* flip followed by a *V* flip is the same as a *U* rotation alone. (The dotted arrow shows the reverse side of the card.)

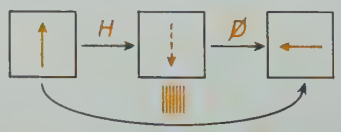


What single rotation could replace these flips?

A *H* followed by
B *V* followed by *V*
C *D* followed by *V*
D *D* followed by
E *H* followed by
F *V* followed by

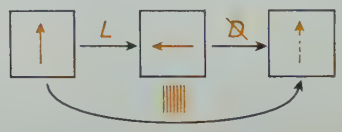
4. Suppose ★ denotes the operation "followed by" for both rotations and flips. Complete each statement and equation.

A



An *H* flip followed by a flip is the same as a rotation alone.
We write: *H* ★ =

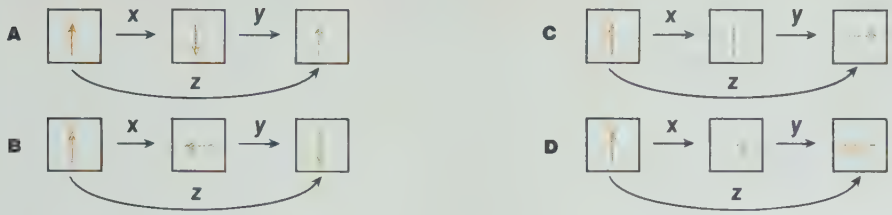
B



An *L* rotation followed by a flip is the same as a flip alone.
We write: *L* ★ =

Using the Ideas

1. Give the rotation or flip for **x** and **y**. Then give the rotation or flip for **z** that gives the same result as **x** followed by **y**.



2. Write equations for each part of Exercise 1 using flip and rotation symbols. The solution for Part A is $\not R \star L = H$.

3. Copy and complete the table for the operation "followed by." In the table, the *IRUL* system has already been completed for you.

★	I	R	U	L	H	V	$\not R$	$\not\varnothing$
I	I	R	U	L				
R	R	U	L	I				
U	U	L	I	R				
L	L	I	R	U				
H								
V								
$\not R$								
$\not\varnothing$								

4. Suppose we call the system consisting of the set of eight elements, $\{I, R, U, L, H, V, \not R, \not\varnothing\}$, and the binary operation ★, the **octic mathematical system**. Which element in the octic system is the **identity element**?

5. Use the table in Exercise 3 to show that ★ is **not** a commutative operation for every pair of elements in the octic system.

6. In the octic system, the element I is the identity element. Show that each element in the system has an inverse element.

   EXAMPLES: Since $\not R \star \not R = I$, the inverse of $\not R$ is $\not R$.
   Since $R \star L = I$, the inverse of R is L.

7. Solve each equation in the octic system.

   A $(U \star H) \star V = x$
   B $U \star (H \star V) = x$
   C $\not R \star (V \star \not\varnothing) = x$
   D $(\not R \star V) \star \not\varnothing = x$
   E $(V \star V) \star R = x$
   F $V \star (V \star R) = x$

8. Do you think that the operation ★ is associative in the octic system? (HINT: Look at the results of Exercise 7.)

★ 9. Solve the equation $I \star R \star L \star U \star V \star H \star \not R \star \not\varnothing = x$.

UTILIZATION

Using the four rotations of a square, the four flips about the lines of symmetry of a square, and the operation "followed by" (★), we construct a mathematical system of eight elements. This eight-element system, called the *octic system*, has a structure that is quite like the structure of the *IRUL* system developed in the previous lesson. The important features of the structure of the octic system are these:

1. The element I is the identity element.
2. The operation ★ is associative.
3. Each element in the octic system has an inverse element.

The same structure was observed in the *IRUL* system, but there is one important difference between the *IRUL* system and the octic system: In the *IRUL* system the operation ★ is found commutative, but in the octic system the operation ★ is not commutative.

In Exercise 5 note that, in order to show that an operation is noncommutative, one needs to show but a single counterexample to commutativity. Thus, since

$$R \star \not R = H \text{ and } \not R \star R = V,$$

the system is noncommutative.

EXTENSION

To provide additional appropriate exercises, make assignments from *Workbook* page 126.

Enrichment: Some students may find Research Projects B and C interesting to pursue at this time.

Assignments

Minimum 1–4. Average 1–7. Maximum 1–9.

Objective

The student can locate points for vectors in the co-ordinate plane and can compute the magnitude of a given vector.

PREPARATION

Materials: Graph paper or co-ordinate grid paper (*Duplicator Masters*, pages 83, 87).

Location of points in the co-ordinate plane and the Pythagorean Theorem play important roles in this lesson. You should review these two concepts with the students before assigning the Investigation.

INVESTIGATION

The Investigation should require only a few minutes for students to complete. The required arrows have endpoints at $(^-3, ^-2)$, $(3, ^-4)$ and $(1, 4)$ for $\vec{a}$, $\vec{b}$, and $\vec{c}$, respectively.

DISCUSSION

The term *arrow* is used as the name for a picture of a vector. A *vector* in the plane (that is, a two-space vector) is an ordered pair of real number (x, y). The *magnitude* or length of a vector (x, y) is $\sqrt{x^2 + y^2}$. In standard texts on vectors, letters in boldface type, such as **a, b, r,** and **v,** are often used to denote vectors. In this intuitive introduction, we use letters with an arrow above them to denote vectors. The context in which the symbols are used will preclude any confusion between symbols that represent vectors and those that represent geometric rays.

To provide additional practice in finding lengths of vectors, have students find the lengths of each vector shown by the arrows in the Investigation. At this point the student can apply the Pythagorean Theorem to show that vector $\vec{a} = (3, 2)$ and its opposite, $\vec{a}^1 = (^-3, ^-2)$, have the same length:

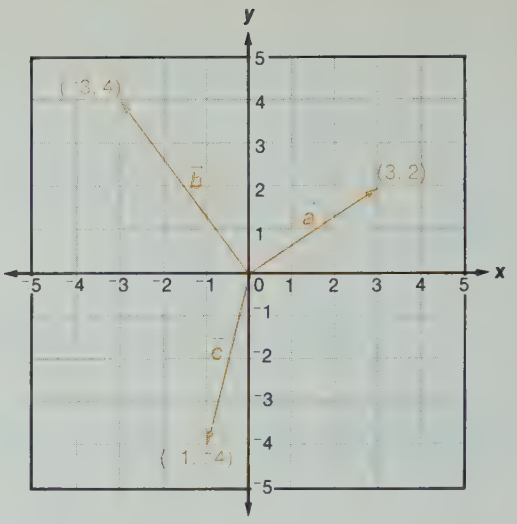
$\ell(\vec{a}) = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\ell(\vec{a}^1) = \sqrt{(^-3)^2 + (^-2)^2} = \sqrt{13}$

Vectors

Investigating the Ideas

Three arrows are drawn from the origin to the points whose co-ordinates are (3, 2), $(^-3, 4)$, and $(^-1, ^-4)$. The arrows are denoted by $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively.



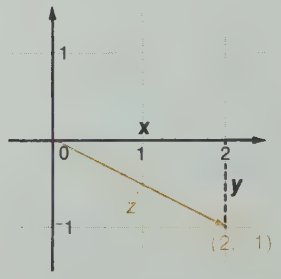
Can you show three more arrows on graph paper that are the same length as these arrows but point in the opposite direction?

Discussing the Ideas

1. What are the co-ordinates of the tips of the arrows you drew? Compare these co-ordinates with the original co-ordinates of ***a, b,*** and ***c.***

2. The mathematical term for an arrow is **vector.** Each vector can be named by giving the co-ordinates of its endpoint (the arrow tip). Thus, $\boldsymbol{a} = (3, 2)$. How would $\vec{b}$ and $\vec{c}$ be designated?

3. The **length** or **magnitude** of a vector can be found by applying the Pythagorean Theorem. The length of $\vec{z} = (2, ^-1)$ is found as follows:



$(\text{length } \vec{z})^2 = \mathbf{x}^2 + \mathbf{y}^2 = 2^2 + (^-1)^2 = 4 + 1 = 5$;

length $\vec{z} = \sqrt{5}$.

What is the length of each of these vectors?

A $\vec{a} = (3, 4)$ B $\vec{b} = (^-2, ^-3)$ C $\vec{c} = (^-5, ^-2)$

4. What vector would have the same length as $\vec{z}$ in Exercise 3 but point in an opposite direction?

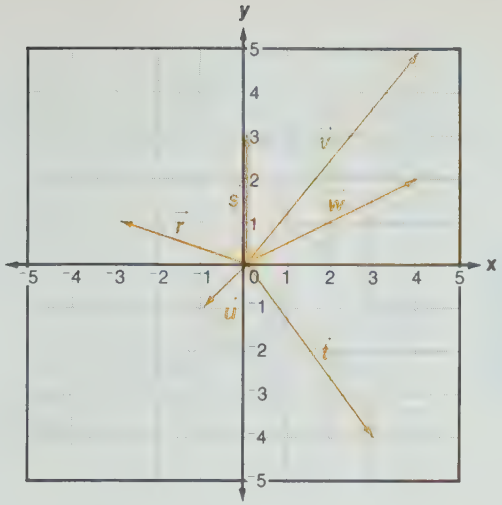
5. The special vector (0, 0) is called the **zero vector** or **null vector**. What is its length?

Using the Ideas

1. Give the ordered pair that names each vector.

   A $\vec{r}$ C $\vec{s}$ E $\vec{t}$

   B $\vec{u}$ D $\vec{v}$ F $\vec{w}$



2. Find the length of each vector in Exercise 1.

3. Show the arrow for each vector on graph paper.

   A $\vec{p} = (^-4, ^-1)$ E $\vec{r} = (0, ^-2)$

   B $\vec{t} = (1, ^-1)$ F $\vec{n} = (^-2, 5)$

   C $\vec{q} = (4, 0)$ G $\vec{s} = (3, 3)$

   D $\vec{k} = (3, ^-4)$ H $\vec{r} = (^-5, 0)$

4. Find the length of each vector in Exercise 3.

5. Which of the two vectors is longer? Compute their lengths, then check by drawing arrows.

   A $\vec{a} = (2, 3)$ or $\vec{b} = (1, 4)$

   B $\vec{c} = (1, 1)$ or $\vec{d} = (2, 0)$

   C $\vec{e} = (6, 8)$ or $\vec{f} = (^-7, 7)$

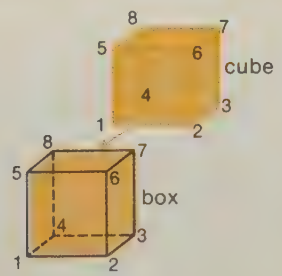
   D $\vec{g} = (5, 0)$ or $\vec{h} = (0, ^-4)$

6. Prove that $\vec{v} = (^-8, 6)$ is twice as long as $\vec{u} = (4, ^-3)$.

7. What vector is three times as long as $\vec{b} = (2, 1)$ and points in the same direction?

think

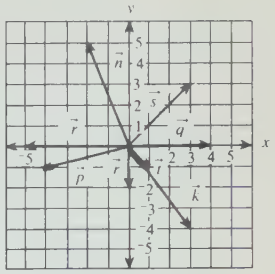
Suppose a cube and an open-top box, into which the cube will just fit, have their vertices numbered as shown. How many different ways can the cube be placed into the box?



UTILIZATION

Solutions, Exercises 3 and 6

3.



6. $\ell(\vec{u}) = \sqrt{(4)^2 + (^-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

   $\ell(\vec{v}) = \sqrt{(^-8)^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$

   Hence $\ell(\vec{v}) = 2 \times \ell(\vec{u})$

EXTENSION

Enrichment: For your more able students, pose the following challenge problem.

Each of these vectors, except one, has a length of 1 unit. Can you find the exception?

A. $\left(\frac{3}{4}, \frac{^-4}{5}\right)$ D. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

B. $(0, ^-1)$ E. $\left(\frac{^-1}{2}, \frac{1}{2}\right)$

C. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ F. $\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$

Solution: E has length $\sqrt{\left(\frac{^-1}{2}\right)^2 + \left(\frac{^-1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} \neq 1$.

Think Solution

Each of the six faces of the cube can fit on the bottom of the box. If the cube is rotated, each face can be turned in four ways. Thus, there are $6 \cdot 4$, or 24, ways to place the cube in the box.

Assignments

Minimum 1–3. Average 1–5. Maximum 1–7.

Objectives

The student can find the sum of two vectors graphically or algebraically.

The student can compare properties of addition of vectors with properties of addition of real numbers.

PREPARATION

Materials: Graph paper or co-ordinate grid paper (*Duplicator Masters*, pages 83, 87).

Review the notation for representation of vectors and give students practice in locating specific vectors on a co-ordinate grid.

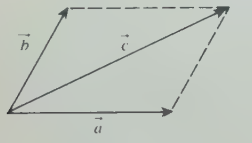
INVESTIGATION

Allow students sufficient time to make the construction called for in the Investigation. They should be careful to make the opposite sides of the quadrilateral parallel to each other so that the completed figure is a parallelogram.

DISCUSSION

In discussing Exercise 1, have students describe any special method they used to make certain that the opposite sides were parallel.

Exercise 2 defines addition of vectors by the "parallelogram" method. If $\vec{a}$ and $\vec{b}$ are vectors, then $\vec{a} \oplus \vec{b} = \vec{c}$, if $\vec{c}$ forms the diagonal of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$. The three vectors emanate from the same point, in this case, the origin $(0, 0)$.



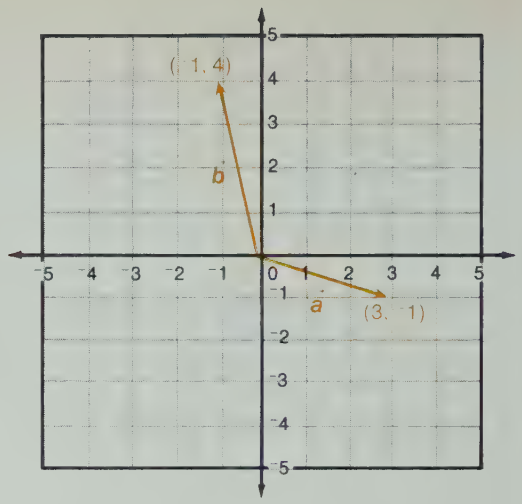
Exercise 3 presents an algebraic method of adding two vectors using ordered pairs to represent vectors.

Addition of Vectors

Investigating the Ideas

Draw vectors $\vec{a}$ and $\vec{b}$ on graph paper. Suppose $\vec{a}$ and $\vec{b}$ are two sides of a parallelogram.

Can you draw the other two sides of the parallelogram?



Discussing the Ideas

1. What are the co-ordinates of the other vertex of the parallelogram in the Investigation?

2. Draw an arrow $\vec{c}$ to the vertex of the parallelogram you found. The vector $\vec{c}$ is called the **sum** of vectors $\vec{a}$ and $\vec{b}$. We denote this as $\vec{a} \oplus \vec{b} = \vec{c}$. The symbol $\oplus$ is used to distinguish it from the addition symbol for numbers. Do you see any numerical relation between $\vec{a} = (3, -1)$, $\vec{b} = (-1, 4)$, and the ordered pair for $\vec{c}$? Describe the relationship.

3. Lisa said that she found the sum of two vectors by adding the numbers in the ordered pairs together.

$\vec{a} = (-3, 2)$

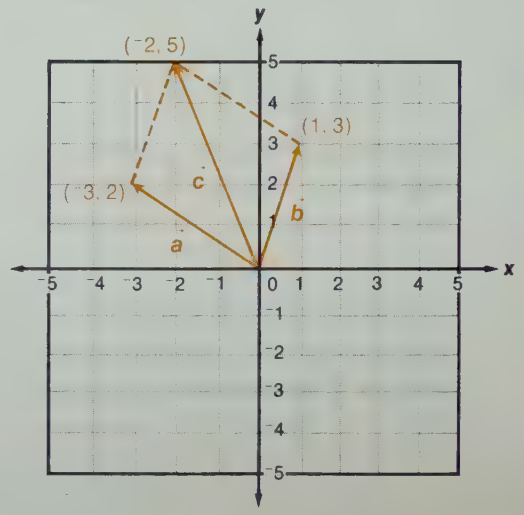
$\vec{b} = (1, 3)$

$\vec{c} = (-3 + 1, 2 + 3) = (-2, 5)$

Try Lisa's method for these sums.

A $\vec{r} = (4, -3)$, $\vec{s} = (1, 5)$. Find $\vec{r} \oplus \vec{s}$.

B $\vec{j} = (1, -1)$, $\vec{k} = (-2, -4)$. Find $\vec{j} \oplus \vec{k}$.



Using the Ideas

1. Sketch the arrow for each vector.
   - A $\vec{r} = (2, 2)$
   - B $\vec{s} = (3, -1)$
   - C $\vec{t} = (-2, 2)$
   - D $\vec{u} = (-1, -3)$

2. Use the vectors in Exercise 1 to find these sums.
   - A $\vec{r} \oplus \vec{s}$
   - B $\vec{s} \oplus \vec{t}$
   - C $\vec{t} \oplus \vec{u}$
   - D $\vec{s} \oplus \vec{u}$
   - E $\vec{u} \oplus \vec{r}$
   - F $\vec{t} \oplus \vec{r}$

3. The **null vector** or **zero vector** is $(0, 0)$. If $\vec{p} = (-7, 3)$, what is the sum of $\vec{p}$ and the zero vector?

4. Is there an identity element for addition of vectors? That is, for every vector $\vec{j}$, is there a vector whose sum with $\vec{j}$ is vector $\vec{j}$? What is the vector?

5. A What is the sum $\vec{a} \oplus \vec{b}$ where $\vec{a} = (7, -3)$ and $\vec{b} = (-7, 3)$?
   - B What is the sum $\vec{b} \oplus \vec{a}$?
   - C Do you think addition of vectors is commutative?

6. If the sum of two vectors is the null vector, each vector is called the **additive inverse** of the other.
   - A What is the additive inverse of $\vec{t} = (6, -5)$?
   - B Does every vector have an additive inverse vector?

7. Let $\vec{d} = (2, 6)$, $\vec{e} = (-3, -2)$, and $\vec{f} = (4, -3)$.
   - A Find $\vec{d} \oplus \vec{e}$.
   - B Find $\vec{e} \oplus \vec{f}$.
   - C Find $(\vec{d} \oplus \vec{e}) \oplus \vec{f}$.
   - D Find $\vec{d} \oplus (\vec{e} \oplus \vec{f})$.

8. Do you think addition of vectors is associative?

★ 9. Let $\vec{n} = (-3, -7)$ and $\vec{m} = (5, 4)$. Find the ordered pair for $\vec{x}$ if $\vec{m} \oplus \vec{x} = \vec{n}$.

think

The number 145 can be expressed as the sum of two different squares in two different ways.

$$145 = 9^2 + 8^2 = 12^2 + 1^2$$

Can you find a smaller number with the same property?

What is the **smallest** number that can be expressed as the sum of two different squares in two different ways?

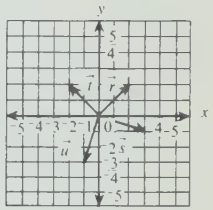
UTILIZATION

Exercise 2 provides practice in addition. Students should use the graphical method to check each sum.

Exercises 3 through 8 involve analysis of the properties of addition of vectors.

Exercise 9 suggests that it is possible to define subtraction for pairs of vectors. Thus the set of vectors with the operation of addition also forms a mathematical system similar to the *IRUL* system and the modulo 4 system.

Solution, Exercise 1



EXTENSION

See *Workbook* page 127 and *Duplicator Masters* page 82 for further practice exercises.

Enrichment: Each vector can be multiplied by a real number to give a new vector. The relevant definition is as follows:

If r is a real number and $\vec{v}$ is a vector such that $\vec{v} = (x, y)$ then $r\vec{v} = (rx, ry)$.

Example: Multiply $\vec{v} = (-2, 3)$ by 3.

Solution: $3\vec{v} = (3 \cdot -2, 3 \cdot 3) = (-6, 9)$

Ask able students to solve these problems:

1. Multiply each vector by 4. Show the original vector and the product on the same grid.
   A. $(-1, 2)$ B. $(2, -2)$ C. $(3, 0)$
2. Multiply each vector by $\frac{1}{2}$. Show both vectors graphically.
   A. $(6, 8)$ B. $(-4, 7)$ C. $(-10, -4)$

Think Solution

The problem must be solved by trial and error. Students could begin with $1^2 + 2^2 = 5$, $1^2 + 3^2 = 10$, $2^2 + 3^2 = 13$, . . . , and work until they discover a number expressed in two different ways. The smallest number is 65:

$1^2 + 8^2 = 65$, and $4^2 + 7^2 = 65$

Assignments

Minimum 1–2. Average 1–6. Maximum 1–9.

Objective

The student can solve problems related to switches in series in an electrical circuit.

PREPARATION

Materials: Model door bell, 1.5 volt dry cell, two knife switches, connecting wires.

If possible, obtain the necessary materials for the simple electrical circuit shown in the Investigation. It is quite likely that your science laboratory contains the necessary material and the science teachers may help you set up a suitable demonstration.

INVESTIGATION

If you have the circuit constructed as suggested in the Preparation, show how switches *A* and *B* are located and demonstrate that the only way to make the bell ring is to close both *A* and *B*. There are three other possible positions for *A* and *B*: *A* and *B* both open; *A* open, *B* closed; and *A* closed, *B* open. However, none of these will cause current to flow to the bell to make it ring.

DISCUSSION

Move through the Discussion slowly, making certain that students understand that in order for the bell to operate, there must be an unbroken circuit from the battery through the bell and back to the battery.

Exercise 2 introduces a mathematical model for a series circuit. Since a switch can be only open or closed, we designate open by 0 and closed by 1; the logical connector **and** is designated by $\otimes$ in series circuits. Then if *A* is closed and *B* is open, the resulting circuit is the same as a single open switch in a circuit. Hence, the equation $1 \otimes 0 = 0$ is a mathematical description of the circuit.

You may want to continue immediately and complete the table accompanying Exercise 1 on page H–89 as a group discussion activity. If so, use the electrical circuit model built for the Investigation.

Series Switches

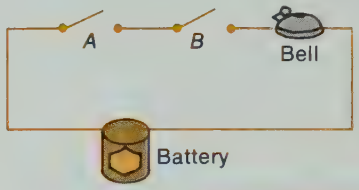
Investigating the Ideas

Electrical circuits have **switches** which can be opened or closed to determine whether or not current will flow through the circuit.

AN **OPEN** SWITCH
No current flows through the switch.

A **CLOSED** SWITCH
Current will flow through the switch.

Switches *A* and *B* are connected in **series**. The bell will not ring if both switches are in open positions. Show the circuits for other possible positions for switches *A* and *B*.



Can you decide for which circuits the bell will ring?

Discussing the Ideas

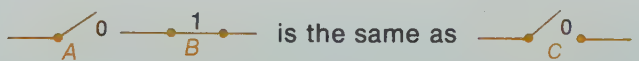
1. **A** Will the bell ring in the series circuit if only switch *A* is closed?
   B Will the bell ring if only switch *B* is closed?
   C Will the bell ring if both switches *A* and *B* are closed?

2. Suppose an **open switch** is designated by **0** and a **closed switch** is designated by **1**.

A is open **and** *B* is open. One open switch

We can describe this by the equation $0 \otimes 0 = 0$ where $\otimes$ means **and**.

What equations can you write using the symbols 0, 1, and $\otimes$ for *A*, *B*, and *C*?

A 

A open **and** *B* closed — One open switch

B

A closed **and** *B* open — One open switch

C

A closed **and** *B* closed — One closed switch

Using the Ideas

1. Use the results of Discussion Exercise 2 to complete the table for the circuits.

⊗	0	1
0		
1		

2. Use the table in Exercise 1 to solve the equations.

A $0 \otimes 1 = x$ **C** $1 \otimes z = 0$

B $1 \otimes y = 0$ **D** $t \otimes 1 = 1$

3. The circuit has two switches, A and B, connected in series. The bell is ringing.

A In what position are switches A and B?

B Write an equation for the circuit.

4. The bell is not ringing in the circuit. Switch A is closed.

A In what position is B?

B Write an equation for the circuit.

5. A, B, and C are three switches in series.

A Will current flow through the circuit?

B Which single switch could replace these three switches, an open switch or a closed switch?

C Solve the equation $(0 \otimes 1) \otimes 0 =$ ▮.

6. Solve the equations.

A $(0 \otimes 0) \otimes 1 =$ ▮ **C** $(0 \otimes 1) \otimes 1 =$ ▮

B $(1 \otimes 0) \otimes 1 =$ ▮ **D** $(1 \otimes 1) \otimes 1 =$ ▮

7. Can you show that the operation ⊗ is commutative?

8. Is ⊗ an associative operation?

UTILIZATION

Be certain that everyone completes the table in Exercise 1 correctly before proceeding with the other exercises. Students are not expected to memorize this table and should be allowed to refer to it freely as they do the exercises.

Exercise 5 introduces a circuit containing three switches in series. Only when all three switches are closed will current flow through the circuit.

Exercises 7 and 8 investigate the mathematical properties of the operation ⊗. This operation on the elements 0 and 1 forms a small mathematical system that is commutative, associative, and has an identity element (1) for ⊗.

Assignments

Minimum 1–4. Average 1–5. Maximum 1–8.

Objective

The student can solve problems related to switches in parallel in electrical circuits.

PREPARATION

Materials: Model door bell, 1.5 volt dry cell, 2 knife switches, connecting wires (as in previous lesson).

It will be extremely helpful to obtain the necessary material and set up a demonstration of the switches in parallel. Again, your science laboratory and science teacher may be able to aid you in this.

INVESTIGATION

Demonstrate the circuit shown in the Investigation if you have the needed material available; otherwise, read through the Investigation carefully with the students, making certain that they understand that for all combinations of open or closed positions of the switches, the bell will ring except when both *A* and *B* are open. This can be made clear by pointing out that, if at least one switch is closed, there is a completed circuit through which the current can flow. Thus if *A* is open (0) and *B* is closed (1) the current will flow through switch *B* to cause the bell to ring.

DISCUSSION

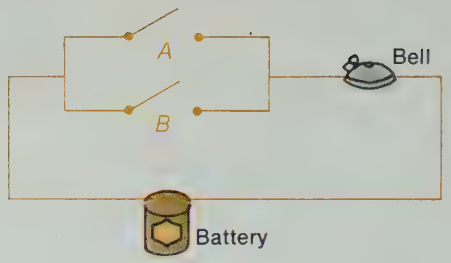
The symbols 0 and 1 are again used to denote open and closed switches, as in the previous lesson, and the symbol ⊕ is used as the logical connector **or** to denote that the switches are in parallel arrangement. With these agreements, a mathematical model of the circuit is constructed. By analyzing the mathematics, we can make conclusions that are valid for the electrical circuit.

Make certain that all students correctly complete the table for the operation ⊕ in Exercise 3. This table should be used as students do the exercises on page H–89.

Parallel Switches

Investigating the Ideas

Two electrical switches A and B are shown in **parallel**. Current will flow through the circuit and the bell will ring when **at least one** of the switches, A or B, is closed. Describe all the possible positions for switches A and B.

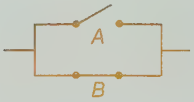


Can you decide in which positions the switches must be in order for the bell to ring?

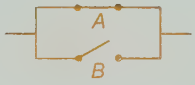
Discussing the Ideas

1. What single switch, open or closed, is equivalent to each of these parallel switches?

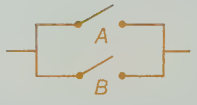
A



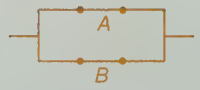
B



C



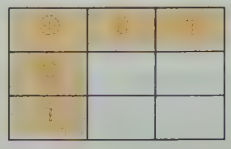
D



2. Suppose that 0 denotes an open switch, 1 denotes a closed switch, and ⊕ denotes switches in parallel. Exercise 1**A** can be described by the equation $0 \oplus 1 = 1$ because if A is open and B is closed, current will flow through the circuit.

   Write equations for the switches in parts **B**, **C**, and **D** of Exercise 1.

3. Use the results of Exercise 2 to copy and complete the table for parallel circuits.



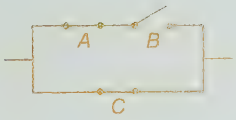
Using the Ideas

1. Use the table in Discussion Exercise 3 to solve these equations for parallel switches.

A $0 \oplus 1 = x$ **C** $1 \oplus 1 = z$ **E** $0 \oplus 0 = t$

B $y \oplus 0 = 0$ **D** $1 \oplus 0 = r$ **F** $0 \oplus n = 1$

2. A circuit contains two switches in series which are connected in parallel with a third switch. To determine if a current will flow through the circuit, we can write



Series Parallel

$A \quad B \quad C \quad A \text{ and } B \quad C$

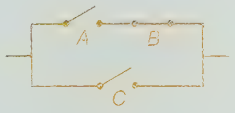
$(1 \otimes 0) \oplus 1 = 0 \oplus 1 = 1$

Therefore, the circuit is equivalent to one closed switch and current will flow through the circuit.

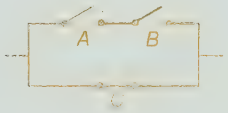
Write and solve similar equations for the switches below.

A

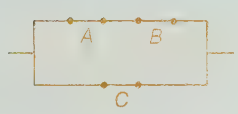
B



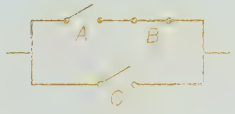
C



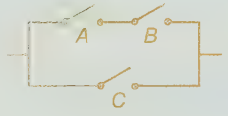
D



E



F



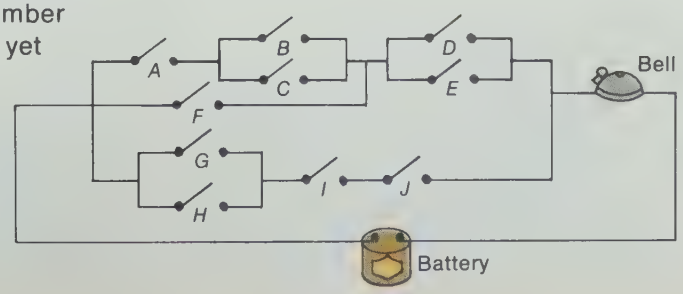
3. Solve the equations. Each contains $\otimes$ and $\oplus$.

A $(1 \otimes 0) \oplus 1 =$ ▮ **C** $(0 \otimes 0) \oplus 1 =$ ▮ **E** $(1 \oplus 1) \otimes 0 =$ ▮

B $(1 \oplus 0) \otimes 1 =$ ▮ **D** $(0 \otimes 1) \oplus 1 =$ ▮ **F** $(1 \otimes 1) \oplus 0 =$ ▮

★ **4.** **A** For the circuit below, what is the fewest number of switches that can be closed to make the bell ring?

B What is the greatest number that can be closed and yet the bell will not ring?



UTILIZATION

Assign the exercises as independent work. Plan to discuss the exercises after students have completed them. There are no specific skills to be mastered in this lesson.

Solutions, Exercise 4

A. If switch F and one of the switches D or E is closed, a complete circuit will be formed and the bell will ring.

B. If A, F, and one of the switches I or J is left open, there can be no completed circuit. Hence, seven of the ten switches can be closed without causing the bell to ring.

EXTENSION

Enrichment: Some students may be interested in undertaking Research Project D at this time.

Circuits as complicated as the one shown for Exercise 4 may be quite difficult to analyze mathematically. For Exercise 4, we have the equation:

$$\{[A \otimes (B \oplus C) \oplus F] \otimes (D \oplus E)\} \oplus [(G \oplus H) \otimes (I \otimes J)] = x.$$

If 0 and 1 are assigned to switches A to J, the value of x will be either 0 or 1. Hence, the ten switches in any combination of open or closed positions can always be replaced by a single switch which is either open or closed.

Assignments

Minimum 1–2. Average 1–3. Maximum 1–4.

Objective

The student can demonstrate the ability to work with the concepts presented in this module.

UTILIZATION

The review exercises can be assigned as independent work, with an oral discussion following their completion. Students should refer back to the lessons in the module as they do the exercises; they are not expected to have memorized the tables of operations for the various mathematical systems presented in this module.

Additional exercises that review the significant ideas presented in this module are provided on page 128 of the *Workbook*.

REVIEWING THE IDEAS

1. Complete the addition table for the modulo 4 system.

$\oplus_4$				
			0	1

2. Solve the equations in the modulo 4 system.

A $3 \oplus_4 1 = x$ C $0 \oplus_4 1 = z$

B $2 \oplus_4 3 = y$ D $2 \oplus_4 2 = t$

3. Complete the multiplication table for the modulo 4 system.

$\otimes_4$				
				2
				1

4. Using the table shown, solve the equations in the *IRUL* system.

★	*I*	*R*	*U*	*L*
I	*I*	*R*	*U*	*L*
R	*R*	*U*	*L*	*I*
U	*U*	*L*	*I*	*R*
L	*L*	*I*	*R*	*U*

A $R \star x = U$

B $I \star x = L$

C $R \star U = x$

D $x \star L = I$

E $L \star L = x$

F $L \star x = U$

5. Solve the equations in the octic system.

A $\not{D} \star x = I$

B $x \star V = R$

C $(U \star x) \star V = V$

D $(V \star L) \star \not{D} = x$

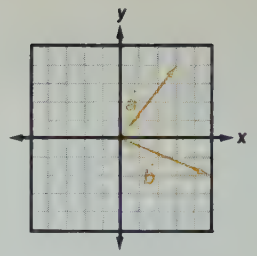
E $(R \star \not{D}) \star x = H$

F $(\not{D} \star \not{D}) \star x = I$

6. A Give the co-ordinates of $\vec{a}$.

B Find the length of $\vec{a}$.

C What vector has the same length as $\vec{a}$ but is opposite in direction?



7. What vector is $\vec{a} \oplus \vec{b}$? (See drawing for Exercise 6.)

8. If $\vec{r} = (^-2, 3)$ and $\vec{s} = (7, ^-1)$, what vector is $\vec{r} \oplus \vec{s}$?

9. What vector is the additive inverse of $\vec{v} = (6, ^-2)$?

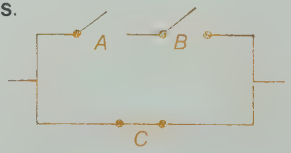
10. If $\vec{j} = (5, 6)$, $\vec{k} = (3, ^-2)$, and $\vec{m} = (^-4, 0)$, what is $(\vec{j} \oplus \vec{k}) \oplus \vec{m}$?

11. Tell whether the switches are in series or in parallel.

A

B

12. Will current flow through this circuit? A, B, and C are switches.



13. Solve the equations for series $\otimes$ and parallel $\oplus$ circuits.

A $(1 \otimes 0) = x$ E $(1 \otimes 0) \oplus 0 = x$

B $(1 \oplus 0) = x$ F $(1 \oplus 1) \otimes 1 = x$

C $(1 \otimes 1) = x$ G $(0 \oplus 1) \otimes 1 = x$

D $(1 \oplus 1) = x$ H $0 \oplus (1 \otimes 1) = x$

TEST YOURSELF

1. Solve the modulo 4 addition equations.

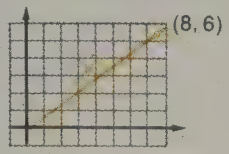
   A $3 \oplus_4 3 = x$

   B $2 \oplus_4 1 = y$

2. Solve the modulo 4 multiplication equations.

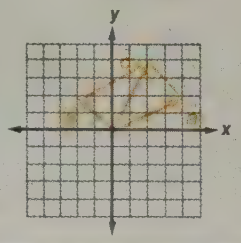
   A $2 \otimes_4 3 = x$ B $3 \otimes_4 3 = y$

3. Find the length of $\vec{v}$.



4. What vector is the additive inverse of $\vec{r} = (1, -3)$?

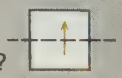
5. What addition equation for vectors does the drawing suggest?



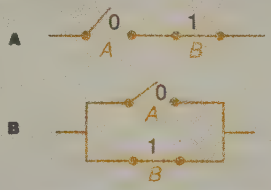
6. If $\vec{c} = (9, 9)$ *and* $\vec{d} = (-1, 4)$, what is $\vec{c} \oplus \vec{d}$?

7. In the octic system, $R \star R =$ ▥.

8. In the octic system, which motion is the inverse of the flip H?



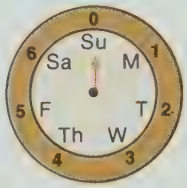
9. What single switch, **open** or **closed**, could replace the two switches in each circuit?



10. Solve the circuit equation. $(1 \otimes 1) \oplus 0 = x$

RESEARCH PROJECTS

A Invent a modular arithmetic based on the days of the week. Make an "addition" table and a "multiplication" table for the system.



B Invent a mathematical system based on the rotations and flips of an equilateral triangle around its centre and its lines of symmetry.

C There is an algebra of "tire switching" on an automobile. Find out about this nonmathematical system. (See *Algebra: A New Way of Looking at* Weybright and Talley, 1968, pp. 32-54; Musson Book Company.)

D Invent more complicated circuits. Analyze the circuits using the symbols 0, 1, $\otimes$, and $\oplus$ as developed in this module. (See *Mathematics and the Modern World* by Mario F. Triola; Menlo Park, California: Cummings Publishing Company, 1973, pp. 39-48; Addison-Wesley.)

TEST YOURSELF

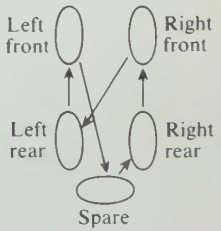
After completing the self-evaluation test, students should check their answers against the answer key and rework any problems that gave them difficulty. Encourage them to refer back to the appropriate lessons of the module as an aid in correcting their own mistakes.

RESEARCH PROJECTS

An excellent reference source for Research Project A is the booklet *Secret Codes and Remainder Arithmetic* by Lyman C. Peck (Washington, D.C.: National Council of Teachers of Mathematics, 1971).

A mathematical system of six elements (Project B) can be constructed using rotations about the centroid of an equilateral triangle plus reflections about the three lines of symmetry.

For Project C, students can invent their own systems of rotating tires. For example, one scheme for rotating a 5-tire system is illustrated below.



The topic of Research Project D might lead students to do some research into Boolean algebra or into some elementary phases of mathematical logic.

Objective

The student can demonstrate the ability to work with the major concepts presented in this learning unit.

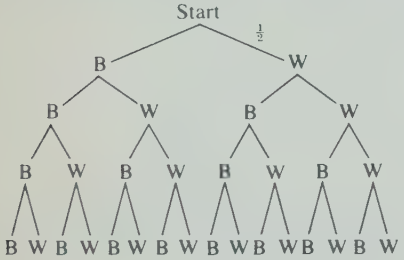
UTILIZATION

The Cumulative Review for Unit H covers the basic concepts and skills developed in the unit. The exercises are grouped so that one module at a time may be reviewed.

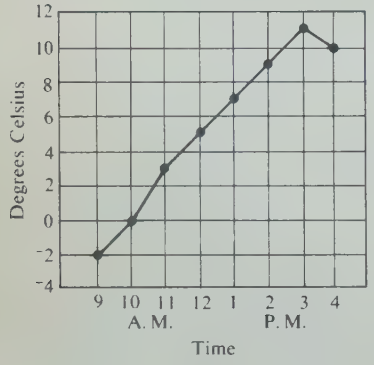
Exercises 1–9: Module 1
Exercises 10–16: Module 2
Exercises 17–26: Module 3
Exercises 27–33: Module 4

Answers, Exercises 5 and 14

5.



14.



CUMULATIVE REVIEW

1. Six names are placed in a hat. Two of the names are boys' names. What are the chances of drawing a girl's name from the hat?

2. A probability experiment has six equally likely outcomes. What is the probability of each outcome?

3. A probability experiment has only two outcomes, A and B. If $P(A) = \frac{3}{8}$, what is $P(B)$?

4. A penny and a nickel are tossed. Give the probability of each event.
   - **A** Two heads
   - **B** Two tails
   - **C** A head and a tail
   - **D** At least one tail

5. Two black marbles and two white marbles are placed in a box. The marbles are withdrawn one at a time until all have been withdrawn. Draw a tree diagram that will show all possible outcomes for this experiment.

6. How many different arrangements (permutations) are there for the letters A, B, C, and D?

7. How many combinations can be made with the letters A, B, C, D, E, and F if three letters are taken at a time?

8. Compute. $\dfrac{7!}{5! \cdot 2!}$

9. A die is tossed. Let E be the event of getting a 3 or more and let F be the event of getting a 1 or 2.
   - **A** What is $P(E)$?
   - **B** What is $P(F)$?
   - **C** What is $P(E \cap F)$?
   - **D** What is $P(E \cup F)$?

10. **A** What is the range of the heights?
    B What is the arithmetic mean of the heights?

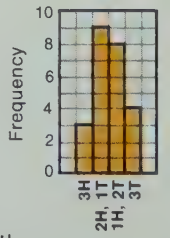
Name	Height
Ann	141 cm
Dave	158 cm
Ruth	173 cm
Paul	180 cm

11. On five tests Kitty had scores of 91, 82, 88, 94, and 85. What was her median score?

12. The table shows the number of points scored by each player on a basketball team.
    - **A** What is the median score?
    - **B** What is the mean?
    - **C** What is the mode?

Name	Points
Jones	21
Willis	10
Marto	8
Mills	0
Ford	2
Baker	6
Evans	0
Leeds	1

13. The histogram shows the results of an experiment of tossing three coins.
    - **A** What was the most frequent outcome?
    - **B** What outcome occurred the fewest number of times?
    - **C** How many trials of the experiment does the graph show?

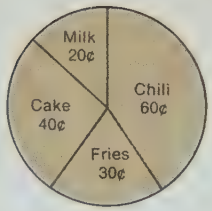


14. Construct a line-segment graph of these hourly temperatures.

9 A.M.	-2°C	1 P.M.	7°C
10 A.M.	0°C	2 P.M.	9°C
11 A.M.	3°C	3 P.M.	11°C
12 Noon	5°C	4 P.M.	10°C

15. **A** What percent of the lunch was the chili?

B How large is the central angle for the fries?



Lunch $1.50

16. A sample of voters in a city shows that 53% favor candidate A and 47% favor candidate B. If 20 000 people vote in the election, about how many votes can each candidate expect to get based on this sample?

17. If $x^2 = 2$, then $x =$ ▨ or $x =$ ▨.

18. Which decimal is the best approximation for $\sqrt{2}$? 1.4 or 1.5?

19. Locate approximately on a number line the points which correspond to $\sqrt{3}$, $\sqrt{21}$, and $\sqrt{20}$.

20. Give the best choices of integers for a and b.

A $a < \sqrt{10} < b$ **C** $a < \sqrt{95} < b$

B $a < \sqrt{40} < b$ **D** $a < \sqrt{601} < b$

21. Find the decimal approximation for the following numbers to the nearest tenth.

A $\sqrt{48}$ **B** $\sqrt{930}$ **C** $\sqrt{5625}$

22. Write a repeating decimal for $\frac{5}{9}$.

23. Write a fraction for each decimal.

A 0.32 **B** 0.071 **C** $0.\overline{77}$

24. Complete each sentence with **repeating** or **nonrepeating.**

A Every rational number has a ? decimal representation.

B Every irrational number has a ? decimal representation.

25. Find each product.

A $\sqrt{15} \cdot \sqrt{15}$ **C** $5\sqrt{7} \cdot 3 \cdot \sqrt{7}$

B $\sqrt{7} \cdot 2 \cdot \sqrt{7}$ **D** $\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{2} \cdot \sqrt{3}$

26. Give a decimal approximation (to the nearest hundredth) for each expression. Use $\sqrt{5} \approx 2.236$ and $\sqrt{2} \approx 1.414$.

A $\sqrt{5} + \sqrt{2}$ **C** $\sqrt{5} - \sqrt{2}$

B $\sqrt{5} \cdot \sqrt{2}$ **D** $\sqrt{5} \div \sqrt{2}$

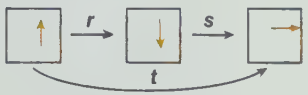
27. Solve the addition equations in the modulo 4 system.

A $1 \oplus_4 3 = x$ **B** $2 \oplus_4 3 = y$

28. Solve the multiplication equations in the modulo 4 system.

A $0 \otimes_4 3 = x$ **B** $2 \otimes_4 2 = y$

29. Give the rotations for r and s and then give the combined rotation for t.



30. In the octic system, $R \bigstar R =$ ▨.

31. If $\vec{v} = (7, 4)$ and $\vec{t} = (2, 2)$, what is $\vec{v} \oplus \vec{t}$?

32. What is the inverse vector for the vector $\vec{r} = (^-2, 6)$?

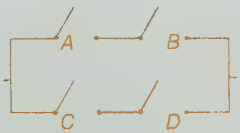
33. Which switches must be closed before current will flow through the circuit?

A Only A and C.

B Either A or C.

C Both A and B.

D Both A and D.



MATHEMATICAL RECREATION

In order to explore this application of game theory, students will need counters (20 per player) and four index cards.

This game can provide an exciting introduction to game theory. The players must develop strategies that will give them optimal chances of winning. The particular game illustrated has a mixed strategy. It can be shown that player A should select card X half of the time and card Y the other times. However, player A should do this on a random basis in order to keep B from discovering his strategy. Player B, on the other hand, should select card X one time out of 6 and card Y five times out of 6 for his optimum strategy.

Students might be encouraged to develop other *payoff matrices* (the table shown) for games modeled on this one.

MATHEMATICAL RECREATIONS

GAME THEORY

A recent and interesting branch of mathematics is known as "Game Theory." The theory of games began with the work of the mathematician John von Neumann (1903–1957). Game theory not only applies to simple games but also has important applications in business, psychology, and the social sciences. The game described below is a simple application of game theory.

Try playing this game with a classmate.

Each player starts a game with 20 counters.
Each player has two cards, one labelled X and the other Y.
Each player chooses one of his cards and both players turn up the cards simultaneously.

The table below shows how the players win counters according to their choices.

		Player B	
		Card X	Card Y
Player A	Card X	A wins 5 counters	B wins 5 counters
	Card Y	B wins 1 counter	A wins 1 counter

The game is completed when one player wins all of the other player's counters.

Try to develop some strategies that may help you win the game.

MATHEMATICAL RECREATIONS

TEST YOURSELF Answers

1. A $x = 2$; B $y = 3$ 2. A $x = 2$; B $y = 1$ 3. 10 units
4. $(^-1, 3)$ 5. $\vec{a} \oplus \vec{b} = \vec{c}$ 6. $\vec{c} \oplus \vec{d} = (8, 13)$
7. U 8. H 9. A open; B closed 10. $x = 1$

Supplementary Exercises

SET 1

1. Write each product using an exponent.

A $5 \cdot 5$ **C** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ **E** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

B $8 \cdot 8 \cdot 8$ **D** $95 \cdot 95 \cdot 95 \cdot 95$ **F** $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

2. Write each product as a power of ten.

A $10^2 \cdot 10^1$ **C** $10^3 \cdot 10^2$ **E** $10^2 \cdot 10^4$ **G** $10^5 \cdot 10^0$

B $10^2 \cdot 10^2$ **D** $10^4 \cdot 10^1$ **F** $10^4 \cdot 10^3$ **H** $10^7 \cdot 10^2$

3. Write each quotient as a power of ten.

A $10^3 \div 10^1$ **C** $10^5 \div 10^2$ **E** $10^9 \div 10^3$ **G** $10^6 \div 10^0$

B $10^2 \div 10^1$ **D** $10^6 \div 10^1$ **F** $10^6 \div 10^4$ **H** $10^4 \div 10^4$

4. Write each base-ten numeral in expanded notation.

A 38 **C** 802 **E** 2035 **G** 14 035 **I** 687 524

B 621 **D** 1259 **F** 8001 **H** 81 253 **J** 300 050

SET 2

1. Round each number to the nearest ten.

A 72 **C** 129 **E** 1253 **G** 15 633 **I** 200 008

B 85 **D** 635 **F** 7507 **H** 48 106 **J** 555 555

2. Round each number to the nearest hundred.

A 385 **C** 1258 **E** 9050 **G** 23 975 **I** 843 705

B 649 **D** 6845 **F** 17 883 **H** 105 105 **J** 503 507

3. Round each number to the nearest thousand.

A 1295 **C** 5765 **E** 20 801 **G** 458 358

B 2603 **D** 15 976 **F** 135 053 **H** 755 555

4. Round **5 607 384** to the nearest

A ten. **C** thousand. **E** hundred thousand.

B hundred. **D** ten thousand. **F** million.

SET 3

1. Use multiples of 10, 100, or 1000 for each ▨.

A To estimate 526 + 385, find the sum 500 + ▨.

B To estimate 1285 − 938, find the difference ▨ − 900.

C To estimate 493 · 32, find the product 500 · ▨.

D To estimate 3207 ÷ 376, find the quotient 3200 ÷ ▨.

E To estimate 543 · 208, find the product ▨ · 200.

F To estimate 6388 + 5892, find the sum 6000 + ▨.

G To estimate 41 850 ÷ 68, find the quotient ▨ ÷ 70.

2. Give an estimate for each of the following problems.

A 336 + 520	**D** 1538 ÷ 48	**G** 2436 − 498	**J** 7437 + 15 805
B 1640 − 1099	**E** 327 · 62	**H** 3208 ÷ 825	**K** 526 · 879
C 873 · 6	**F** 9051 + 832	**I** 3128 · 168	**L** 6172 · 983

SET 4

1. Find the sums.

A 56 +33	**B** 35 +57	**C** 158 + 49	**D** 351 +493	**E** 1205 + 999	**F** 15 873 + 3 059
G 42 67 +95	**H** 153 48 + 7	**I** 506 721 +273	**J** 942 267 +307	**K** 3841 1762 +4211	**L** 76 437 8 649 + 3 001

2. Find the sums.

A 82 + 9 + 105	**G** 364 + 721 + 480	**M** 99 + 999 + 9999
B 86 + 157 + 19	**H** 6300 + 503 + 59	**N** 76 095 + 2649 + 850
C 135 + 49 + 63	**I** 78 + 151 + 35 + 103	**O** 45 083 + 30 906 + 1285
D 350 + 35 + 3500	**J** 756 + 8005 + 10 700	**P** 850 721 + 37 842 + 7655
E 943 + 17 + 306	**K** 3700 + 37 000 + 370	**Q** 937 210 + 653 213
F 77 + 103 + 520	**L** 538 + 5603 + 809	**R** 8 340 756 + 4 750 095

SET 5

1. Find the differences.

A 87 − 54	**B** 72 − 39	**C** 100 − 53	**D** 248 − 86	**E** 963 − 277	**F** 801 − 352
G 1576 − 347	**H** 2834 − 1843	**I** 6912 − 6853	**J** 5350 − 4795	**K** 9500 − 6306	**L** 10 000 − 591

2. Find the differences.

A 93 − 27	**F** 1586 − 958	**K** 51 753 − 1845	**P** 100 000 − 10 000
B 150 − 69	**G** 4081 − 3982	**L** 67 595 − 39 008	**Q** 100 000 − 1000
C 302 − 124	**H** 7800 − 839	**M** 30 000 − 3000	**R** 100 000 − 100
D 950 − 666	**I** 9040 − 150	**N** 67 805 − 1800	**S** 343 000 − 24 793
E 1500 − 875	**J** 3897 − 2999	**O** 97 031 − 950	**T** 956 000 − 955 311

SET 6

1. Find the products.

A 52 · 6	**D** 129 · 4	**G** 1576 · 2	**J** 12 511 · 3
B 73 · 8	**E** 456 · 7	**H** 4609 · 6	**K** 37 059 · 5
C 87 · 9	**F** 903 · 5	**I** 8080 · 9	**L** 90 903 · 7

2. Find the products.

A 54 · 18	**D** 253 · 35	**G** 1200 · 69	**J** 15 000 · 18
B 26 · 67	**E** 125 · 88	**H** 5306 · 81	**K** 37 112 · 23
C 46 · 22	**F** 605 · 76	**I** 7953 · 44	**L** 53 025 · 85

3. Find the products.

A 179 · 201	**D** 535 · 806	**G** 1734 · 213	**J** 6305 · 833
B 368 · 555	**E** 490 · 359	**H** 5877 · 303	**K** 5950 · 970
C 843 · 347	**F** 993 · 865	**I** 7951 · 537	**L** 8005 · 508

SET 7

1. Find the quotients and remainders.

A $5\overline{)74}$	**D** $6\overline{)384}$	**G** $3\overline{)501}$	**J** $6\overline{)1522}$	**M** $8\overline{)34\,300}$
B $7\overline{)105}$	**E** $4\overline{)149}$	**H** $5\overline{)419}$	**K** $4\overline{)1905}$	**N** $9\overline{)51\,355}$
C $8\overline{)125}$	**F** $9\overline{)570}$	**I** $7\overline{)959}$	**L** $2\overline{)9999}$	**O** $7\overline{)61\,499}$

2. Find the quotients and remainders.

A $42\overline{)183}$	**D** $54\overline{)1568}$	**G** $74\overline{)2075}$	**J** $19\overline{)51\,300}$
B $63\overline{)516}$	**E** $27\overline{)2245}$	**H** $68\overline{)8346}$	**K** $34\overline{)29\,806}$
C $36\overline{)172}$	**F** $49\overline{)1677}$	**I** $81\overline{)8000}$	**L** $99\overline{)81\,565}$

3. Find the quotients and remainders.

A $189\overline{)4160}$	**D** $351\overline{)4593}$	**G** $235\overline{)10\,815}$	**J** $262\overline{)37\,470}$
B $304\overline{)2463}$	**E** $705\overline{)6615}$	**H** $324\overline{)22\,043}$	**K** $936\overline{)105\,100}$
C $611\overline{)5500}$	**F** $931\overline{)9200}$	**I** $654\overline{)54\,321}$	**L** $677\overline{)200\,000}$

SET 8

1. Evaluate $2x - 1$ using the numbers in the set $\{1, 2, 3, 4\}$.

2. Find the missing numbers for each part.

A If $a = 6$, then $6 + a =$ ▒.	**F** If $f = 4$, then $5f - 3 =$ ▒.
B If $b = 2$, then $17 - b =$ ▒.	**G** If $g = 8$, then $(16 - g) + 2 =$ ▒.
C If $c = 3$, then $6c =$ ▒.	**H** If $h = 3$, then $15 - 3h =$ ▒.
D If $d = 42$, then $d \div 6 =$ ▒.	**I** If $k = 5$, then $4k \div 2 =$ ▒.
E If $e = 17$, then $e - 5 =$ ▒.	**J** If $j = 10$, then $j^2 =$ ▒.

3. Suppose $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is the replacement set for $2x + 3$.
What is the replacement for x that will make the value of the expression

A 11? **B** 5? **C** 3? **D** 15? **E** 9?

SET 9

1. Solve the equations. Use {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} as the replacement set.

A $x - 3 = 4$

B $5a = 20$

C $81 \div r = 9$

D $6 + q = 15$

E $3y = 18$

F $19 - 2n = 11$

G $3t + 1 = 16$

H $36 \div 2m = 2$

I $4d - 6 = 22$

J $(13 - a) + 4 = 15$

K $16 + 3c = 16$

L $2h \div 5 = 2$

2. Find the solution set for each inequality. Use {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} as the replacement set.

A $x < 5$

B $y > 9$

C $a + 5 < 12$

D $d - 2 > 5$

E $2w > 4$

F $15 - t > 9$

G $3g + 1 < 10$

H $t^2 > 16$

I $5y - 2 < 19$

J $q^2 - 1 < 25$

K $x^2 + x < 50$

L $d^3 + 1 < 100$

SET 10

1. Solve the equations.

A $7 + c = 13$

B $21 - x = 17$

C $72 = r + 23$

D $8t = 40$

E $38 + y = 52$

F $t \div 9 = 4$

G $42 = r + 19$

H $21 - t = 16$

I $17x = 51$

J $7r = 42$

K $54 \div y = 54$

L $40w = 120$

M $t + 19 = 19$

N $28 \div x = 4$

O $15y - 3 = 27$

2. Give the number for the ▒. Then give the number for **x**.

A $3x + 4 = 10$
$3x =$ ▒
$x =$ **?**

B $2x - 3 = 9$
$2x =$ ▒
$x =$ **?**

C $4x + 2 = 14$
$4x =$ ▒
$x =$ **?**

D $2x + 7 = 15$
$2x =$ ▒
$x =$ **?**

E $5x - 4 = 6$
$5x =$ ▒
$x =$ **?**

F $7x + 6 = 27$
$7x =$ ▒
$x =$ **?**

G $4x - 6 = 14$
$4x =$ ▒
$x =$ **?**

H $8x + 5 = 5$
$8x =$ ▒
$x =$ **?**

SET 11

1. Give the number for the ▥. Then find the number for x.

A $15 + (x - 1) = 15$

$x - 1 =$ ▥

$x = ?$

B $3(x - 1) = 3$

$x - 1 =$ ▥

$x = ?$

C $6x + 23 = 23$

$6x =$ ▥

$x = ?$

D $3x + 2x = 45$

▥$x = 45$

$x = ?$

E $4x - 2x = 14$

▥$x = 14$

$x = ?$

F $5x + 3x - 4 = 20$

▥$x = 24$

$x = ?$

2. Solve the equations.

A $2x + 3x = 10$

B $6x - 2x = 12$

C $11x - 3x = 24$

D $(7x - 2x) + 4x = 36$

E $2x + (x + 3x) = 12$

F $4 + (3x - x) = 10$

3. Solve the equations.

A $(7x + 2x) - 5 = 40$

B $7 + (9x - 3x) = 37$

C $(5x + 2x) + x = 48$

D $(7x - 2x) - 3x = 6$

E $(8x - 4x) + 4 = 20$

F $(6x + 3x) - 2 = 7$

G $3 + (8x - 3x) = 13$

H $(10x - 7x) - 10 = 11$

I $(x + 2x) + 3x = 54$

SET 12

Write and solve an equation for each problem.

1. The product of a certain number and 8 is 72. What is the number?

2. Six times a certain number minus 2 is 34. What is the number?

3. When the sum of a certain number and 8 is multiplied by 6, the product is 66. What is the number?

4. When the sum of a certain number and 5 is divided by 7, the quotient is 13. What is the number?

5. Maria bought 3 cans of soup for 57 cents. How much did she pay for each can?

6. Ricardo had 93 cents. After he bought 3 pens, he had 6 cents left. How much for each pen?

7. Jan is 3 years older than Sam. In 5 years the sum of their ages will be 33.

A What is the sum of their ages now?

B How old is each person?

SET 13

1. Complete the table of numbers for the equation $x - y = 3$.

x	8	7	6	5	4	3
y	5					

2. Make a graph of the equation $x - y = 3$ using the pairs (x, y) from the table in Exercise 1.

3. Complete the table for the equation $2y - 4 = x$.

y	2	3	4	5	6	7
$2y$	4	6				
x	0					

4. Graph the equation $2y - 4 = x$ using the pairs (x, y) from the table in Exercise 3.

5. Make a table of pairs (x, y) for each equation. Then graph the equation.

   A $x + y = 8$ B $2x + y = 12$ C $y + 3 = x$

SET 14

1. Factor each number completely.

A 30	D 18	G 27	J 105	M 125	P 305
B 56	E 72	H 42	K 103	N 136	Q 489
C 24	F 44	I 51	L 110	O 150	R 512

2. Use exponents to give the prime factorization of each number.

A 15	E 54	I 336	M 308	Q 632	U 750
B 18	F 60	J 252	N 820	R 198	V 900
C 27	G 125	K 500	O 704	S 810	W 960
D 36	H 140	L 512	P 612	T 350	X 980

SET 15

1. Find the greatest common factor (GCF) of each pair.

A 6 and 18	**F** 24 and 36	**K** 28 and 42	**P** 90 and 120
B 11 and 12	**G** 14 and 35	**L** 30 and 70	**Q** 68 and 95
C 15 and 25	**H** 24 and 42	**M** 28 and 48	**R** 92 and 93
D 24 and 6	**I** 60 and 100	**N** 75 and 125	**S** 150 and 270
E 10 and 35	**J** 14 and 49	**O** 48 and 72	**T** 360 and 450

2. What is the GCF of the numerator and denominator of each fraction?

A $\frac{9}{12}$	**D** $\frac{20}{25}$	**G** $\frac{18}{27}$	**J** $\frac{60}{75}$	**M** $\frac{75}{300}$	**P** $\frac{240}{400}$
B $\frac{10}{40}$	**E** $\frac{20}{24}$	**H** $\frac{21}{31}$	**K** $\frac{25}{150}$	**N** $\frac{95}{85}$	**Q** $\frac{270}{280}$
C $\frac{44}{55}$	**F** $\frac{14}{44}$	**I** $\frac{27}{45}$	**L** $\frac{60}{150}$	**O** $\frac{33}{105}$	**R** $\frac{525}{2500}$

SET 16

1. Find the least common multiple (LCM) of each pair.

A 2 and 5	**F** 4 and 5	**K** 6 and 15	**P** 30 and 40
B 6 and 9	**G** 15 and 12	**L** 9 and 33	**Q** 10 and 13
C 10 and 4	**H** 20 and 25	**M** 10 and 12	**R** 18 and 30
D 2 and 7	**I** 15 and 20	**N** 12 and 16	**S** 16 and 24
E 8 and 3	**J** 2 and 17	**O** 21 and 15	**T** 90 and 60

2. Find the LCM of each group of numbers.

A 2, 6, and 10	**F** 4, 6, and 12	**K** 2, 7, and 5
B 3, 5, and 10	**G** 2, 4, and 7	**L** 5, 6, and 15
C 4, 6, and 15	**H** 2, 4, and 9	**M** 3, 8, and 12
D 10, 12, and 15	**I** 5, 10, and 15	**N** 10, 20, and 30
E 2, 3, and 5	**J** 8, 12, and 15	**O** 8, 12, and 16

SET 17

1. Find the sums.

A $^{-}3 + 2$	**F** $^{-}16 + 9$	**K** $^{-}50 + 70$	**P** $120 + {}^{-}119$
B $4 + {}^{-}6$	**G** $14 + {}^{-}12$	**L** $95 + {}^{-}30$	**Q** $^{-}98 + {}^{-}100$
C $^{-}8 + 5$	**H** $^{-}13 + 17$	**M** $72 + {}^{-}91$	**R** $175 + {}^{-}223$
D $^{-}9 + 9$	**I** $^{-}15 + {}^{-}8$	**N** $^{-}83 + {}^{-}17$	**S** $^{-}106 + {}^{-}106$
E $0 + {}^{-}7$	**J** $^{-}21 + {}^{-}21$	**O** $^{-}16 + 32$	**T** $^{-}935 + 0$

2. Find the sums.

A $6 + 4 + {}^{-}7$	**F** $15 + {}^{-}13 + 7$	**K** $(30 + 30) + (30 + {}^{-}30)$
B $6 + {}^{-}4 + {}^{-}7$	**G** $^{-}5 + 19 + {}^{-}3$	**L** $(30 + {}^{-}30) + (30 + {}^{-}30)$
C $6 + {}^{-}4 + 7$	**H** $26 + {}^{-}20 + {}^{-}8$	**M** $(30 + 30) + ({}^{-}30 + {}^{-}30)$
D $^{-}6 + {}^{-}4 + 7$	**I** $^{-}37 + {}^{-}1 + 35$	**N** $({}^{-}30 + 30) + ({}^{-}30 + {}^{-}30)$
E $^{-}6 + 4 + {}^{-}7$	**J** $^{-}86 + {}^{-}10 + {}^{-}1$	**O** $({}^{-}30 + {}^{-}30) + ({}^{-}30 + {}^{-}30)$

SET 18

1. Solve the equations. Think about missing addends.

A $0 - 3 = n$	**F** $2 - 6 = n$	**K** $5 - 9 = n$	**P** $7 - {}^{-}3 = n$
B $1 - 3 = n$	**G** $6 - {}^{-}2 = n$	**L** $^{-}5 - 3 = n$	**Q** $14 - {}^{-}6 = n$
C $0 - {}^{-}3 = n$	**H** $^{-}6 - 2 = n$	**M** $^{-}8 - {}^{-}2 = n$	**R** $^{-}4 - {}^{-}6 = n$
D $1 - {}^{-}3 = n$	**I** $^{-}6 - {}^{-}2 = n$	**N** $7 - {}^{-}6 = n$	**S** $^{-}12 - 3 = n$
E $^{-}1 - {}^{-}3 = n$	**J** $^{-}2 - {}^{-}6 = n$	**O** $^{-}6 - 9 = n$	**T** $^{-}12 - {}^{-}3 = n$

2. Find the differences.

A $17 - 8$	**G** $0 - {}^{-}5$	**M** $25 - 28$	**S** $38 - {}^{-}58$
B $5 - 13$	**H** $^{-}4 - {}^{-}14$	**N** $^{-}35 - {}^{-}3$	**T** $^{-}105 - 3$
C $0 - 19$	**I** $26 - {}^{-}19$	**O** $48 - {}^{-}5$	**U** $^{-}225 - {}^{-}1$
D $^{-}6 - 15$	**J** $^{-}22 - 13$	**P** $^{-}53 - 12$	**V** $^{-}65 - {}^{-}65$
E $20 - 30$	**K** $^{-}5 - {}^{-}8$	**Q** $10 - {}^{-}33$	**W** $543 - {}^{-}17$
F $^{-}6 - {}^{-}9$	**L** $12 - {}^{-}5$	**R** $75 - 47$	**X** $418 - 814$

SET 19

1. Find the products.

A $2 \cdot {}^{-}10$	**F** ${}^{-}12 \cdot {}^{-}6$	**K** $20 \cdot {}^{-}14$	**P** ${}^{-}1 \cdot 87$	**U** $200 \cdot {}^{-}8$
B ${}^{-}4 \cdot {}^{-}6$	**G** ${}^{-}100 \cdot 0$	**L** ${}^{-}15 \cdot {}^{-}8$	**Q** ${}^{-}1 \cdot {}^{-}87$	**V** ${}^{-}500 \cdot 0$
C $20 \cdot {}^{-}3$	**H** ${}^{-}1 \cdot {}^{-}53$	**M** ${}^{-}1 \cdot 43$	**R** $10 \cdot {}^{-}87$	**W** ${}^{-}35 \cdot {}^{-}4$
D ${}^{-}3 \cdot {}^{-}16$	**I** ${}^{-}30 \cdot {}^{-}3$	**N** ${}^{-}50 \cdot {}^{-}30$	**S** ${}^{-}10 \cdot {}^{-}87$	**X** ${}^{-}56 \cdot 22$
E ${}^{-}10 \cdot 15$	**J** $0 \cdot {}^{-}27$	**O** ${}^{-}17 \cdot {}^{-}2$	**T** ${}^{-}10 \cdot 87$	**Y** ${}^{-}30 \cdot {}^{-}45$

2. Find the products.

A $3 \cdot 3 \cdot 3$	**F** ${}^{-}5 \cdot {}^{-}4 \cdot {}^{-}1$	**K** $({}^{-}5 \cdot 4) \cdot ({}^{-}5 \cdot 4)$
B $3 \cdot 3 \cdot {}^{-}3$	**G** ${}^{-}6 \cdot 5 \cdot {}^{-}3$	**L** $({}^{-}5 \cdot {}^{-}4) \cdot (5 \cdot 4)$
C $3 \cdot {}^{-}3 \cdot {}^{-}3$	**H** ${}^{-}10 \cdot {}^{-}10 \cdot {}^{-}8$	**M** $({}^{-}5 \cdot {}^{-}5) \cdot (4 \cdot 4)$
D $3 \cdot {}^{-}3 \cdot 3$	**I** ${}^{-}3 \cdot 15 \cdot 4$	**N** $({}^{-}5 \cdot {}^{-}4) \cdot ({}^{-}5 \cdot 4)$
E ${}^{-}3 \cdot {}^{-}3 \cdot {}^{-}3$	**J** ${}^{-}12 \cdot 3 \cdot {}^{-}4$	**O** $({}^{-}5 \cdot {}^{-}4) \cdot ({}^{-}5 \cdot {}^{-}4)$

SET 20

1. Find the missing factor.

A $27 \div 3 = n$	**E** ${}^{-}22 \div {}^{-}2 = n$	**I** ${}^{-}72 \div {}^{-}8 = n$	**M** ${}^{-}100 \div 10 = n$
B $15 \div {}^{-}5 = n$	**F** $49 \div {}^{-}7 = n$	**J** ${}^{-}27 \div {}^{-}27 = n$	**N** ${}^{-}1000 \div {}^{-}10 = n$
C ${}^{-}18 \div 3 = n$	**G** ${}^{-}54 \div {}^{-}6 = n$	**K** $36 \div {}^{-}12 = n$	**O** ${}^{-}125 \div {}^{-}25 = n$
D $19 \div {}^{-}1 = n$	**H** ${}^{-}73 \div {}^{-}1 = n$	**L** ${}^{-}45 \div 45 = n$	**P** $0 \div {}^{-}128 = n$

2. Find the quotients.

A ${}^{-}15 \div 3$	**F** $55 \div {}^{-}11$	**K** ${}^{-}300 \div 60$	**P** $\frac{18}{{}^{-}9}$	**S** $\frac{({}^{-}15 + 3)}{{}^{-}6}$
B ${}^{-}18 \div {}^{-}2$	**G** ${}^{-}48 \div 16$	**L** ${}^{-}472 \div {}^{-}8$		
C $21 \div 3$	**H** ${}^{-}65 \div {}^{-}5$	**M** ${}^{-}1000 \div {}^{-}25$	**Q** $\frac{{}^{-}24}{{}^{-}8}$	**T** $\frac{({}^{-}10 + {}^{-}20)}{5 \cdot {}^{-}1}$
D $42 \div {}^{-}7$	**I** ${}^{-}33 \div {}^{-}1$	**N** ${}^{-}1000 \div {}^{-}10$		
E ${}^{-}13 \div 13$	**J** ${}^{-}120 \div {}^{-}40$	**O** ${}^{-}3500 \div 70$	**R** $\frac{(4 - 6)}{(16 - 15)}$	**U** $\frac{({}^{-}4 - {}^{-}6)}{(4 - 8)}$

SET 21

1. What is represented by each of the following?

A $|3|$ C $|5|$ E $|^{-}42|$ G $|4+2|$ I $|^{-}4+2|$ K $|^{-}4-^{-}2|$

B $|^{-}3|$ D $|^{-}8|$ F $|0|$ H $|4-2|$ J $|^{-}4+^{-}2|$ L $|4-^{-}2|$

2. Solve the equations.

A $x=|^{-}8|$ D $|^{-}3\cdot 1|=x$ G $|^{-}14-^{-}7|=x$ J $|5-x|=2$

B $|3|=x$ E $|^{-}5\cdot 0|=x$ H $|21\div^{-}3|=x$ K $|x-^{-}6|=30$

C $x=|4-2|$ F $|^{-}8+^{-}3|=x$ I $|35-50|=x$ L $|^{-}9+x|=20$

3. Give the correct symbol ($<$, $=$, or $>$) for each ▒.

A $^{-}5$ ▒ $|^{-}5|$ C 5 ▒ $|^{-}5|$ E $|3|-|6|$ ▒ $3-6$

B $^{-}5$ ▒ 5 D $|^{-}5|$ ▒ $|0-5|$ F $|^{-}3|-|^{-}6|$ ▒ $|^{-}6|-|^{-}3|$

SET 22

1. Give the missing number for each ▒.

A If $4x+6=2$, then $4x=$ ▒, and $x=$ ▒.

B If $3x+16=13$, then $3x=$ ▒, and $x=$ ▒.

C If $2x-6=10$, then $2x=$ ▒, and $x=$ ▒.

2. Give the missing integers in each equation.

A $3x+4=^{-}5 \Rightarrow 3x=$ ▒ $\Rightarrow x=$ ▒.

B $2x-6=14 \Rightarrow 2x=$ ▒ $\Rightarrow x=$ ▒.

C $4x+5=^{-}3 \Rightarrow 4x=$ ▒ $\Rightarrow x=$ ▒.

D $3(x+2)=9 \Rightarrow x+2=$ ▒ $\Rightarrow x=$ ▒.

E $2(x+14)=20 \Rightarrow x+14=$ ▒ $\Rightarrow x=$ ▒.

3. Solve the equations.

A $3+x=^{-}2$ D $x\div 6=^{-}13$ G $^{-}4x+3=15$ J $^{-}4(x+1)=16$

B $^{-}5x=45$ E $(x+5)-3=^{-}8$ H $^{-}2x-8=10$ K $5(x+2)=^{-}35$

C $^{-}17+x=^{-}56$ F $3x+1=^{-}5$ I $^{-}5x\div 3=^{-}15$ L $3(x+6)=0$

SET 23

1. Write two fractions equivalent to each given fraction.

A $\frac{1}{4}$	**D** $\frac{2}{3}$	**G** $\frac{2}{5}$	**J** $\frac{5}{6}$	**M** $\frac{10}{9}$	**P** $\frac{5}{9}$	**S** $\frac{8}{5}$	**V** $\frac{1}{13}$
B $\frac{3}{4}$	**E** $\frac{3}{8}$	**H** $\frac{3}{2}$	**K** $\frac{7}{3}$	**N** $\frac{3}{20}$	**Q** $\frac{10}{11}$	**T** $\frac{1}{15}$	**W** $\frac{4}{21}$
C $\frac{1}{6}$	**F** $\frac{7}{10}$	**I** $\frac{5}{8}$	**L** $\frac{5}{12}$	**O** $\frac{4}{7}$	**R** $\frac{4}{4}$	**U** $\frac{3}{16}$	**X** $\frac{4}{30}$

2. Give the lowest-terms fraction for each given fraction.

A $\frac{14}{16}$	**D** $\frac{6}{21}$	**G** $\frac{36}{40}$	**J** $\frac{21}{27}$	**M** $\frac{18}{81}$	**P** $\frac{12}{45}$	**S** $\frac{48}{100}$	**V** $\frac{45}{55}$
B $\frac{15}{18}$	**E** $\frac{30}{40}$	**H** $\frac{15}{24}$	**K** $\frac{12}{25}$	**N** $\frac{30}{42}$	**Q** $\frac{16}{40}$	**T** $\frac{12}{90}$	**W** $\frac{24}{64}$
C $\frac{9}{9}$	**F** $\frac{5}{30}$	**I** $\frac{14}{40}$	**L** $\frac{15}{35}$	**O** $\frac{21}{7}$	**R** $\frac{60}{72}$	**U** $\frac{44}{55}$	**X** $\frac{95}{150}$

SET 24

1. Give the missing number for each ▥. Then tell whether the fractions are equivalent or not.

A $\frac{3}{5}$ $\frac{5}{8}$ $3 \cdot 8 = ▥$, $5 \cdot 5 = ▥$

B $\frac{5}{6}$ $\frac{20}{24}$ $5 \cdot 24 = ▥$, $6 \cdot 20 = ▥$

C $\frac{4}{9}$ $\frac{20}{36}$ $4 \cdot 36 = ▥$, $9 \cdot 20 = ▥$

D $\frac{7}{5}$ $\frac{42}{20}$ $7 \cdot 20 = ▥$, $5 \cdot 42 = ▥$

2. Give the correct symbol ($=$ or $\neq$) for each ◍.

A $\frac{2}{5}$ ◍ $\frac{5}{2}$	**D** $\frac{10}{15}$ ◍ $\frac{4}{6}$	**G** $\frac{2}{15}$ ◍ $\frac{10}{90}$	**J** $\frac{5}{9}$ ◍ $\frac{25}{45}$	**M** $\frac{54}{64}$ ◍ $\frac{7}{8}$
B $\frac{3}{4}$ ◍ $\frac{27}{32}$	**E** $\frac{7}{8}$ ◍ $\frac{28}{32}$	**H** $\frac{7}{6}$ ◍ $\frac{18}{21}$	**K** $\frac{21}{35}$ ◍ $\frac{10}{15}$	**N** $\frac{27}{63}$ ◍ $\frac{9}{27}$
C $\frac{11}{12}$ ◍ $\frac{55}{60}$	**F** $\frac{8}{56}$ ◍ $\frac{2}{14}$	**I** $\frac{20}{24}$ ◍ $\frac{5}{6}$	**L** $\frac{7}{5}$ ◍ $\frac{42}{25}$	**O** $\frac{25}{90}$ ◍ $\frac{5}{16}$

3. Find the integer for x in each equation.

A $\frac{2}{3} = \frac{x}{18}$	**C** $\frac{3}{5} = \frac{x}{35}$	**E** $\frac{x}{3} = \frac{20}{15}$	**G** $\frac{5}{27} = \frac{15}{x}$	**I** $\frac{3}{8} = \frac{x}{104}$
B $\frac{6}{5} = \frac{x}{20}$	**D** $\frac{x}{7} = \frac{24}{56}$	**F** $\frac{7}{8} = \frac{x}{48}$	**H** $\frac{9}{10} = \frac{72}{x}$	**J** $\frac{5}{16} = \frac{x}{400}$

SET 25

1. Find the sums.

A $\frac{5}{9}+\frac{1}{3}$ **D** $\frac{5}{5}+\frac{2}{3}$ **G** $\frac{-4}{9}+\frac{-5}{6}$ **J** $\frac{2}{3}+\frac{-3}{4}$ **M** $\frac{3}{20}+\frac{-1}{4}$

B $\frac{3}{8}+\frac{1}{4}$ **E** $\frac{1}{2}+\frac{5}{8}$ **H** $\frac{5}{12}+\frac{3}{4}$ **K** $\frac{-1}{6}+\frac{-3}{8}$ **N** $\frac{4}{7}+\frac{5}{8}$

C $\frac{3}{4}+\frac{1}{8}$ **F** $\frac{4}{5}+\frac{8}{15}$ **I** $\frac{9}{10}+\frac{-3}{4}$ **L** $\frac{1}{9}+\frac{5}{6}$ **O** $\frac{-3}{4}+\frac{3}{10}$

2. Find the sums.

A $22\frac{3}{8}+17\frac{1}{16}$ **B** $36\frac{1}{2}+9\frac{1}{4}$ **C** $53\frac{5}{6}+26\frac{2}{3}$ **D** $8\frac{5}{6}+14\frac{1}{2}$ **E** $76\frac{1}{2}+24\frac{3}{5}$ **F** $106\frac{3}{10}+321\frac{3}{5}$

G $33\frac{1}{4}+44\frac{1}{2}+9\frac{1}{3}$ **H** $8\frac{1}{9}+10\frac{1}{5}+6\frac{2}{3}$ **I** $31\frac{1}{10}+15\frac{1}{100}+8\frac{1}{50}$ **J** $31\frac{4}{15}+19\frac{3}{10}+15\frac{2}{5}$ **K** $18\frac{1}{6}+14\frac{1}{9}+38\frac{5}{12}$ **L** $155\frac{1}{20}+73\frac{4}{5}+128\frac{7}{10}$

SET 26

1. Find the differences.

A $\frac{5}{8}-\frac{1}{4}$ **D** $\frac{11}{10}-\frac{4}{5}$ **G** $\frac{1}{20}-\frac{-3}{10}$ **J** $\frac{-3}{5}-\frac{-3}{5}$ **M** $\frac{11}{12}-\frac{3}{4}$

B $\frac{7}{12}-\frac{1}{6}$ **E** $\frac{3}{8}-\frac{3}{8}$ **H** $\frac{5}{7}-\frac{1}{3}$ **K** $\frac{2}{3}-\frac{5}{9}$ **N** $\frac{1}{5}-\frac{-1}{9}$

C $\frac{2}{3}-\frac{1}{2}$ **F** $\frac{9}{10}-\frac{1}{4}$ **I** $\frac{9}{15}-\frac{-1}{5}$ **L** $\frac{3}{4}-\frac{-1}{9}$ **O** $\frac{3}{7}-\frac{-1}{3}$

2. Find the differences.

A $6\frac{1}{2}-4\frac{1}{3}$ **B** $8\frac{1}{3}-5\frac{1}{6}$ **C** $16\frac{7}{10}-9\frac{9}{20}$ **D** $25\frac{3}{4}-18\frac{1}{6}$ **E** $63\frac{5}{8}-7\frac{1}{5}$ **F** $50\frac{1}{20}-8\frac{1}{10}$

G $79\frac{5}{6}-35\frac{9}{10}$ **H** $125\frac{3}{8}-95\frac{3}{4}$ **I** $100\frac{1}{100}-1\frac{1}{10}$ **J** $534\frac{7}{15}-150\frac{4}{5}$ **K** $750\frac{7}{15}-339\frac{3}{10}$ **L** $635\frac{19}{25}-49\frac{17}{20}$

SET 27

1. Find the products.

A $\frac{1}{4} \cdot \frac{1}{5}$	**D** $\frac{3}{4} \cdot \frac{4}{3}$	**G** $2 \cdot \frac{3}{8}$	**J** $\frac{2}{3} \cdot 45$	**M** $\frac{11}{12} \cdot \frac{3}{3}$
B $\frac{3}{4} \cdot \frac{1}{2}$	**E** $\frac{1}{2} \cdot \frac{4}{5}$	**H** $\frac{3}{5} \cdot \frac{10}{6}$	**K** $\frac{7}{3} \cdot \frac{9}{8}$	**N** $\frac{7}{15} \cdot \frac{3}{8}$
C $\frac{2}{5} \cdot \frac{1}{4}$	**F** $\frac{3}{5} \cdot \frac{5}{9}$	**I** $\frac{1}{16} \cdot 8$	**L** $0 \cdot \frac{5}{6}$	**O** $\frac{8}{25} \cdot \frac{5}{6}$

2. Find the products.

A $\frac{-1}{3} \cdot \frac{1}{2}$	**D** $-1 \cdot \frac{1}{9}$	**G** $\frac{-1}{3} \cdot \frac{4}{5}$	**J** $-28 \cdot \frac{3}{4}$	**M** $\frac{15}{2} \cdot \frac{-9}{25}$
B $\frac{1}{5} \cdot \frac{-1}{3}$	**E** $5 \cdot \frac{-1}{10}$	**H** $\frac{-3}{4} \cdot \frac{-1}{5}$	**K** $\frac{-1}{3} \cdot \frac{-1}{3}$	**N** $\frac{-4}{7} \cdot 0$
C $\frac{-1}{7} \cdot \frac{-1}{3}$	**F** $-4 \cdot \frac{-1}{12}$	**I** $\frac{2}{9} \cdot \frac{-3}{8}$	**L** $\frac{-3}{5} \cdot \frac{15}{18}$	**O** $\frac{-7}{9} \cdot \frac{3}{21}$

3. Write each mixed numeral numeral as an improper fraction. Then find the product.

A $1\frac{1}{4} \cdot \frac{1}{3}$	**D** $\frac{2}{3} \cdot -2\frac{3}{4}$	**G** $\frac{-7}{8} \cdot -1\frac{5}{9}$	**J** $-10 \cdot 3\frac{5}{6}$	**M** $4\frac{7}{8} \cdot -5\frac{2}{3}$
B $2\frac{1}{2} \cdot 7$	**E** $\frac{-1}{2} \cdot -3\frac{1}{3}$	**H** $1\frac{1}{2} \cdot 1\frac{2}{3}$	**K** $8 \cdot -9\frac{3}{4}$	**N** $-15 \cdot 8\frac{1}{10}$
C $\frac{-4}{5} \cdot 1\frac{1}{2}$	**F** $3 \cdot -4\frac{1}{6}$	**I** $4\frac{1}{2} \cdot 3\frac{1}{3}$	**L** $\frac{-9}{10} \cdot 6\frac{2}{3}$	**O** $\frac{-9}{25} \cdot -2\frac{1}{4}$

SET 28

1. Find the quotients.

A $\frac{3}{4} \div \frac{1}{2}$	**D** $\frac{7}{6} \div \frac{7}{8}$	**G** $\frac{-5}{6} \div \frac{-1}{8}$	**J** $\frac{3}{4} \div -9$	**M** $\frac{-5}{8} \div -10$
B $\frac{7}{8} \div \frac{1}{4}$	**E** $\frac{1}{3} \div \frac{4}{5}$	**H** $1 \div \frac{7}{6}$	**K** $\frac{-8}{3} \div \frac{-5}{4}$	**N** $\frac{-9}{10} \div \frac{3}{8}$
C $\frac{4}{9} \div \frac{2}{3}$	**F** $\frac{-2}{3} \div \frac{2}{5}$	**I** $\frac{-3}{8} \div \frac{-4}{9}$	**L** $\frac{5}{8} \div \frac{-1}{3}$	**O** $\frac{-3}{4} \div \frac{-4}{15}$

2. Write each mixed numeral as an improper fraction. Then find the quotient.

A $2\frac{1}{3} \div \frac{1}{6}$	**D** $\frac{1}{4} \div 3\frac{1}{2}$	**G** $-2\frac{3}{4} \div \frac{1}{2}$	**J** $-25 \div 8\frac{1}{3}$	**M** $4\frac{1}{2} \div -18$
B $3\frac{1}{5} \div \frac{3}{10}$	**E** $2\frac{1}{3} \div 5\frac{1}{4}$	**H** $-14 \div 4\frac{2}{3}$	**K** $-4\frac{2}{3} \div -1\frac{3}{4}$	**N** $-3\frac{9}{10} \div 4\frac{1}{3}$
C $4\frac{1}{2} \div 15$	**F** $3\frac{1}{5} \div 1\frac{1}{7}$	**I** $-5 \div \frac{-5}{6}$	**L** $-3\frac{1}{3} \div 16\frac{2}{3}$	**O** $33 \div -5\frac{1}{2}$

SET 29

Write and solve an equation for each problem.

1. Carl is 12 years old. Sam is $\frac{3}{2}$ Carl's age. How old is Sam?

2. Marie bought a pair of shoes at a "$\frac{1}{2}$ off" sale. If the shoes were regularly priced at $15, what was the sale price of the shoes?

3. A furniture store is having a "$\frac{1}{3}$ off" sale. What is the sale price of a chair regularly priced at $105?

4. Jody played volleyball for $\frac{3}{4}$ hour yesterday and $\frac{2}{3}$ hour today. How much time did she play in two days?

5. If Mrs. Jackson drives 90 kilometres in $1\frac{1}{5}$ hours, what is her average speed in kilometres per hour?

★ 6. A football team won $\frac{3}{4}$ of its games. It won 4 more games than it lost. How many games did the team play?

SET 30

1. Find the sums.

A 6.5 + 1.43	E 0.366 + 6.2 + 0.18	I 4.7 + 32.643 + 825.6
B 12.32 + 0.471	F 243.2 + 3.61 + 4.438	J 55.89 + 5.589 + .5589
C 243.2 + 3.65	G 36 + 2.64 + 0.735	K 0.85 + 0.937 + 0.9539
D 0.538 + 8.469	H 0.7305 + 69.1 + 0.88	L 326.043 + 82.62 + 1.9

2. Find the differences.

A 6.28 − 3.7	E 35.8 − 6.45	I 30.2 − 4.86	M 84.27 − 62.99
B 15.75 − 9.39	F 4.85 − 0.769	J 7.06 − 2.49	N 500 − 65.37
C 38.6 − 27.7	G 10 − 1.58	K 23.56 − 17.742	O 830.7 − 179.03
D 0.937 − 0.66	H 18.6 − 5.935	L 326.1 − 44.78	P 5000 − 493.971

3. Find the sums.

A 3.75 + $^{-}$2.49	E $^{-}$0.97 + $^{-}$0.58	I 93 + $^{-}$76.7 + $^{-}$1.35
B 73.1 + $^{-}$69.5	F 0.935 + $^{-}$0.769	J $^{-}$9.5 + $^{-}$0.95 + 95
C $^{-}$5.34 + $^{-}$8.09	G $^{-}$28.9 + 33.73	K 83.7 + $^{-}$0.75 + $^{-}$78
D $^{-}$69.5 + 37.42	H $^{-}$0.0639 + 0.0115	L 60.8 + $^{-}$25 + 7.904

SET 31

1. Find the products.

A 4.72×6
B 32×0.4
C 43×0.07
D 22.1×7.1
E 52.6×4.2
F 3.86×0.4
G 76.4×0.34
H 8.26×0.65
I 436.2×0.03
J 2.51×0.361
K 64.3×2.66
L 1.43×0.953
M 23.7×0.564
N 8.63×22.6
O 92.7×0.365
P 9.731×42.56
Q 0.0631×5.2
R 504.6×7.31
S 32.2×0.3742
T 0.653×0.641

2. Find the products.

A $0.3 \times {}^{-}0.9$
B ${}^{-}2.1 \times 6$
C ${}^{-}3.5 \times 0.8$
D ${}^{-}1.7 \times {}^{-}2.5$
E ${}^{-}8.3 \times 0.43$
F ${}^{-}43.9 \times {}^{-}0.15$
G $5.96 \times {}^{-}1.7$
H ${}^{-}12.4 \times 68.4$
I ${}^{-}8.56 \times {}^{-}3.501$
J $143.6 \times {}^{-}340$
K $205.3 \times {}^{-}8.61$
L ${}^{-}95.1 \times {}^{-}37.05$
M $9.421 \times {}^{-}5.631$
N $0.075 \times {}^{-}0.99$
O ${}^{-}3.109 \times {}^{-}65.13$

SET 32

1. Represent each number in scientific notation.

A 400
B 3000
C 6300
D 20 000
E 54 000
F 350 000
G 9 000 000
H 45 600 000 000
I 8 030 000 000
J 7 905 000 000 000
K 0.5
L 8.49
M 0.14
N 0.07
O 0.003
P 0.0702
Q 0.00009
R 0.000453
S 0.0000008
T 0.0008002

2. Give the ordinary decimal for each number.

A 4.2×10^{4}
B 5.61×10^{2}
C 3.7×10^{6}
D 9.22×10^{1}
E 4.05×10^{5}
F 3.2×10^{9}
G 9.55×10^{0}
H 1.07×10^{8}
I 9.005×10^{4}
J 3.54×10^{7}
K 6×10^{-1}
L 8×10^{-3}
M 1.5×10^{-1}
N 3.4×10^{-2}
O 6.91×10^{-4}
P 7×10^{-8}
Q 4.4×10^{-10}
R 3.9×10^{-6}
S 8.01×10^{-7}
T 1.392×10^{-11}

SET 33

1. Find the quotients.

A $6\overline{)14.4}$	E $59\overline{)207.09}$	I $0.3\overline{)19.02}$	M $0.73\overline{)17.155}$
B $8\overline{)507.2}$	F $81\overline{)25.920}$	J $1.8\overline{)6.498}$	N $0.048\overline{)0.01728}$
C $15\overline{)139.5}$	G $1.3\overline{)0.3296}$	K $0.35\overline{)2.275}$	O $8.17\overline{)8.0883}$
D $26\overline{)98.80}$	H $542\overline{)504.06}$	L $4.05\overline{)1.4985}$	P $4.96\overline{)0.56544}$

2. Find the quotients. Annex zeros when necessary.

A $5\overline{)4.1}$	F $0.8\overline{)92.4}$	K $160\overline{)24}$	P $0.604\overline{)2.4764}$
B $12\overline{)61.8}$	G $0.55\overline{)22}$	L $8.8\overline{)1188}$	Q $5.56\overline{)4.0588}$
C $18\overline{)6.21}$	H $7.2\overline{)28.8}$	M $0.516\overline{)23.22}$	R $7.6\overline{)4.028}$
D $32\overline{)33.6}$	I $0.25\overline{)8.07}$	N $4.68\overline{)2.574}$	S $0.021\overline{)0.0714}$
E $95\overline{)5.89}$	J $1.35\overline{)621}$	O $0.38\overline{)2.47}$	T $9.012\overline{)67.69}$

SET 34

1. Round each number to one significant digit. Use scientific notation and estimate the products.

A 4264×7536	F 21.6×0.047	K 0.647×3846
B 567.3×384.5	G 8376×5.66	L $54\,230 \times 57.62$
C 62.7×218.8	H 325.7×478	M 7.06×0.407
D 0.736×0.38	I 0.731×537.9	N 6573×0.0045
E 74.2×6.78	J 865.7×0.338	O 0.00873×0.00531

2. Round each number to one significant digit. Use scientific notation and estimate the quotients.

A $19.76 \div 3.95$	E $88.6 \div 331.2$	I $0.00173 \div 0.0041$
B $382.5 \div 2.04$	F $75.9 \div 0.384$	J $0.083 \div 0.00437$
C $605.3 \div 28.4$	G $0.207 \div 0.0039$	K $0.44285 \div 0.21756$
D $86{,}730 \div 5.99$	H $6327 \div 4.578$	L $0.000872 \div 0.00285$

SET 35

1. Find the decimal approximation for $\frac{8}{15}$ to the nearest

   A tenth. **B** hundredth. **C** thousandth.

2. Find the decimal approximation of each number to the nearest hundredth.

   A $\frac{7}{8}$ **B** $\frac{3}{8}$ **C** $\frac{1}{9}$ **D** $\frac{4}{11}$ **E** $\frac{7}{18}$ **F** $\frac{20}{23}$

3. Find the decimal approximation of each number to the nearest thousandth.

   A $\frac{11}{16}$ **B** $\frac{8}{9}$ **C** $\frac{15}{16}$ **D** $\frac{32}{45}$ **E** $\frac{41}{71}$ **F** $\frac{38}{39}$

SET 36

1. Solve the proportions.

   A $\frac{3}{5} = \frac{x}{40}$ **C** $\frac{4}{9} = \frac{x}{45}$ **E** $3:4 = x:36$ **G** $x:9 = 16:24$

   B $\frac{5}{6} = \frac{x}{54}$ **D** $\frac{x}{24} = \frac{15}{40}$ **F** $5:7 = 30:x$ **H** $9:x = 36:60$

2. Write and solve a proportion for each problem.

   A Apples sell at 6 for 43 cents. How much for 18 apples?

   B The ratio of boys to girls is 5:7. If there are 42 girls, how many boys are there?

   C If a car averages 8 kilometres per litre of gasoline, how many kilometres can it go on 23 litres?

   D A basketball player made 7 free throws out of every 10 attempts. How many free throws did he make in 130 attempts?

   E On a map, 3 centimetres represents 75 kilometres. How many kilometres of actual distance is represented by 6.4 centimetres on the map?

SET 37

Solve the following problems involving scale drawings.

1. Measure the length of the bolt in centimetres. What is the actual length of the bolt, if the scale is

A 1:4? B 1:8? C 2:1? D 1:7.5?

2. If the scale on a map is 1 cm = 90 km, what is the actual distance between two cities whose distance apart on the map is 3.1 cm?

3. The actual distance between two cities is 720 km. What is the distance on the map if the scale for the map is 1 cm = 40 km?

4. A lot for Mr. Hoppe's new house measures 80 metres wide by 150 metres long. What would be the dimensions of a scale drawing of the lot using the scale 1 cm = 20 metres?

SET 38

1. Write a decimal for each percent.

A 18%	C 10%	E 5%	G 100%	I 2%	K 7.5%	M 1.5%
B 33%	D 9%	F 70%	H 1%	J 75%	L 4%	N 33.3%

2. Write a percent for each decimal.

A 0.38	C 0.2	E 1.5	G 1.30	I 0.093	K 16.3	M 0.017
B 0.65	D 0.03	F 0.15	H 2.05	J 5.5	L 0.178	N 0.309

3. Give the lowest-terms fraction for each percent.

A 37%	C 40%	E 75%	G 2%	I 15%	K 28%	M 520%
B 50%	D 30%	F 5%	H 1%	J 16%	L 8%	N 64%

4. Find a decimal for each fraction to the nearest hundredth, when necessary. Then express the decimal as a percent.

A $\frac{59}{100}$	C $\frac{1}{100}$	E $\frac{17}{50}$	G $\frac{9}{25}$	I $\frac{7}{18}$	K $\frac{5}{7}$	M $\frac{9}{15}$
B $\frac{3}{100}$	D $\frac{3}{4}$	F $\frac{11}{40}$	H $\frac{9}{16}$	J $\frac{5}{12}$	L $\frac{7}{23}$	N $\frac{27}{29}$

SET 39

Solve these percent problems.

1. Price of sweater: $15.75.
   Sale: 20% off.
   - A Find the amount of discount.
   - B What is the sale price?

2. Class: 35 students.
   Today: 28 present.
   What percent is present?

3. Earnings: $650 per month.
   Deductions: 26%.
   - A How much deducted?
   - B How much take-home pay?

4. Price of 10-speed bicycle: $155.
   Sales tax: 6%.
   - A What is the amount of sales tax?
   - B What is the total cost of the bicycle?

5. Purchase price of house: $38 000.
   Selling price: 120% of original purchase price.
   What is the selling price?

6. Which costs less? how much less?
   - [A] $25 coat at 20% off.
   - [B] $30 coat at 30% off.

SET 40

1. Write a decimal for each percent.

A 270%	C 1000%	E 1250%	G 0.3%	I $\frac{1}{2}$%	K 0.06%
B 850%	D 795%	F 3500%	H $\frac{7}{10}$%	J 0.35%	L $\frac{1}{8}$%

2. Find the amounts.

A 300% of 650	D 1200% of 525	G 0.6% of 750	J 0.04% of 3000
B 425% of 400	E 750% of 300	H $\frac{1}{2}$% of 988	K $\frac{3}{20}$% of 1500
C 170% of 1600	F 903% of 8000	I $\frac{1}{5}$% of 1280	L $\frac{1}{8}$% of 5000

3. Solve each percent problem.
   - A Population in 1940: 76 500.
     Population in 1970: 225% more.
     What is the population in 1970?
   - B Length of material: 3 metres.
     Percent of shrinkage: 0.25%.
     Find the greatest amount of shrinkage.

SET 41

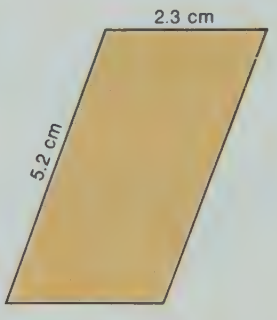
1. Express each measure in metres.

A 300 cm	**D** 15 km	**G** 4.6 km	**J** 0.1 km	**M** 0.1 cm
B 1300 mm	**E** 40 dm	**H** 40 dam	**K** 0.1 hm	**N** 0.1 mm
C 5 km	**F** 74 cm	**I** 530 mm	**L** 0.1 dam	**O** 0.003 km

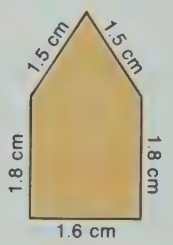
2. Find the perimeter of each region.

A

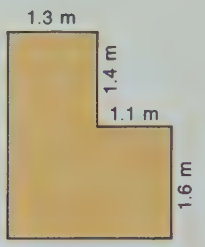
B



C



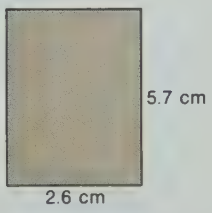
D



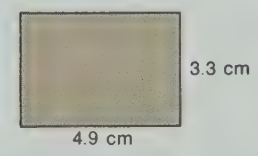
SET 42

1. Find the area of each rectangular region. Use the formula $A = lw$.

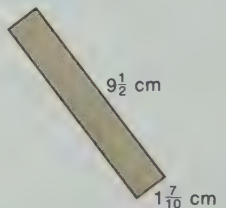
A



B



C

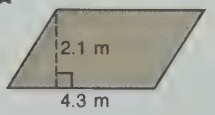


D $l = 65$ cm
$w = 32.1$ cm

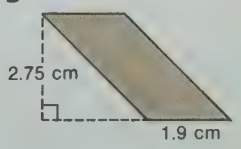
E $l = 8.75$ m
$w = 3.09$ m

2. Find the area of each parallelogram. Use the formula $A = bh$.

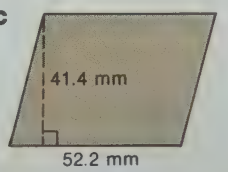
A



B



C



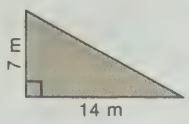
D $b = 48.3$ mm
$h = 56.1$ mm

E $b = 3.88$ dm
$h = 1.36$ dm

SET 43

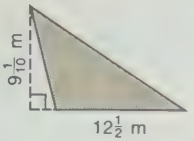
1. Find the area of each triangular region.

A

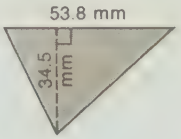


B

C



D



2. Find the area of each triangular region. Use the formula $A = \frac{1}{2}bh$.

A $b = 150$ cm
$h = 93$ cm

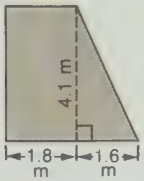
B $b = 8.5$ m
$h = 4.9$ m

C $b = 8\frac{1}{2}$ m
$h = 4\frac{3}{10}$ m

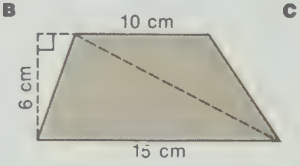
D $b = 9.54$ dm
$h = 9.07$ dm

3. Find the area of each region.

A

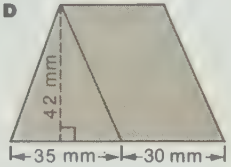


B



C

D



SET 44

1. Find the **sums** of the angle measures.

A $38°\,15'\,14''$
$27°\,33'\,42''$

B $22°\,34'\,47''$
$35°\,13'\,35''$

C $45°\,22'\,16''$
$53°\,49'\,33''$

D $62°\,15'\,28''$
$32°\,44'\,40''$

E $87°\,26'\,54''$
$43°\,15'\,\ 7''$

F $55°\,36'\,45''$
$57°\,48'\,48''$

G $73°\,48'\,16''$
$16°\,11'\,44''$

H $145°\,36'\,52''$
$38°\,14'\,18''$

2. Find the **differences** of the angle measures.

A $36°\,38'\,25''$
$14°\,12'\,19''$

B $48°\,37'\,15''$
$21°\,13'\,28''$

C $86°\,25'\,39''$
$35°\,41'\,28''$

D $123°\ 9'\,37''$
$95°\,48'\,50''$

E $136°\,38'\,14''$
$74°\,52'\,36''$

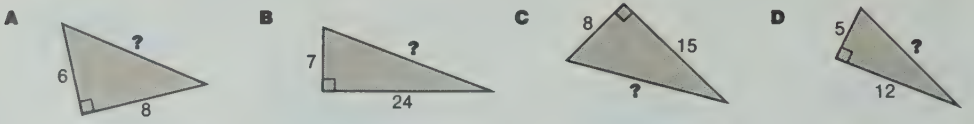
F $112°\,56'\,22''$
$88°\,55'\,28''$

G $75°\,48'$
$49°\,59'\,11''$

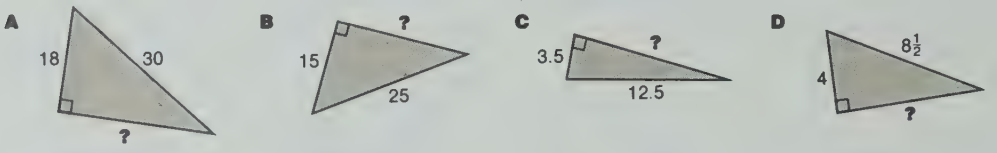
H $90°$
$43°\,52'\,47''$

SET 45

1. Find the length of the hypotenuse of each right triangle.



2. Find the length of the third side of each right triangle.



3. Find the length of the third side of each right triangle.

A $a = 15, b = 20$

B $a = 10, b = 24$

C $a = 18, b = 24$

D $a = 21, b = 28$

E $a = 8, \ c = 15$

F $b = 48, c = 52$

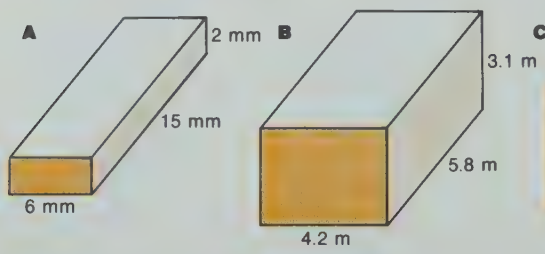
G $a = 16, \ c = 34$

H $b = 4.4, c = 5.5$

I $a = 6, \ c = 6.8$

SET 46

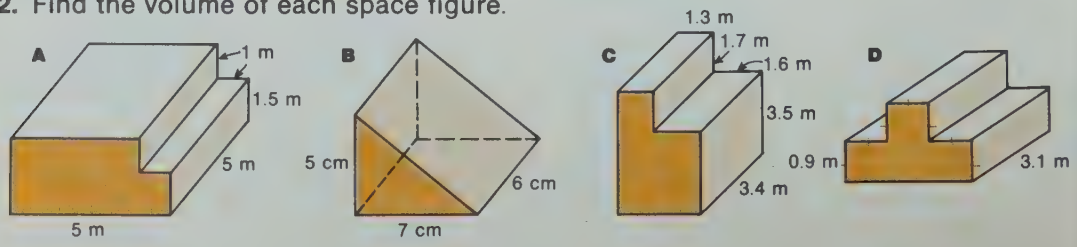
1. Find the volume of each rectangular solid. Use the formula $V = lwh$.



D $l = 220$ mm, $w = 175$ mm, $h = 30$ mm

E $l = 8.6$ m, $w = 4.9$ m, $h = 7.05$ m

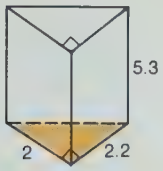
2. Find the volume of each space figure.



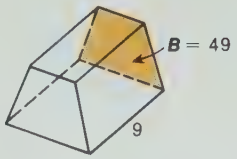
SET 47

1. Find the volume of each prism.

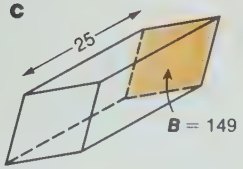
A



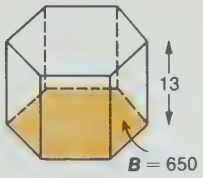
B



C

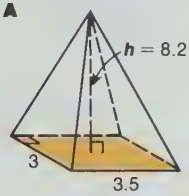


D

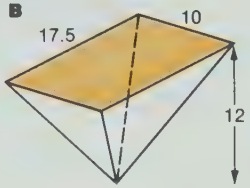


2. Find the volume of each pyramid.

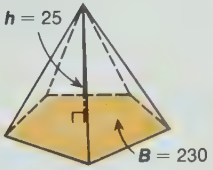
A



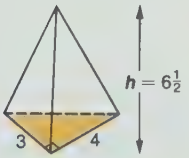
B



C

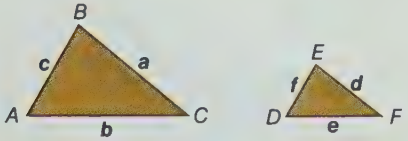


D



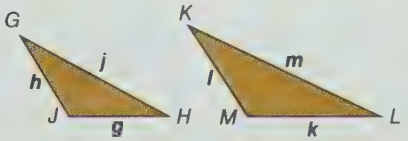
SET 48

1. $\triangle ABC \sim \triangle DEF$. The letters a, b, c, d, e, and f represent the lengths of the sides.



A Find b, if $a = 2$, $e = 9$, $d = 6$.
B Find f, if $c = 16$, $e = 3$, $b = 6$.
C Find a, if $d = 8$, $b = 24$, $e = 6$.
D Find d, if $a = 21$, $c = 18$, $f = 6$.
E Find c, if $a = 42$, $d = 14$, $f = 13$.
F Find e, if $a = 6.4$, $b = 4.2$, $d = 3.2$.
G Find b, if $e = 1\frac{1}{2}$, $f = 1\frac{3}{4}$, $c = 3\frac{1}{2}$.
H Find d, if $a = 0.9$, $c = 0.8$, $f = 0.4$.

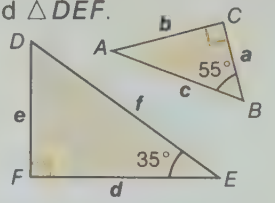
2. $\triangle GHJ \sim \triangle KLM$. The letters g, h j, k, l, and m represent the lengths of the sides.



A Find g, if $j = 8$, $k = 7$, $m = 14$.
B Find m, if $j = 8$, $l = 18$, $h = 12$.
C Find j, if $m = 24$, $g = 12$, $k = 32$.
D Find h, if $l = 12$, $j = 15$, $m = 36$.
E Find k, if $g = 6$, $m = 72$, $j = 9$.
F Find l, if $h = 10$, $k = 48$, $g = 16$.
G Find g, if $h = 5.5$, $l = 11$, $k = 7.8$.
H Find j, if $h = 3\frac{1}{3}$, $l = 10$, $m = 12$.

SET 49

1. Refer to $\triangle ABC$ and $\triangle DEF$.

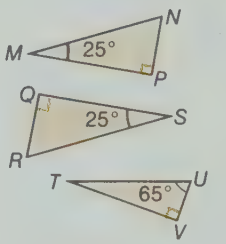


   A $\angle A \cong \angle \underline{?}$

   B $\frac{a}{b} =$ [illegible]

   C $\triangle ABC \sim \triangle \underline{?}$

2. Refer to $\triangle MNP$, $\triangle QRS$, and $\triangle TUV$.



   A $\frac{\overline{MN}}{\overline{NP}} = \frac{RS}{[illegible]} = \frac{[illegible]}{[illegible]}$

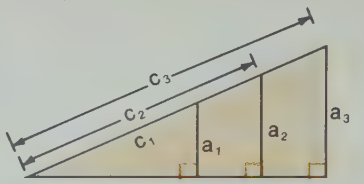
   B $\angle Q \cong \angle \underline{?} \cong \angle \underline{?}$

   C $\triangle MNP \sim \triangle \underline{?} \sim \triangle \underline{?}$

3. Suppose $\frac{a_1}{c_1} = \frac{2}{5}$. Give the ratios.

   A $\frac{a_2}{c_2}$

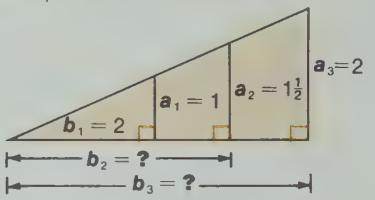
   B $\frac{a_3}{c_3}$



4. Suppose $\frac{a_1}{b_1} = \frac{1}{2}$. Find these lengths.

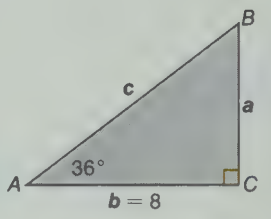
   A b_2

   B b_3



SET 50

1. In right $\triangle ABC$, $b = 8$, $\tan 36° \approx 0.73$, and $\cos 36° \approx 0.81$. Find a and c.



   A $\tan 36° = \frac{a}{8}$

   $\frac{a}{8} \approx 0.73$

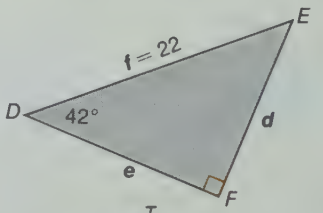
   $a \approx$ [illegible]

   B $\cos 36° = \frac{8}{c}$

   $\frac{8}{c} \approx 0.81$

   $c \approx$ [illegible]

2. In right $\triangle DEF$, $f = 22$, $\sin 42° \approx 0.67$, and $\cos 42° \approx 0.74$. Find d and e.



   A $\sin 42° = \frac{d}{22}$

   $\frac{d}{22} \approx 0.67$

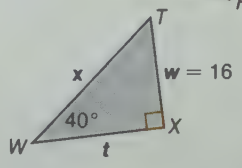
   $d \approx$ [illegible]

   B $\cos 42° = \frac{e}{22}$

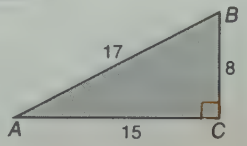
   $\frac{e}{22} \approx 0.74$

   $e \approx$ [illegible]

3. In right $\triangle TWX$, $w = 16$, $\sin 40° \approx 0.64$, $\tan 40° \approx 0.84$. Find t and x.



4. Use the figure to compute the trigonometric ratios for $\angle A$ to the nearest hundredth.



   A $\tan A$ B $\sin A$ C $\cos A$

SET 51

1. Find the circumference of each circle with the given diameter. Use the formula $C = \pi d$ and the fact $\pi \approx 3.14$.

A

B

C $d = 14$ cm

D $d = 23$ cm

E $d = 310$ mm

F $d = 1.6$ m

G $d = 17.3$ cm

H $d = 0.9$ km

2. Find the circumference of each circle with the given radius. Use the formula $C = 2\pi r$ and the fact $\pi \approx 3.14$.

A

B

C $r = 22$ mm

D $r = 440$ m

E $r = 52\frac{1}{2}$ cm

F $r = 8.3$ m

G $r = 0.35$ mm

H $r = 6.15$ m

SET 52

1. Find the area of each circular region. Use the formula $A = \pi r^2$ and the fact $\pi \approx 3.14$.

A

B

C $r = 20$ m

D $r = 12$ cm

E $r = 1.5$ m

F $r = 3\frac{1}{10}$ m

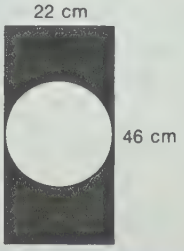
G $r = 3.2$ dm

H $r = 0.4$ mm

2. A circular hole is cut in the rectangular board at the right.

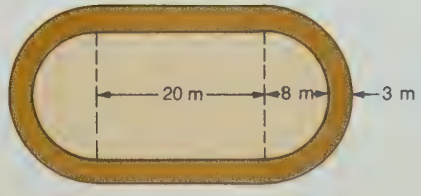
A If the diameter of the circle is the same as the width of the board, what is the area of the circle?

B What is the area of the rectangular region after the circular region is removed?



3. **A** Find the area of the inner region of the track.

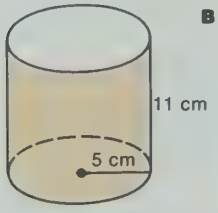
B Find the area of the track.



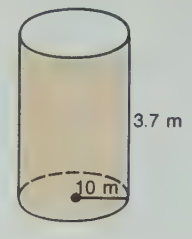
SET 53

1. Find the volume of each right circular cylinder. Use the formula $V = \pi r^2 h$ and the fact $\pi \approx 3.14$.

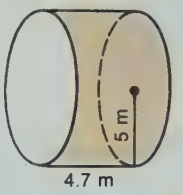
A



B



C



D $r = 10$ m, $h = 20$ m

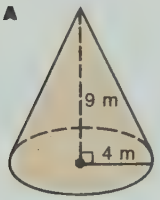
E $r = 12$ cm, $h = 15$ cm

F $r = 3.1$ m, $h = 5.2$ m

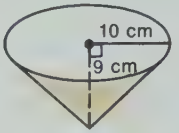
G $r = 2.2$ m, $h = 1.53$ m

2. Find the volume of each right circular cone. Use the formula $V = \frac{1}{3}\pi r^2 h$ and the fact $\pi \approx 3.14$.

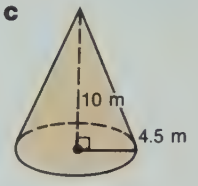
A



B



C



D $r = 5$ cm, $h = 7.5$ cm

E $r = 40$ m, $h = 60$ m

F $r = 1.02$ m, $h = 4.2$ m

G $r = 0.8$ m, $h = 1.8$ m

SET 54

1. Suppose the cube has the letters **A**, **B**, **B**, **C**, **C**, and **C** on its faces. Give the probability that each event will appear.

A $P(\mathbf{A})$ **B** $P(\mathbf{B})$ **C** $P(\mathbf{C})$

2. Refer to the spinner. Give the probability that each event will occur.

A P(even) **C** P(multiple of 3)

B P(odd) **D** P(multiple of 4)

3. An experiment of tossing a die has a sample space $\{1, 2, 3, 4, 5, 6\}$. Give the probability that each event will occur.

A $P(x = 1)$ **D** $P(x > 5)$ **G** $P(1 < x < 4)$

B $P(x > 1)$ **E** $P(x < 5)$ **H** $P(x$ is even$)$

C $P(x \leq 3)$ **F** $P(x \leq 4)$ **I** $P(x > 6)$

SET 55

1. Compute each of the following:

A $4!$ **B** $6!$ **C** $7!$ **D** $2!\,4!$ **E** $3!\,5!$ **F** $6! \div 3!$

2. How many different arrangements of the letters in the word CART are possible? List them to verify your answer.

3. How many different arrangements of the letters in the word CANDY are possible?

4. How many different ways can 8 students be arranged in a row of 8 desks?

5. How many different ways can 9 pictures be arranged in a straight line on a wall?

6. Solve the equations. Give your answer as a factorial, when possible.

A $7 \cdot 6! = n$
B $9! \cdot n = 10!$
C $5! \cdot n = 7!$
D $6! \div 30 = n$
E $5! \div 5 = n$
F $8! \div n = 8$
G $7! \cdot 8 \cdot 9 = n$
H $6! \div 3! = n$
I $6! \div 5! = n$

SET 56

1. From the set $\{3, 4, 5, 6, 7, 8\}$, a 2-digit numeral is to be selected.

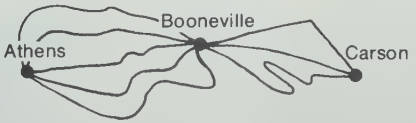
A How many different digits can be used for the tens' digit?

B How many different digits can be used for the ones' digit?

C How many 2-digit numerals are possible?

2. Susan has 6 sweaters and 4 skirts. How many different outfits does she have if any sweater can be worn with any skirt?

3. There are 5 roads from Athens to Booneville and 3 roads from Booneville to Carson. How many different routes are there from Athens to Carson?



4. From a menu of 8 items, you can choose any 3 of them. How many choices do you have?

5. Compute the number of combinations indicated.

A ${}_4C_2$ **B** ${}_6C_3$ **C** ${}_7C_5$ **D** ${}_8C_5$ **E** ${}_7C_3$ **F** ${}_9C_6$

SET 57

1. Suppose an experiment consists of tossing a die. The sample space is $\{1, 2, 3, 4, 5, 6\}$.

   A Let E be the event of getting a 1 or a 6. What is $P(E)$?
   B Let F be the event of getting a 2, 3, or 5. What is $P(F)$?
   C Find $E \cap F$. What is $P(E \cap F)$?
   D Find $E \cup F$. What is $P(E \cup F)$?

2. A spinner is divided into 8 parts of equal size and numbered 1 through 8.

   A Let E be the event of getting a 1, 2, or 3. What is $P(E)$?
   B Let F be the event of getting an odd number. What is $P(F)$?
   C What is $P(E \cap F)$? D What is $P(E \cup F)$?

3. An experiment consists of drawing two cards from those labeled E, F, G, H, I. Let E be the event of getting a vowel and let F be the event of getting a consonant.
   A What is $P(E)$? B What is $P(F)$? C What is $P(E \cup F)$?

SET 58

1. What is the average (arithmetic mean) of each set of numbers?
   A 25, 27, 29
   B 66, 65, 70, 67
   C 253, 250, 213, 248
   D 123, 101, 132, 130, 114
   E 208, 206, 199, 244, 235, 216
   F 4500, 4498, 4603, 4555

2. What is the average (arithmetic mean) to the nearest whole number?
   A 38, 50, 49, 61, 56
   B 305, 299, 316, 278, 366
   C 135, 164, 173, 122, 155, 90
   D 675, 701, 699, 693, 711, 700
   E 1266, 1313, 1288, 1295
   F 102, 99, 75, 87, 96, 92, 67
   G 58.2, 56.7, 52.5, 54.3
   H 48.73, 53.71, 59.77, 49.99

SET 59

1. Find the median of each set of numbers.

A 46, 48, 49, 51, 55

B 89, 84, 88, 80

C 120, 123, 128, 127

D 38, 35, 41, 42, 39, 44

E 1200, 1300, 1400, 1500

F 8, 8, 9, 10, 10, 11, 12, 12, 12

G 21, 19, 15, 17, 18, 19, 23, 20

H 303, 305, 333, 306, 325, 321

I 1560, 1545, 1598, 1534, 1555

J 72.6, 68.1, 93.1, 79.4

Solve the problems.

2. Test scores:
63, 52, 72, 75, 81, 76, 80, 77, 65
What is the median score?

3. Bowling scores:
156, 143, 152, 118, 121, 125, 104, 123, 118, 126, 142
What is the median score?

4. Masses of 6 teen-agers:
58.2, 56.7, 52.5, 54.3, 57.2, 55.4

A What is the median?

B What is the mean?

C Which shows the more "average" of the masses, the mean or the median?

SET 60

1. In 120 tosses of a die, a class found the frequency for each outcome as given in the table. What is the mode for the experiment?

Outcome						
Frequency	14	16	11	35	22	21

2. A student survey of the number of books read in a month is shown in the table. How many books did the greatest number of students read?

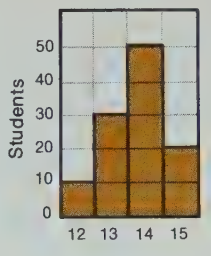
Books read	1–2	2–3	3–4	4–5	5–6
Frequency	12	18	10	6	4

3. The table shows the temperatures of some cities on a certain day in March. What is the mode?

Temperature °C	1–5	6–10	11–15	16–20	21–25	26–30
Frequency	15	26	19	32	8	10

SET 61

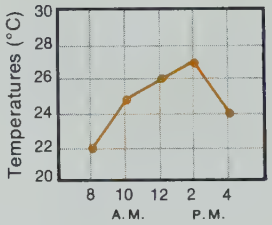
1. **A** How many students are represented by this histogram?



B What percent are 13 years old?

C What percent are over 13 years old?

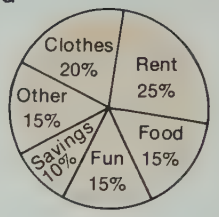
2. **A** What is the highest temperature?



B What is the lowest temperature?

C How many degrees did the temperature rise between 8 A.M. and noon?

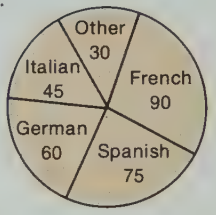
3. The circle graph shows how Marcia accounted for her monthly earnings of $600.



A How much was spent for rent?

B How much was spent for food?

4. The circle graph shows how 300 students are taking foreign language classes.



A What percent take French?

B What percent take Italian?

SET 62

1. Find the positive square root of each number.

A 169	**E** 1156	**I** 3136	**M** 98.01	**Q** 46.24
B 529	**F** 5041	**J** 7396	**N** 77.44	**R** 14.44
C 441	**G** 2916	**K** 6889	**O** 70.56	**S** 0.3259
D 256	**H** 2209	**L** 8836	**P** 37.21	**T** 0.1849

2. Find an approximation of each number to the nearest tenth.

A $\sqrt{14}$	**D** $\sqrt{150}$	**G** $\sqrt{500}$	**J** $\sqrt{12.5}$	**M** $\sqrt{51.9}$
B $\sqrt{23}$	**E** $\sqrt{425}$	**H** $\sqrt{9100}$	**K** $\sqrt{26.5}$	**N** $\sqrt{0.55}$
C $\sqrt{69}$	**F** $\sqrt{485}$	**I** $\sqrt{3000}$	**L** $\sqrt{41.3}$	**O** $\sqrt{0.11}$

Mathematical Symbols

$=$	Is equal to
$\neq$	Is not equal to
$>$	Is greater than
$<$	Is less than
$\leq$	Is less than or equal to
$\approx$	Is approximately
$\cong$	Is congruent to
$\sim$	Is similar to
$\{a, b, c\}$	Set with elements a, b, c
ϕ	Empty set
$\cup$	Union
$\cap$	Intersection

$\overleftrightarrow{AB}$	Line through points A and B
$\overrightarrow{AB}$	Ray AB with endpoint A
$\overline{AB}$	Segment with endpoints A and B
$\angle ABC$	Angle ABC with vertex B
$m\angle ABC$	Measure of angle ABC
$\triangle ABC$	Triangle ABC
$\overset{\frown}{RS}$	Arc with endpoints R and S
$35^\circ\ 14'\ 20''$	Thirty-five *degrees*, fourteen *minutes*, twenty *seconds*
%	Percent
π	Pi

Metric System Prefixes

kilo-	one thousand
hecto-	one hundred
deca-	ten
deci-	one tenth
centi-	one hundredth
milli-	one thousandth

Formulae

$P = a + b + c$	Perimeter of triangle
$P = 2 \cdot l + 2 \cdot w$	Perimeter of rectangle
$A = l \cdot w$	Area of rectangle
$A = b \cdot h$	Area of parallelogram
$A = \frac{1}{2} \cdot b \cdot h$	Area of triangle
$a^2 + b^2 = c^2$	Pythagorean Theorem
$V = l \cdot w \cdot h$	Volume of rectangular solid
$C = \pi \cdot d$	Circumference of circle
$A = \pi \cdot r^2$	Area of circle
$V = \frac{4}{3} \cdot \pi \cdot r^3$	Volume of sphere
$V = \pi \cdot r^2 \cdot h$	Volume of cylinder
$V = \frac{1}{3} \cdot \pi \cdot r^2 h$	Volume of cone
$V = B \cdot h$	Volume of prism (B = base area)
$V = \frac{1}{3} \cdot B \cdot h$	Volume of pyramid

Table of Powers and Roots

Number n	Square n^2	Square root $\sqrt{n}$	Square root of 10 times n $\sqrt{10 \times n}$	Number n	Square n^2	Square root $\sqrt{n}$	Square root of 10 times n $\sqrt{10 \times n}$
1	1	1.000	3.162	51	2601	7.141	22.583
2	4	1.414	4.472	52	2704	7.211	22.804
3	9	1.732	5.477	53	2809	7.280	23.022
4	16	2.000	6.325	54	2916	7.348	23.238
5	25	2.236	7.071	55	3025	7.416	23.452
6	36	2.449	7.746	56	3136	7.483	23.664
7	49	2.646	8.367	57	3249	7.550	23.875
8	64	2.828	8.944	58	3364	7.616	24.083
9	81	3.000	9.487	59	3481	7.681	24.290
10	100	3.162	10.000	60	3600	7.746	24.495
11	121	3.317	10.488	61	3721	7.810	24.698
12	144	3.464	10.954	62	3844	7.874	24.900
13	169	3.606	11.402	63	3969	7.937	25.100
14	196	3.742	11.832	64	4096	8.000	25.298
15	225	3.873	12.247	65	4225	8.062	25.495
16	256	4.000	12.649	66	4356	8.124	25.690
17	289	4.123	13.038	67	4489	8.185	25.884
18	324	4.243	13.416	68	4624	8.246	26.077
19	361	4.359	13.784	69	4761	8.307	26.268
20	400	4.472	14.142	70	4900	8.367	26.458
21	441	4.583	14.491	71	5041	8.426	26.646
22	484	4.690	14.832	72	5184	8.485	26.833
23	529	4.796	15.166	73	5329	8.544	27.019
24	576	4.899	15.492	74	5476	8.602	27.203
25	625	5.000	15.811	75	5625	8.660	27.386
26	676	5.099	16.125	76	5776	8.718	27.568
27	729	5.196	16.432	77	5929	8.775	27.749
28	784	5.292	16.733	78	6084	8.832	27.928
29	841	5.385	17.029	79	6241	8.888	28.107
30	900	5.477	17.321	80	6400	8.944	28.284
31	961	5.568	17.607	81	6561	9.000	28.461
32	1024	5.657	17.889	82	6724	9.055	28.636
33	1089	5.745	18.166	83	6889	9.110	28.810
34	1156	5.831	18.439	84	7056	9.165	28.983
35	1225	5.916	18.708	85	7225	9.220	29.155
36	1296	6.000	18.974	86	7396	9.274	29.326
37	1369	6.083	19.235	87	7569	9.327	29.496
38	1444	6.164	19.494	88	7744	9.381	29.665
39	1521	6.245	19.748	89	7921	9.434	29.833
40	1600	6.325	20.000	90	8100	9.487	30.000
41	1681	6.403	20.248	91	8281	9.539	30.166
42	1764	6.481	20.494	92	8464	9.592	30.332
43	1849	6.557	20.736	93	8649	9.644	30.496
44	1936	6.633	20.976	94	8836	9.695	30.659
45	2025	6.708	21.213	95	9025	9.747	30.822
46	2116	6.782	21.448	96	9216	9.798	30.984
47	2209	6.856	21.679	97	9409	9.849	31.145
48	2304	6.928	21.909	98	9604	9.899	31.305
49	2401	7.000	22.136	99	9801	9.950	31.464
50	2500	7.071	22.361	100	10 000	10.000	31.62

GLOSSARY

abacus A device used for calculating, usually involving sliding beads or counters along a wire.

absolute value The undirected distance from a point on the number line to the origin. The absolute value of a number x is either x or ^{-}x, whichever is nonnegative. We denote absolute value with vertical lines: $|^{-}5| = 5$; $|5| = 5$; $|0| = 0$.

acute angle An angle smaller than a right angle.

addend Any one member of a set of numbers to be added. In the equation $7 + 9 = 16$, 7 and 9 are addends.

addition An operation that combines a first number (*addend*) and a second number (*addend*) to give another number (this number is called the *sum* of the two addends).

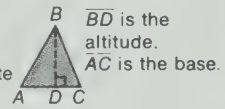
additive inverse Each of two numbers whose sum is zero is said to be the additive inverse of the other.

adjacent angles Two angles with a common vertex, a common side, and no common interior points. In the figure, $\angle 1$ and $\angle 2$ are adjacent angles.

adjacent sides Two sides of a polygon with a common vertex.

algorithm (algorism) A computation procedure. For example, the algorithm for finding the quotient of two numbers.

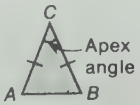
altitude (of a triangle) The segment from any vertex perpendicular to the opposite side. Also, see *height*.



angle The union of two rays with a common endpoint (vertex).

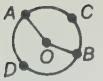
angle bisector The ray which divides an angle into two congruent angles. In the figure $\overline{DB}$ bisects $\angle ADC$.

apex angle The angle of an isosceles triangle whose vertex is the intersection of the two congruent sides of the triangle.



approximation A number that is suitably "close" to another number. For example, a decimal approximation for $\frac{2}{9}$ is 0.22. The symbol "$\approx$" denotes "is approximately"; for example, $\pi \approx 3.14$.

arc If $\angle AOB$ is a central angle of a circle, arc ACB ($\overset{\frown}{ACB}$) is the set of points on the circle which are interior to $\angle AOB$ or on $\angle AOB$. $\overset{\frown}{ACB}$ denotes a *minor arc*. $\overset{\frown}{ADB}$ denotes a *major arc*.



area The measure of a plane region in terms of a chosen unit, usually a square.

arithmetic mean The quotient of the sum of numbers in a set divided by the number of numbers in that set (Often called the "average.")

associative (grouping) principle The principle which states that the sum (or product) of three or more numbers is the same regardless of grouping:
$(a + b) + c = a + (b + c)$
or $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

average See *arithmetic mean*.

axes (x-axis, y-axis) The coordinate axes in a plane are perpendicular number lines used to match each point in the plane with an ordered pair of numbers. Usually, the x-axis is the horizontal number line and is directed to the right while the y-axis is the vertical number line and is directed upward.

bar graph See *histogram*.

base (of numeration) The term "base" refers to the type of grouping involved in a system of numeration. For example, in base eight: $25_{(8)}$ means 2 eights and 5; $346_{(8)}$ means 3 sixty-fours, 4 eights, and 6.

base (of a polygon) Any side of a polygon may be referred to as a base. See figure for *altitude*.

base (of a space figure) See examples below.

bisect To divide into two congruent parts.

Celsius temperature Temperature based on the Celsius scale in which water freezes at 0° and boils at 100°

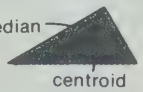
centre A (fixed) point of symmetry for a geometric figure. See the figure for *circle*.

centi- A prefix meaning *one hundredth*.

centimetre A unit of length that is $\frac{1}{100}$ of a metre.

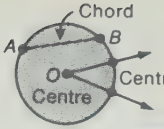
central angle An angle with its vertex at the centre of the circle. See the figure for *circle*.

centroid (of a triangle) The point of intersection of the three medians of a triangle.



chord A segment with both endpoints on a circle. See the figure for *circle*.

circle The set of all points in a plane which are a specified distance from a fixed point called the *centre*.

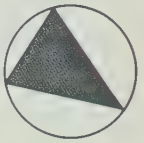


circle graph A diagram used to show statistical information, particularly to show how a given quantity is divided into fractional parts.

circumcentre The point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle circumscribing the triangle.

circumference The limit of the perimeters of regular inscribed polygons in a circle. The circumference of a circle is the product of its diameter and the number π. The formula is $c = \pi d$.

circumscribed circle A circle drawn through the vertices of a polygon. A circle can be circumscribed about any triangle but only about certain other polygons.



circumscribed polygon If the sides of a polygon are tangent to a circle, then the polygon is circumscribed about the circle. Also, see *inscribed circle*.

closed curve Intuitively, a curve is closed if you can begin at any point and trace the curve until you return to the starting point.

closure A property of a subset of a set A relative to an operation defined on the set A. For example, the operation of addition on the set of whole numbers is closed relative to the subset of even whole numbers, for the sum of any two evens is even. It is not closed upon the subset of odd whole numbers, for the sum of two odds is not odd. In the set of rational numbers the subset of integers is closed under subtraction, but the subset of whole numbers is not.

collinear (points) Two or more points lying on the same line.

combination Essentially, the word combination means the same as "subset." We say that from a set of 4 elements, 6 combinations of 2 elements can be formed, meaning that the set of 4 has 6 subsets of 2. We denote this as ${}_4C_2$.

common factor A number that is a factor of each of two or more given numbers. For example, 3 is a common factor of 6 and 9.

common multiple A number that is a multiple of each of two or more given numbers. For example, 18 is a common multiple of 2, 3, and 9.

commutative (order) principle The principle which states that the sum (or product) of any two numbers is the same regardless of the order in which they are added (or multiplied):
$a + b = b + a$ or $a \cdot b = b \cdot a$

complementary angles Two angles whose measures have a sum of 90°.

complex fraction A fraction that has a fraction for the numerator or the denominator, or both.

composite number Any whole number greater than 1 which is not prime. Composite numbers have more than two factors.

concave figure See *convex figure*.

cone A space figure formed by a closed plane curve and all line segments from a point not in the plane of the curve to all the points of the curve.

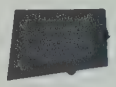
congruent angles Angles which have the same measure.

congruent figures Two figures are congruent if one of the figures is the image of the other by one of the *rigid motions* (reflection, rotation, translation) or by any combination of motions.

congruent integers Two integers a and b are congruent, modulo m if the difference $a - b$ is divisible by m. We write $a \equiv b \pmod{m}$.
$23 \equiv 7 \pmod 4$ because $23 - 7$ is divisible by 4.
$5 \equiv {}^{-}13 \pmod 3$ because $5 - {}^{-}13$ is divisible by 3.

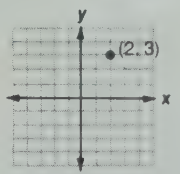
congruent segments Segments whose endpoints are equally far apart; segments which have the same length.

convex figure A figure is convex if every segment joining any two points on the boundary of the figure contains only points on the figure or in its interior.

convex figure 

non-convex or **concave** figure

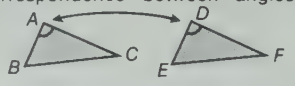
coordinate axes Two intersecting perpendicular number lines used for graphing ordered number pairs (*coordinates*).



coordinates An ordered number pair matched with a point in the coordinate plane. See the figure for *coordinate axes*.

correspondence A matching of elements in one set with those of another set.

corresponding angles Two angles matched by any one-to-one correspondence between angles of two triangles. For example, $\angle A$ and $\angle D$.



corresponding points Two points matched by any one-to-one correspondence between points of two triangles. For example, point C corresponds to point F above.

corresponding segments Two segments matched by any one-to-one correspondence between segments of two figures. For example, $\overline{AB}$ and $\overline{DE}$ are corresponding segments above.

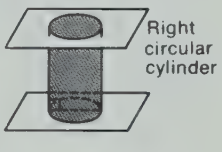
cosine See *trigonometric ratios*.

cross section The intersection of a space figure and a plane.

cube (geometry) A regular polyhedron each of whose six faces is a square. It is also a regular hexahedron.

cube (numeration) A number raised to the third power. 8 is the cube of 2 because $2^3 = 8$. Also, to raise a number to the third power.

cylinder The space figure formed by two congruent curves in parallel planes and the parallel segments connecting corresponding points of the curves.



deca- A prefix meaning *ten*.

decametre A unit of length that is 10 metres.

deci- A prefix meaning *one tenth*.

decimal (numeral) Any base-ten numeral written using a decimal point.

decimal system of numeration A system of numeration in which powers of ten and place value are used.

decimetre A unit of length that is $\frac{1}{10}$ of a metre, or 10 centimetres.

degree A unit angle that is $\frac{1}{90}$ of a right angle.

denominator For each fraction $\frac{a}{b}$, $b \neq 0$, b is the denominator.

diagonal A segment connecting two nonconsecutive vertices of a polygon.

diameter Any chord that contains the centre of a circle.

difference The number resulting from subtraction.

digits The basic symbols used in a place-value system of numeration. In base ten, the symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

distributive principle The fundamental principle connecting addition and multiplication.
$a \cdot (b + c) = a \cdot b + a \cdot c$.

dividend The number to be divided in a division problem.

$$\begin{array}{r} 3 \leftarrow \text{Quotient} \\ 5\overline{)17} \leftarrow \text{Dividend} \\ \text{Divisor} \rightarrow \underline{15} \\ 2 \leftarrow \text{Remainder} \end{array}$$

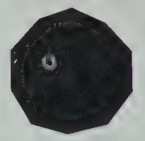
divisible A given number is divisible by a second number if the remainder is zero. If $a = b \cdot c$ with $b \neq 0$ and $c \neq 0$, then a is divisible by either b or c.

division The inverse of the operation of multiplication.

$$\begin{array}{ccc} F & F & P \\ 3 \cdot 4 = 12 & \begin{array}{l} \rightarrow \\ \rightarrow \end{array} & \begin{array}{ccc} P & F & F \\ 12 \div 4 = 3 \\ 12 \div 3 = 4 \end{array} \end{array}$$

divisor See *dividend*.

dodecahedron A polyhedron which has 12 faces. A regular dodecahedron has 12 congruent pentagonal faces.



domain The set of all input numbers for a function.

edge A segment that is a side of any face of a space figure. A cube has 12 edges.

element A member of a set.

empty set The set containing no elements. It is denoted by "{ }" or the symbol ϕ.

equal sets Sets containing the same elements. They may be listed in different order.

equality The relation of identity. A statement of equality asserts that two symbols name the same object or number.

equally likely outcomes Outcomes or results that have the same chance of occurring.

equation A mathematical sentence using the equality symbol (=). $7 + n = 9$ is an equation.

equiangular triangle A triangle all of whose angles have the same measure.

equilateral triangle A triangle having all sides the same length.

equivalent decimals Decimals which represent the same number. For example, 0.5, 0.50, and 0.500 are equivalent decimals.

equivalent fractions Fractions which represent the same number, such as $\frac{1}{2}$, $\frac{5}{10}$ and $\frac{50}{100}$.

equivalent ratios Two ratios which represent the same rational number. For example, the ratio 5:10 is the same as 1:2.

equivalent sets Sets having the same number of elements.

estimate An approximation for a given number. Often used in the sense of a rough calculation.

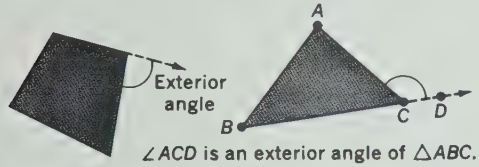
even number A whole-number multiple of 2. The set of even numbers is $\{0, 2, 4, 6 \ldots\}$

event Any subset of a sample space for a probability experiment. If a die is tossed, the event of getting a number greater than 3 has occurred if any number in the subset $\{4, 5, 6\}$ results from the toss.

expanded notation A representation of a number as a sum of powers of ten such as:
$3424 = 3 \cdot 10^3 + 4 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$.

exponent In a symbol such as 5^4, the numeral 4 is an *exponent* and 5 is the *base*. Positive exponents indicate how many times the base is to be used as a factor; $5^4 = 5 \cdot 5 \cdot 5 \cdot 5$. Negative exponents are used to indicate reciprocals of powers of numbers; $6^{-3} = \frac{1}{6^3}$, $10^{-4} = \frac{1}{10^4}$.

exterior angle of a polygon An angle formed by extending any one side of a polygon. A triangle has six exterior angles.



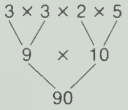
$\angle ACD$ is an exterior angle of $\triangle ABC$.

extremes The first and last terms in a proportion written in the form $a:b = c:d$. The numbers a and d are extremes and b and c are called the *means*.

face Any one of the bounding polygonal regions of a space figure. See the figure for *edge*.

factor Any one member of a set of numbers to be multiplied. In the equation $5 \cdot 7 = 35$, 5 and 7 are factors.

factor tree A diagram suggestive of a tree showing the prime factorization of a number.



factorial If n is a positive whole number, factorial n denoted by $n!$, is defined to be
$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-2) \cdot (n-1) \cdot n$.
$0!$ is defined to be 1.

flow chart A diagram which gives instructions in a logical order.

formula A general fact or rule expressed by using symbols. For example, the area any parallelogram with base b and height h is given by $b \cdot h$. The formula is usually presented with the equation $A = b \cdot h$.

fraction A symbol for a fractional number of the form $\frac{a}{b}$ such that $b \neq 0$.

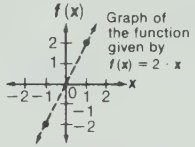
fractional number The set of non-negative rational numbers.

frequency The number of times an outcome occurs in a probability experiment or the number of times a bit of data occurs in a collection of data.

function A correspondence that associates each element in a first set with a unique element of a second set. Different elements in a first set may possibly be matched with the same element of the second set.

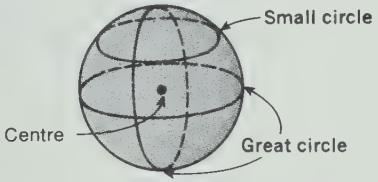
gram A unit of mass. The mass of one cubic centimetre of water at 4°C.

graph A set of points associated with a given set of numbers or number pairs showing a relation or function.



graph (statistical) A picture used to illustrate a given collection of data. It may be in the form of a bar graph, a circle graph, a line-segment graph, or a pictograph.

great circle A circle formed by the intersection of a sphere and a plane containing the centre of the sphere.



greater than A number x is greater than a number y (written $x > y$) if $x = y + p$ and p is positive. $^-3$ is greater than $^-5$, for $^-3 = {}^-5 + 2$. We write, $^-3 > {}^-5$.

greatest common factor (GCF) The largest, or greatest, number that is a factor of each of a set of numbers.

greatest possible error Half of the basic unit in which a measurement is given. For example, if a length is given as 18 centimetres to the nearest centimetre, then the basic unit is the centimetre and the greatest possible error is 0.5 centimetre.

hecto- A prefix meaning *one hundred*.

height (of a prism) The perpendicular distance between the planes of the parallel faces.

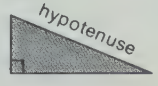
heptagon A 7-sided polygon.

hexagon A 6-sided polygon.

Hindu-Arabic numeral A numeral for a whole number formed from the basic Hindu-Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

histogram A bar graph showing the frequency distribution of a set of data.

hypotenuse The side of a right triangle opposite the right angle.



icosahedron A polyhedron of 20 faces. A regular icosahedron has equilateral triangles for its faces.

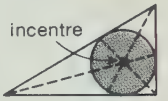
identity element For addition, zero is the identity element since $a + 0 = a$ for any number a. Similarly, for multiplication, 1 is the identity element since $a \cdot 1 = a$.

identity principles For any number a, $a + 0 = a$ and $a \cdot 1 = a$.

image The figure obtained after a rigid motion of a given figure. From a functional viewpoint, y is the image of x if there is a function f such that y is the output when x is the input; that is, $f(x) = y$.

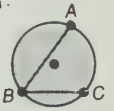
improper fraction A fraction whose numerator is greater than or equal to its denominator.

incentre The point of intersection of the angle bisectors of each angle of a triangle. The centre of the circle inscribed in a triangle.



inequality A mathematical statement using either of the symbols $<$ or $>$ such as $\frac{3}{4} < \frac{7}{8}$.

inscribed angle An angle that contains three points of the circle and has its vertex on the circle.



inscribed polygon See *circumscribed circle*.

integers The set consisting of the natural numbers, the negatives of the natural numbers, and zero; $\{\ldots {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, \ldots\}$.

interpolation When we wish to find a number between two entries in a table, we interpolate. For example, if a table of square roots shows

$$\sqrt{7} = 2.646 \quad \text{and} \quad \sqrt{8} = 2.828$$

then a good approximation for $\sqrt{7.5}$ can be interpolated between these numbers by choosing the number midway between them:

$\sqrt{7.5}$ is approximately $\dfrac{2.646 + 2.828}{2} = 2.737$.

intersection The intersection of two sets is the set of all elements common to both sets.

inverse (operation) Operations which are opposite in effect such as addition and subtraction or multiplication and division.

irrational number A real number that cannot be expressed as the quotient of two integers; a non-rational number. Irrational numbers have infinite nonrepeating decimal representations.

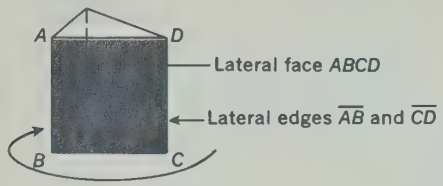
isosceles triangle A triangle with at least two congruent sides.

kilo- A prefix meaning *one thousand*.

kilogram A unit of mass that is 1000 grams.

kilometre A unit of length that is 1000 metres.

lateral edges An edge of a prism connecting the parallel faces.



lateral face One of the faces of a prism containing two lateral edges. (See figure for lateral edges.)

least common denominator The least common multiple of the denominators of two or more fractions. for example, the LCD of $\frac{5}{6}$ and $\frac{3}{4}$ is 12.

least common multiple (LCM) The smallest non-zero number that is a multiple of each of two or more given numbers. The LCM of 4 and 6 is 12.

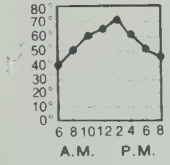
legs (of a right triangle) The perpendicular sides of a right triangle.

length The measure of a segment in terms of a chosen unit.

less than a is less than b (written $a < b$) if b is greater than a. (*See* greater than.)

line of symmetry Line t is a line of symmetry of a set S of points if for each point A of S there is a corresponding point B such that t is the perpendicular bisector of $\overline{AB}$.

line-segment graph A diagram used to show statistical information, particularly that of a continuous nature.



litre The basic unit of capacity in the metric system. 1000 cubic centimetres.

lowest-terms (simplest) fraction A fraction is in lowest terms if the numerator and denominator of the fraction have no common factor other than 1.

mass The amount of matter in a physical object. Except in scientific work, the word "weight" is used synonymously with mass. A rock whose *mass* is 1 kilogram on earth will have a mass of 1 kilogram on the moon, but its weight on the moon will be only about $\frac{1}{6}$ its earth weight.

mean See *arithmetic mean.*

means The second and third terms in a proportion written in the form $a:b = c:d$. See *extremes.*

median (of a set) The middle number of a set containing an odd number of elements that are arranged in order. If a set contains an even number of elements, the average of the two middle numbers is the median.

median (of a triangle) The segment from the vertex to the midpoint of the opposite side of a triangle.

metre The basic unit of length in the metric system. Defined as 1 650 763.73 times the wave length of the orange-red spectral emission line of krypton-86.

metric system A system of measurement in which the metre, the litre, and the kilogram are the basic units of length, capacity, and mass.

midpoint (middle point) A point which divides a segment into two congruent segments.

milli- A prefix meaning *one thousandth.*

minute A unit of angular measure that is $\frac{1}{60}$ of a degree.

mixed decimal numeral A combination of a decimal and a fraction such as $.4\frac{1}{3}$ and read as "four and one-third tenths."

mixed numeral A numeral such as $4\frac{2}{3}$ indicating the sum of a whole number and a fractional number; $4\frac{2}{3} = 4 + \frac{2}{3}$.

mode The most frequently occurring element of a set. There may be more than one mode of a set.

multiple A first number is a multiple of a second number if there is a whole number that multiplies by the second number to give the first number. For example, 24 is a multiple of 4 and 6 since $4 \times 6 = 24$.

multiplication An operation that combines a first number (*factor*) and a second number (*factor*) to give another number (this result is called the *product* of the two factors).

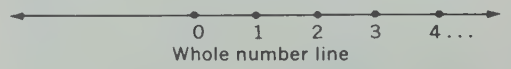
multiplicative inverse See *reciprocal.*

negative integers The set of numbers $\{^{-}1, ^{-}2, ^{-}3, \ldots\}$.

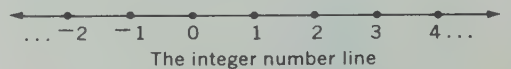
nonagon A 9-sided polygon.

nonrepeating decimal A decimal not formed from any point on by repeating blocks of digits. Each nonrepeating decimal represents an irrational number, while each repeating decimal represents a rational number.

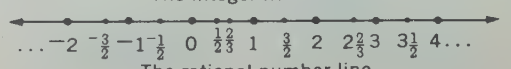
number line A line with a subset of its points matched with a subset of the real numbers. We say that the rational number line has "holes" in it because some points are not matched with rational numbers. The real number line is said to be "complete" because each point is matched with some real number. (*See* real number line.)



Whole number line



The integer number line



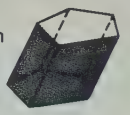
The rational number line

number pair An ordered pair of numbers such as (3, 4); often, co-ordinates of a point in a plane.

numeral A symbol for a number.

numerator For each fraction $\frac{a}{b}$, a is the numerator.

oblique prism A prism in which the base is not perpendicular to its lateral edges.

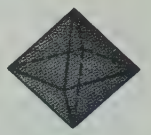

Oblique prism

Right prism

obtuse angle An angle greater than a right angle and smaller than a straight angle.

octagon An 8-sided polygon.

octahedron A polyhedron having eight faces. Each face of a regular octahedron is an equilateral triangle.



odd number Any number in the set $\{1, 3, 5, 7, \ldots\}$.

one principle For any number a, $a \cdot 1 = a$. See *identity principle* for multiplication.

one-to-one correspondence Two sets S and T are in one-to-one correspondence if there is a matching so that each element in S is matched with exactly one element in T, and vice versa.

operation A binary operation like "multiplication" associates each order paired of numbers with one number. Usually we speak of the four fundamental operations: addition, subtraction, multiplication, and division.

opposites principle For any integer a, $a + {}^{-}a = 0$.

origin The intersection of the co-ordinate axes; the point associated with the number pair (0, 0).

outcome A possible result in a probability experiment.

parallel lines Lines in the same plane which do not intersect.

parallelepiped A space figure all of whose faces are parallelograms.

parallelogram A quadrilateral whose opposite sides are parallel.

pentagon A 5-sided polygon.

percent Literally means "per hundred". A symbol such as 3% is an abbreviation for "3 per 100" or "$\frac{3}{100}$."

perimeter The distance around a polygon.

period Each set of three digits in a numeral. In the numeral 421 365 761 409, there are 421 billions, 365 millions, 761 thousands, and 409 units.

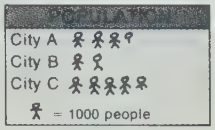
permutation Any ordered subset of a set is referred to as a permutation. For example, six permutations of subsets of 2 can be formed from any set of three elements.
$\{a, b, c\} \longrightarrow (a, b), (b, a), (a, c), (c, a), (b, c), (c, b)$

perpendicular bisector A line which bisects a segment as well as being perpendicular to it.

perpendicular lines Two intersecting lines that form right angles.

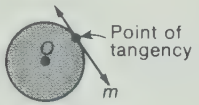
pi (π) The ratio of the circumference of a circle to its diameter. $\pi \approx 3.14159$.

pictograph A diagram used to show statistical information with the use of pictures.



plane figure A set of points in one plane.

point of tangency If a line and circle intersect in just one point, the intersection is the point of tangency.



polygon A closed plane figure whose boundary is made up of segments.

polygonal path A path consisting of a union of segments which are connected end to end.

polyhedron A space figure each of whose faces is a polygonal region.

positive integer Any number in the set $\{1, 2, 3, \ldots\}$.

power In the statement $a = b^n$, a is the n^{th} power of b. For example, for $1000 = 10^3$, 1000 is the third power of 10.

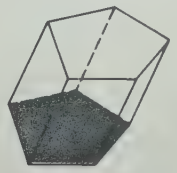
precision One measurement is *more precise* than another if its *relative* error is smaller than that of the other.

prime factorization An expression of a composite number as a product of prime factors.

prime number A whole number greater than 1 whose only factors are itself and 1.

prism A 3-dimensional (space) figure whose bases are congruent polygonal regions in parallel planes and whose faces are parallelograms.

Triangular prism


Pentagonal prism

probability The ratio of the number of times a certain outcome can occur to the number of total possible outcomes.

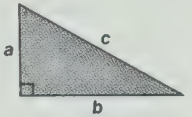
product The number $a \cdot b$ which results from applying the multiplication operation to the numbers a and b.

proportion An equation stating that two ratios are equal: $a:b = c:d$ or $\frac{a}{b} = \frac{c}{d}$.

protractor An instrument for measuring angles.

pyramid A 3-dimensional (space) figure with a polygonal base and triangular lateral faces.

Pythagorean Theorem In any right triangle, the sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.



$a^2 + b^2 = c^2$

quadrilateral A 4-sided polygon.

quotient The number $a \div b$ (or $b \div a$) which results from applying the division operation to the numbers a and b.

radian A unit of angular measure. A central angle of a circle subtended by an arc whose length is equal to the radius; approximately 57.3°; exactly $(180 \div \pi)°$.

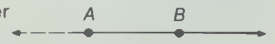
radical sign The sign $\sqrt{\ }$ which indicates that a root is the be extracted. For example, $\sqrt[3]{7}$ is the cube root of 7 and $\sqrt{8}$ is the positive square root of 8.

radius Any segment from the centre of a circle to a point on the circle.

ratio The ratio of two numbers a and b is their quotient, $\frac{a}{b}$.

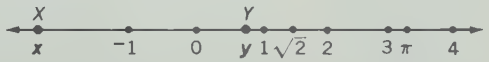
rational number The quotient of two integers, the divisor not being zero.

ray A half-line together with the point determining it (endpoint).



Ray AB or $\overrightarrow{AB}$

real number line Any line with its points matched one to one with the real numbers is called a real number line.

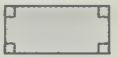


The real number line. A point for each real number and a number for each point.

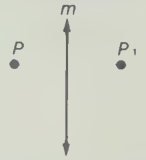
real numbers All numbers, either positive, negative, or zero, having decimal representations. Those real numbers with nonrepeating decimals are the irrational numbers.

reciprocal If $a \cdot b = 1$, each of the numbers a and b is the reciprocal (multiplicative inverse) of the other. Each nonzero real number has a unique reciprocal. Zero has no reciprocal.

rectangle A parallelogram with four right angles.



reflection A rigid motion that maps the points of a plane onto itself. If point P is reflected in a line with a reflection image P_1, then the line is the perpendicular *bisector of* $\overline{PP_1}$.



region All the points in the part of a plane bounded by a simple closed curve.

The interior of a quadrilateral is a region.

This curve forms two regions.

regular polygon A polygon that is equiangular and equilateral; for example, a square.

relative error The ratio of the greatest possible error of a measurement to the measurement itself. For example, if a measurement is given as 10 centimetres to the nearest centimetre and the greatest possible error is 0.5 centimetre, then the relative error is the ratio of 0.5 to 10 or 5%.

remainder In whole-number division, if dividend D is divided by divisor d and the relation $D = (q \cdot d) + r$ is obtained where $0 < r < d$, then r is called the remainder. Note that $r = 0$ if d is a factor of D (see *dividend*). Also, the difference in a subtraction problem.

repeating decimal A decimal whose digits can be grouped from some point on in repeating periods. Examples: $0.333\ldots$; $0.2727\ldots$; $4.273149149\ldots$. In these examples, the repeating decimals have (respectively) periods of 1, 2, and 3 digits and can be written as $0.\overline{3}$, $0.\overline{27}$, and $4.\overline{273149}$.

rhombus An equilateral parallelogram which is not a square.

right angle An angle whose measure is 90°.

rigid motion A transformation of the points of the plane that preserves all distances.

Roman numerals The numerals I, V, X, L, C, D, M, and combinations of the numerals used in the Roman numeration system.

rotation A rigid motion in which the points of the plane are turned about a fixed point.

rounding A process of replacing a number by an approximation (another number) with fewer significant digits. For example, rounding 456 789 to the nearest thousand would be 457 000.

sample space The set of all possible outcomes of a probability experiment.

scale drawing A drawing of an object made so that distances in the drawing are proportional to actual distances. A scale of 1 : 10 indicates that distances in the drawing are $\frac{1}{10}$ of the actual distances.

scalene triangle A triangle with no pair of congruent sides.

scientific notation A notation used for writing any number as the product of a power of ten and a number between 1 and 10 (including 1).

second A unit of angular measure that is $\frac{1}{60}$ of a minute.

segment A set of points consisting of two points A and B and all points between them.

Segment AB or $\overline{AB}$

semicircle An arc that is exactly half a circle. One of two arcs cut off by a diameter.

set A group or collection of objects.

Sieve of Eratosthenes A process of determining the set of all primes that are less than a finite number n.

significant digits All nonzero digits reported in a measurement are significant, but if terminal zeros occur, then it must be specified whether or not they are significant. For example, suppose a population is reported as 43 000.
43 000: If only the 4, 3 are significant, then the population is expected to fall between 42 500 and 43 500.
43 000: If the first zero is also significant, then the population is expected to fall between 42 950 and 43 050.
In scientific notation, all digits in the factor times the power of ten are significant.
2.3×10^4 has 2 significant digits.
2.370×10^4 has 4 significant digits.

similar polygons Two polygons are similar if there is a one-to-one correspondence between the vertices such that corresponding angles are congruent and corresponding sides are proportional. Congruence is a special case of similarity.

sine See *trigonometric ratios.*

skew lines Lines that are not in the same plane. They do not intersect, yet they are not parallel.

space The set of all points. This term usually denotes a 3-dimensional quality. We also refer to a plane as 2-dimensional space and a line as a 1-dimensional space.

space figure A geometric figure whose points are not all in the same plane.

sphere The set of all points in space at a fixed distance from a given point.

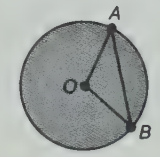
square root If $x^2 = y$, then x is a square root of y. We use the sign $\sqrt{\ }$ to indicate the operation of extracting the positive square root of a number. For example, in the set of integers, 9 has two square roots, 3 and $^-3$. The symbol $\sqrt{9}$ by agreement denotes only the positive square root, 3.

statistics The science of analyzing numerical information.

straightedge An unmarked ruler.

subset Set A is a subset of set B if every element of A belongs to B.

subtend Literally means "to be opposite to." For example, central angles and inscribed angles in a circle are said to subtend the arcs they cut off. Also, chords and arcs in a circle subtend the angles of a circle.



$\overset{\frown}{AB}$ subtends $\angle AOB$ or
$\overline{AB}$ is subtended by $\angle AOB$

subtraction The operation that is the inverse of the addition operation.

$$\begin{matrix} A & A & S \\ 5 & + 6 & = 11 \end{matrix} \quad \begin{matrix} S & A & A \\ 11 & - 5 & = 6 \\ 11 & - 6 & = 5 \end{matrix}$$

sum The number $a + b$ which results when the operation of addition is applied to the numbers a and b.

supplementary angles Two angles whose measures have a sum of 180°.

surface area The total area of the polygonal regions (faces) of a polyhedron.

symmetric figure A plane figure which can be divided into two congruent parts by a line (of symmetry).

tangent A line which intersects a circle in just one point. See also *trigonometric ratios.*

$\overleftrightarrow{AB}$ is tangent to circle O at point A.

terminating decimal A decimal such as 0.5, 1., 1.24, 0.0307, and so on, that represents the quotient of a whole number and a power of 10:

$$0.5 = \tfrac{5}{10},\ 1.24 = \tfrac{124}{10^2},\ 0.0307 = \tfrac{307}{10^4}$$

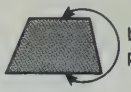
tessellation A repeated pattern of geometric figures which will completely "cover a plane" without any gaps or overlapping.

tetrahedron A polyhedron having 4 faces.

theorem A statement to be proved true by means of basic assumptions and mathematical reasoning.

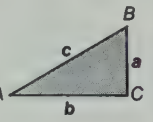
translation A rigid motion in which each point of the plane is moved the same distance and in the same direction.

trapezoid A quadrilateral with a pair of parallel sides.


bases, or parallel sides

triangle A 3-sided polygon.

trigonometric ratios If one of the acute angles of a right triangle is chosen, then each of the six possible ratios that can be formed between two of the sides of the triangle is called a trigonometric ratio of that angle.



$\frac{a}{c}$ is the sine of $\angle A$ (sin A).

$\frac{b}{c}$ is the cosine of $\angle A$ (cos A).

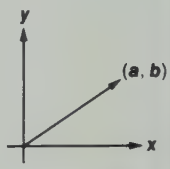
$\frac{a}{b}$ is the trangent of $\angle A$ (tan A).

union If A and B are sets, then the union of A and B ($A \cup B$) is the set consisting of all elements that belong to at least one of the two sets.

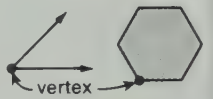
unit The object adopted as a standard of measurement. For length, we choose a *unit* segment; for area, we choose a *unit* square; and for volume, a *cube.*

variable A symbol, usually a letter, used to represent any number in a given set. For example, we may say: "If x is a whole number, then $2 \cdot x$ is an even number."

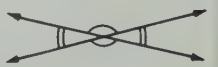
vector A directed line segment. In the co-ordinate plane a vector is often denoted by an arrow emanating from the origin and is denoted by the ordered pair of the point at the tip of the arrow for the vector.



vertex A point that two rays of an angle have in common. Also, the common point of any two sides of a polygon.



vertical angles Two pairs of angles formed by two intersecting lines.



volume The measure of a space figure in terms of a chosen unit.

whole numbers Any number in the set $\{0, 1, 2, 3, \ldots\}$.

zero principle The principle which states that for any number a, $a + 0 = a$. Also, see *identity element* for addition.

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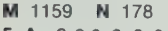
Answers

UNIT E: MODULE 1

E-2 DISCUSSING THE IDEAS

2. **3.** 237

E-3 USING THE IDEAS

1. A 34 **B** 122 **C** 12 304 **D** 230 123 **E** 1 020 262
F 2 200 306 **2. A** 7 **B** 35 **C** 43 **D** 620 **3. A** 27
B 1616 **C** 2877 **4. A** 5 **B** 6 **C** 4 **D** 10 **E** 11
F 9 **G** 60 **H** 40 **I** 600 **J** 400 **K** 1100 **L** 900
M 1159 **N** 178 **O** 1970
5. A  **B**
C CCXLVII **6.** MMMCLXXIX
Think 1776

E-4 DISCUSSING THE IDEAS

1. A 126 **2. A** 125 **B** group **3. A** ten thousands; hundred thousands; millions **B** millions, billions, trillions and so on

E-5 USING THE IDEAS

1. B 900 **C** 4000 **D** 600 000 **E** 5 000 000
2. A 874 **B** 3691 **C** 28 437 **D** 100 101 **E** 67 080
F 3 700 000 **3. A** 6 000 000 000 000 000
B 73 000 000 000 **C** 276 000 000 000 000 000 000 000 000
D 1 000 000 000 000 000 000 000 000 000 000 000
E 439 000 000 000 000 000 000 000 000 000 000 **F** 89 000 000
4. 1 000 000 000 000
Think 61

E-6 DISCUSSING THE IDEAS

? A 0, 15; 1, 14; 2, 13; 3, 10; 4, 11; 6, 10; 6, 9; 7, 8;
B 1.36; 2.18; 3.12; 4.9; 6.6; **1.** 9 **2.** 9 **3.** 6 **4.** 6

E-7 USING THE IDEAS

1. A 14 **B** 13 **C** 25 **D** 80 **E** 200 **F** 1300 **G** 8
H 5 **I** 7 **J** 0 **K** 8 **L** 138 **M** 35 **N** 72 **O** 0 **P** 1
Q 3 **R** 8 **2. A** 9 **B** 7 **C** 3 **D** 9 **E** 7 **F** 90
G 502 **H** 10 **I** 63 **3. A** 729 **B** 779 **C** 786
D 1265 **E** 1295 **F** 1303 **G** 563 **H** 513 **I** 507
J 317 **K** 277 **L** 269
Think *A*, *C*, *D*, *F*

E-9 SOME SPECIAL FUNCTION MACHINE RULES

1. A 24 **B** 120 **C** 200 **2. A** 18 **B** 34 **C** 90
3. A 6 **B** 20 **C** 28 **4. A** 110 **B** 82 **C** 200
5. A 11 **B** 30 **C** 800 **6. A** 21 **B** 45 **C** 39
7. cubeachadd
Think **1.** 4, 15 **2.** 5, 24 **3.** 6, 35 **4.** 10, 99
5. 20, 399 **6.** 100, 9999

E-10 DISCUSSING THE IDEAS

? 512 **1. A** 8 **B** 64 **C** 8 **D** $8 \cdot 8 \cdot 8$ **2. A** 5 **3.** 8^3

E-11 USING THE IDEAS

1. A 7^2 **B** 3^3 **C** 2^5 **D** 10^4 **E** 8^6 **F** 10^5 **G** 5^2
H 9^7 **I** 4^3 **2. A** 3 **B** 2 **C** 6 **D** 7 **E** 19 **F** 50
G 1000 **3. A** 1000 **B** 100 000 **C** 1 000 000
D 100 000 000 **E** 1 000 000 000 **F** 1 000 000 000 000
G 10 **4. A** 10^4 **B** 10^5 **C** 10^5 **D** 10^4 **E** 10^7 **F** 10^6
G 10^8 **H** 10^9 **I** 10^{13} **5. A** 10^2 **B** 10^3 **C** 10^1
D 10^3 **6. A** 2 **B** 3 **C** 4 **7.** D **8. A** $(2 \cdot 10) + (7 \cdot 10^0)$
B $(4 \cdot 10^2) + (7 \cdot 10^1) + (3 \cdot 10^0)$ **C** $(5 \cdot 10^2) + (7 \cdot 10^0)$
D $(9 \cdot 10^3) + (6 \cdot 10^2) + (2 \cdot 10^1) + (5 \cdot 10^0)$ **E** $(8 \cdot 10^4) + (7 \cdot 10^3) + (3 \cdot 10^1) + (4 \cdot 10^0)$ **F** $(4 \cdot 10^6) + (6 \cdot 10^5) + (3 \cdot 10^4) + (2 \cdot 10^3) + (4 \cdot 10^2) + (7 \cdot 10^1)$
Think 2^{1000}

E-12 DISCUSSING THE IDEAS

? A 280 **B** 46 000 **C** 2 400 000 **D** 3200
E 760 000 **F** 54 000 000 **1.** 300 **2.** 3200 **5.** 3000

E-13 USING THE IDEAS

1. A 6000 **B** 6000 **2. A** 2 857 460 **B** 2 857 500
C 2 857 000 **D** 2 860 000 **E** 2 900 000 **3. A** 437 000
B 2 084 000 **C** 1 639 000 **D** 31 000 **4. A** 9 976 000
B 10 000 000 **5. A** 6100 **B** 6000 **6. A** 8 500 000
B 9 000 000
Think 639

E-14 DISCUSSING THE IDEAS

1. A yes **B** yes **C** 390
4. A $200 + 400 + 100 + 400 = 1100$
B $60 \times 600 = 36\,000$ **C** $1600 \div 40 = 40$

E-15 USING THE IDEAS

1. A 600 **B** 700 **C** 80 **D** 70 **E** 700 **F** 4000
G 500 **H** 300 **I** 90 **J** 8000 **K** 400 **2. A** 900
B 6300 **C** 4800 **D** 30 **E** 500 **F** 24 000 **G** 900
H 18 000 **I** 230 **J** 10 000 **K** 280 000 **3. A** 1400
B 1100 **C** 400 **D** 2400 **E** 200 **F** 49 000 **G** 9000
H 200 **I** 100 000 **J** 1000 **K** 11 000 **L** 18 000 **4.** 5
5. $20 000 000 000 **6. A** 10 **B** 2 **C** USSR

E-16 DISCUSSING THE IDEAS

? 1. 13 556 **2.** 144 835 **3.** 18 642 **4.** 376, R39
4. 90 000 **6.** B **7.** 608, R42

E-17 USING THE IDEAS

1. A 525 **B** 2442 **C** 881 188 **D** 69 696 **E** 15 851
F 43 911 934 **2. A** 1089 **B** 1089 **C** 1089 **D** 1089
3. A 4416 **B** 4416 **C** 215 418 **D** 215 418 **4. A** 52
B 3713, R90 **C** 839
5. A

$$\begin{array}{r} 5837 \\ +\ 9678 \\ \hline 15515 \end{array}$$

B

$$\begin{array}{r} 4007 \\ -\ 2989 \\ \hline 1018 \end{array}$$

C

$$\begin{array}{r} 476 \\ 999 \\ \hline 4284 \\ 4284 \\ 4284 \\ \hline 475524 \end{array}$$

D

$$\begin{array}{r} 11 \\ 76\overline{)836} \\ 76 \\ \hline 76 \\ 76 \\ \hline 0 \end{array}$$

E-18 DISCUSSING THE IDEAS

1. No **2. A** $63 = 32 + 16 + 8 + 4 + 2 + 1$
B $31 = 16 + 8 + 4 + 2 + 1$ **C** $15 = 8 + 4 + 2 + 1$
D $7 = 4 + 2 + 1$ **E** $3 = 2 + 1$ **3. A** 6 **B** 5 **C** 4
D 3 **E** 2 **F** 1 **G** 0 **4. A** 100 **B** 15 **C** 53
5. 111011

E-19 USING THE IDEAS

1. A 1000110 **B** 1101111 **C** 0110001 **D** 1010011
E 1111010 **F** 1100010 **2. A** 102 **B** 39 **C** 85 **D** 59

E 44 **F** 126 **3. A** 65 **B** 53 **C** 95 **D** 77 **E** 56
F 117 **4. A** 111 **B** 1101 **C** 11010 **D** 11110
E 101010 **F** 110010 **G** 0111101 **H** 1000011
I 1001001 **J** 1010000 **K** 1100100 **L** 1111101

E-20 DISCUSSING THE IDEAS

? 105 117 1110101 **1.** 2 **2.** 0, 1
3. A $(1\cdot2^1)+(1\cdot2^0)$ **B** $(1\cdot2^2)$
C $(1\cdot2^2)+(1\cdot2^1)+(1\cdot2^0)$ **D** $(1\cdot2^3)+(1\cdot2^1)+(1\cdot2^0)$
E $(1\cdot2^4)+(1\cdot2^2)+(1\cdot2^0)$

E-21 USING THE IDEAS

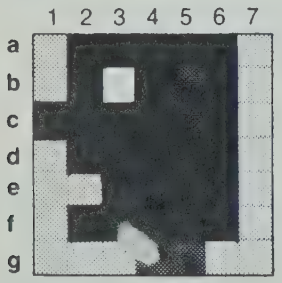
1. A twos, 1×2^1 **B** sixteens, 1×2^4
C sixty-fours, 1×2^6 **2. A** 1 **B** 2 **C** 7 **D** 4 **E** 5
F 6 **G** 3 **H** 12 **I** 9 **J** 10 **K** 11 **L** 15 **3.** 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192 **4. A** 14
B 8 **C** 28 **D** 16 **E** 31 **F** 32 **G** 119 **H** 170
5. A $11_{(2)}$ **B** $101_{(2)}$ **C** $11000_{(2)}$ **D** $101110_{(2)}$
6. A $10011100_{(2)}$ **B** $101110100_{(2)}$ **C** $1000011011_{(2)}$
D $1010101100_{(2)}$ **E** $1110000100_{(2)}$ **F** $1111111111_{(2)}$
G $10011111111_{(2)}$ **H** $11000110000_{(2)}$
Think **A** 23 **B** 133 **C** 225 **D** 511 **E** 433

E-22 DISCUSSING THE IDEAS

1. 00101 **2.** 13 **3. A** Dave **B** Mary **C** Paula
4. 01110 00101 10110 00101 10010; 01111 00100 00100; 01111 10010; 00101 10110 00101 01110; **5.** 16 **6.** 32

E-23 USING THE IDEAS

1. A 41 **B** black **2.** 111110 **3.** 000001
4. 011111, 100000
5.



E-24 DISCUSSING THE IDEAS

?

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

3. 1 **4.** 111 **6.** 1011

E-25 USING THE IDEAS

1. A 11 **B** 101 **C** 110 **D** 10010 **E** 10110
F 1011 **2. A** 10 **B** 1 **C** 1 **D** 10 **E** 11011 **F** 111
3. A 11 **B** 0 **C** 100 **D** 1000 **E** 10000 **F** 1001
G 1111 **H** 110001 **I** 11000 **J** 10101 **K** 100011
L 11110 **4. A** 101 **B** 11 **C** 1011 **D** 1000 **5. A** 10
B 1 **C** 10 **D** 11 **E** 11 **F** 1110 **G** 100 **H** 100 **I** 1

Think

+	0	1	2	3	4	5	6	7	8	9	D	E
0	0	1	2	3	4	5	6	7	8	9	D	E
1	1	2	3	4	5	6	7	8	9	D	E	10
2	2	3	4	5	6	7	8	9	D	E	10	11
3	3	4	5	6	7	8	9	D	E	10	11	12
4	4	5	6	7	8	9	D	E	10	11	12	13
5	5	6	7	8	9	D	E	10	11	12	13	14
6	6	7	8	9	D	E	10	11	12	13	14	15
7	7	8	9	D	E	10	11	12	13	14	15	16
8	8	9	D	E	10	11	12	13	14	15	16	17
9	9	D	E	10	11	12	13	14	15	16	17	18
D	D	E	10	11	12	13	14	15	16	17	18	19
E	E	10	11	12	13	14	15	16	17	18	19	1D

×	0	1	2	3	4	5	6	7	8	9	D	E
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	D	E
2	0	2	4	6	8	D	10	12	14	16	18	1D
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	D	13	18	21	26	2E	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2E	36	41	48	53	5D	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
D	0	D	18	26	34	42	50	5D	68	76	84	92
E	0	E	1D	29	38	47	56	65	74	83	92	D1

E-26 REVIEWING THE IDEAS

1. A 110213 **B** 1668 **C** 1464 **2. A** 3 000 000 000
B 748 000 000 **3.** first 3 on left. **4.** $53-27=n$
5. $234\div9=y$ **6. A** commutative (+) **B** one
C commutative (×) **D** associative (+) **E** zero
7. A 6^3 **B** 4^5 **C** 10^2 **D** 2^7 **8.** 0 **9. A** 10^5 **B** 2^7
C 10^4 **D** 8^3 **E** 7^5 **F** 10^1 **10. A** 7 463 500
B 7 463 000 **C** 7 000 000 **11.** *B* **12.** 500
13. A 18 481 **B** 5638 **C** 399 280 **D** 247, R14
14. 10100 **15. A** 101 **B** 1100 **C** 10011 **D** 11101
E 100000 **F** 1000010 **16. A** 6 **B** 11 **C** 8 **D** 27
E 15 **F** 43 **17 A** 1010 **B** 1001 **C** 100011 **D** 1001

UNIT E: MODULE 2

E-30 DISCUSSING THE IDEAS

? 1. $\{1, 3, 5, 7, \ldots\}$ **2.** ●
3. $\{0, 1, 4, 9, 16, 25, 36, \ldots\}$ **1.** all infinite **2. A** yes
B no **C** no **3.** *B, D* **5.** yes, $\varnothing$

E-31 USING THE IDEAS

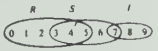
1. A infinite **B** infinite **C** finite **D** finite
2. A true **B** false **C** false **D** false **E** true **F** false **G** true **H** false **I** true **J** false **3.** $\varnothing$, {2, 4, 6}, {2}, {4}, {6}, {2, 4}, {2, 6}, {4, 6} **4.** 16 **5.** $B \subset A$; $C \subset A$; $C \subset B$ **6.** $Q \subset M$; $Q \subset N$; $N \subset M$ **7.** $J \subset L$ **8.** $A \subset C$
Think 18

E-32 DISCUSSING THE IDEAS

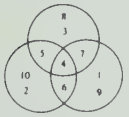
1. B {5, 6, 7, 8, 9, 10, 11, 12} **2. B** {7, 8, 9, 10}
3. X and Z **4. A** P **B** Q

E-33 USING THE IDEAS

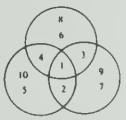
1. A {1, 3, 5, 6, 7, 9, 12} **B** {3, 9}
2. A {0, 1, 2, 3, 4, 5, 6, 7} **B** {0, 1, 2, 3, 4, 5, 7, 8, 9}
C {3, 4, 5, 6, 7, 8, 9} **D** {3, 4, 5} **E** $\varnothing$ **F** {7}
3. A {7} **B** {3, 4, 5, 7, 8, 9} **C** $\varnothing$
D {0, 1, 2, 3, 4, 5, 7} **E** {3, 4, 5} **F** {0, 1, 2, 3, 4, 5, 7}
G {7} **H** {3, 4, 5, 7, 8, 9} **I** {3, 4, 5, 6, 7} **J** {3, 4, 5}
K {3, 4, 5, 7} **L** {3, 4, 5, 7} **4.**



5. A {4, 8, 12, 16, . . .} **B** {6, 12, 18, 24, . . .}
C {12, 24, 36, . . .} **D** common multiples of 4 and 6
Think
1.



2.



E-34 DISCUSSING THE IDEAS

? B 3, 5, 7, 9, 11, 13, 15 **C** 5, 6, 9, 14, 21, 30, 41
3. A y **B** n **C** r^2 **D** z **4. A** 14 **B** 8 **C** 43 **D** 32
E 5 **F** 5 **G** 120 **H** 120

E-35 USING THE IDEAS

1. 10, 11, 12 **2.** 7, 6, 5, 4 **3.** −2, 4, 10, 16 **4. A** 10
B 54 **C** 3 **D** 49 **E** 29 **F** 2 **G** 8 **H** 6 **5.** 8
6. 1, 3, 7, 13, 21 **7. A** 4, 9, 16 **B** 4, 9, 16
Think 20^{20}, then pattern repeats

E-36 DISCUSSING THE IDEAS

B $x = 0$, false; $x = 1$ false; $x = 2$ false; $x = 3$ true
C $x = 0$ false; $x = 1$ false; $x = 2$ false; $x = 3$ true
1. A {3} **B** {3} **2.** {0, 1, 2, 3, 4, 5} **3.** no **4.** B, C

E-37 USING THE IDEAS

1. A 8 **B** 5 **C** 3 **D** 1
E {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} **F** 10 **G** 7 **H** 0
I {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} **2. A** {0, 1, 2, 3}
B {6, 7, 8, 9} **C** {0, 1, 2, 3, 4, 5} **D** {5, 6, 7, 8, 9}
E $\varnothing$ **F** {6, 7, 8, 9} **G** {0, 1, 2, 3, 4, 5, 6}
H {0, 1, 2, 3, 4} **I** {6, 7, 8, 9} **3.** *A, C, D, F, G, H, K* are generalizations **4.** *A, C, D, F, G, H, K*
Think 6

E-38 DISCUSSING THE IDEAS

? A 8 **B** 16 **C** 13 **D** 3 **1. A** addend **2. A** sum
3. A factor **B** factor **4. A** $8y$ **B** $1w$ **C** $(z + 2)7$ **D** $6r$

E-39 USING THE IDEAS

1. A 5, 7 **B** 21, 7 **C** 8, 11 **D** 6, 18 **E** 9, 5
F 13, 7 **2. A** 4 **B** 6 **C** 19 **D** 1 **E** 8 **F** 26 **G** 38
H 7 **I** 90 **J** 10 **K** 12 **L** 42 **M** 24 **N** 19 **O** 3
P 12 **3. A** 8, 4 **B** 20, 5 **C** 20, 10 **D** 18, 6 **E** 14, 7
F 20, 4 **4. A** 5 **B** 10 **C** 3 **D** 0 **E** 1 **F** 6

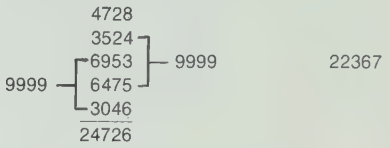
E-40 DISCUSSING THE IDEAS

? A, B, C, D, E **1. F** **2. A** no **B** yes **3.** 0, 5 **4.** 1, 4

E-41 USING THE IDEAS

1. A 0, 1 **B** 1, 2 **C** 0, 2 **D** 1, 4 **E** 0, 11 **F** 7, 4
G 8, 6 **H** 0, 0 **I** 3, 11 **J** 10, 7 **K** 15, 12 **L** 2
2. A 8 **B** 5 **C** 5 **D** 2 **E** 7 **F** 6 **G** 5 **H** 6 **I** 4

Think



E-42 DISCUSSING THE IDEAS

? 0, 5, 10, 15, 20, 25 **1.** multiples of 5 **2. A** yes
B yes **3.** 2, 3, 5 **4. A** 8 **B** 10 **C** 9 **D** 13 **E** 5
F 11 **5. A** $13x$ **B** $17x$ **C** $27x$ **6. A** $2x$ **B** $6x$

E-43 USING THE IDEAS

1. A $12x$ **B** $11x$ **C** $6x$ **D** $12x$ **E** $7x$ **F** $8x$ **G** x
H $9x$ **I** $18x$ **J** $21x$ **K** $23x$ **L** $8x$ **2. A** $3x$ **B** $3x$
C $2x$ **D** $6x$ **E** $9x$ **F** $4x$ **G** $8x$ **H** $8x$ **I** $3x$ **J** $4x$
K $10x$ **L** 0 **3. B** $4x$ **C** $8x$ **D** $7x$ **E** $2x$ **F** $8x$ **G** x
H x **I** $6x$ **J** x **K** x **4. A** 45 **B** 90 **C** 63 **D** 48
E 120 **F** 140 **G** 14 **H** 60 **I** 0 **J** 0 **K** 5 **L** 2
Think SPOMYEEMLATETHTFEUANS

E-44 DISCUSSING THE IDEAS

? $a = 9, b = 8$ **2.** 7

E-45 USING THE IDEAS

1. A 3 **B** 6 **C** 7 **D** 7, 1 **E** 3, 4 **F** 27, 3 **G** 20, 4
H 35, 7 **I** 14, 2 **2. A** 2 **B** 4 **C** 4 **D** 5 **E** 5 **F** 2
G 3 **H** 5 **I** 2 **3. A** 3 **B** 3 **C** 4 **D** 6 **E** 5 **F** 3
G 5 **H** 5 **I** 6
Think 11, 15

E-46 DISCUSSING THE IDEAS

? A $3x - 5 = 7$ **B** $2(t + 12) = 36$ **C** $6n - n = 25$

2. A

B B **3. A** 4 **B** 6 **C** 5

E-47 USING THE IDEAS

1. A $(x + 10)4 = 56$ **B** $6y - 8 = 10$ **C** $5z + z = 18$
2. A 4 **B** 3 **C** 3 **3.** 8 **4.** 13 **5.** 13 **6.** 15 **7.** 7
8. A 33 **B** Beth 19; Tina 14 **9.** Dave 18; Chuck 15

Think

E-48 DISCUSSING THE IDEAS

2. B **3. A** $\{1, 2, 3\}$ **B** $\{3, 4, 5, 6, 7, 8, 9\}$ **4. A** $x = 7$
B $6 < x < 8$ **C** $x \geq 5$ **D** $1 < x < 7$ **E** $3 > x > 5$
F $x \geq 6$

E-49 USING THE IDEAS

1. A $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$; $B = \{6, 7, 8, 9, 10\}$;
$A \cap B = \{6, 7\}$ **B** $C = \{3, 4, 5, 6, 7, 8, 9, 10\}$;
$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$; $C \cap D = \{3, 4, 5, 6, 7, 8\}$
C $E = \{0, 1, 2\}$; $F = \{8, 9, 10\}$; $E \cup F = \{0, 1, 2, 8, 9, 10\}$
D $G = \{6, 7, 8, 9, 10\}$; $H = \{2\}$;
$G \cup H = \{2, 6, 7, 8, 9, 10\}$ **2. A** $2 < x < 10$
B $12 < x < 19$ **C** $0 < x < 11$ **D** $a < x < b$ **E** $x \leq 9$
F $x \geq 3$ **3. A** $\{6, 7, 8, 9\}$ **B** $\{3, 4, 5, 6\}$
C $\{0, 1, 2, 8, 9, 10\}$ **D** $\{7, 8, 9, 10\}$
E $\{0, 1, 2, 3, 4\}$ **F** $\{6, 7, 8, 9, 10\}$
4. A $\{24, 25, 26, 27, 28, 29, 30\}$ **B** $\{4, 5, 6, \ldots, 18, 19\}$
C $\{4998, 4999, 5000, 5001\}$ **D** $\{1, 2, 3\}$
E $\{101, 102, \ldots, 998, 999\}$ **F** $\varnothing$
G $\{244, 245, \ldots, 341, 342\}$ **H** $\{577, 578, \ldots, 674, 675\}$
I $\{27, 28, \ldots, 80, 81\}$ **5. A** $\{9, 10, 11\}$
B $\{0, 1, 2, 3, 4, 5, 6, 7\}$ **C** $\{1, 2, 3, 4\}$ **D** $\{13\}$
E $\{3, 4, 5, 6, 7, 8, 9\}$ **F** $\{14\}$ **G** $\{8, 9, 10, 11, 12\}$
H $\varnothing$ **I** set of whole numbers

E-50 DISCUSSING THE IDEAS

? A 4 **B** 9 **C** 121 **D** 256 **E** 484 **F** 1849
G 9409 **H** 10, 201 **1.** 25, 121 **3.** 3, 6, 10, 15

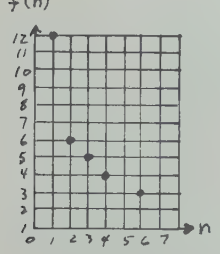
E-51 USING THE IDEAS

1. A 81 **B** 9 **C** 0 **2. A** 42 **B** 72 **C** 216 **3. A** 39
B 54 **C** 69 **4. A** 18 **B** 100 **C** 448 **D** 900 **5. A** 3
B 4 **C** 9 **D** 99 **6. A** 9 **B** 17 **C** 33 **D** 65
7. $10n - 1$
Think 55

E-52 DISCUSSING THE IDEAS

2. A 4, 3, 2

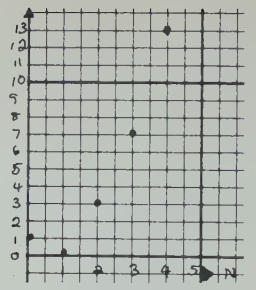
B



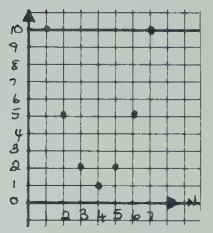
C no

E-53 USING THE IDEAS

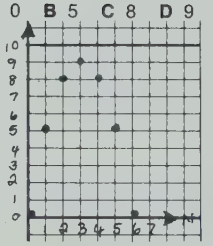
1. A 3 **B** 7 **C** 13



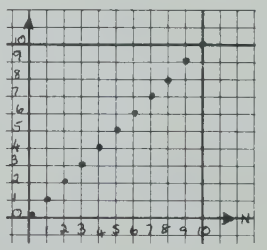
2. A 5 **B** 2 **C** 1 **E** 5



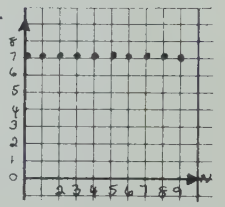
3. A 0 **B** 5 **C** 8 **D** 9 **E** 8 **F** 5 **G** 0



4.



5.



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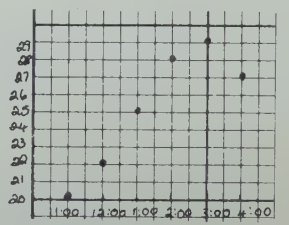
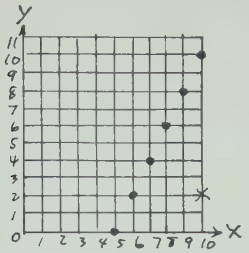


Think 4

E-54 DISCUSSING THE IDEAS

? C **2. A** 14, 4; 16, 6; 18, 8; 20, 10;

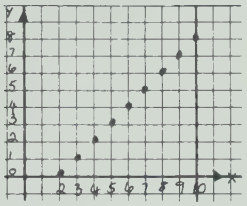
C



E-55 USING THE IDEAS

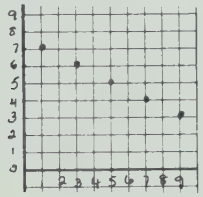
1. 6, 5, 4, 3, 2, 1, 0

2.



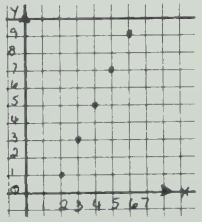
3. 5, 10; 4, 8; 3, 6

4.

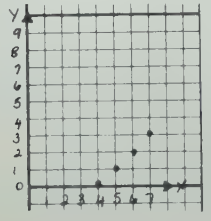


5. 5; 10, 7; 12, 9

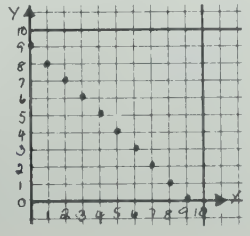
6.



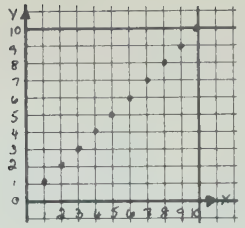
7. A



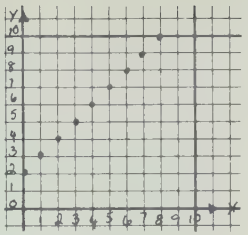
B



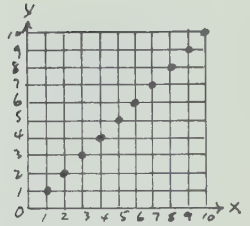
C



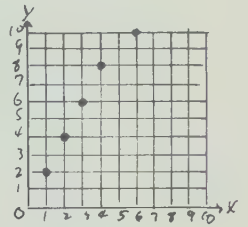
D



E



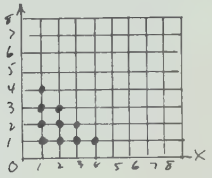
F



Think Chris: 3; Juanita: 6

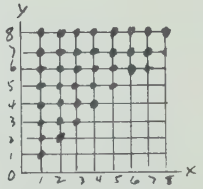
E-56 DISCUSSING THE IDEAS

?

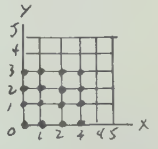


1. 15

2. A

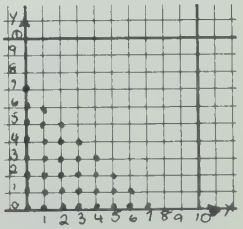


B

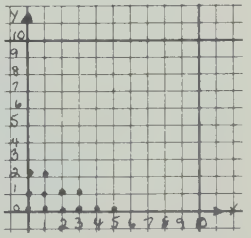


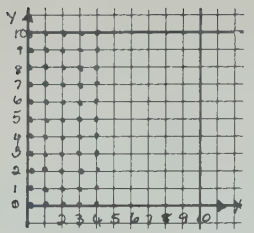
E-57 USING THE IDEAS

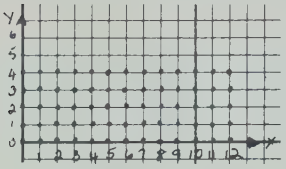
1. A

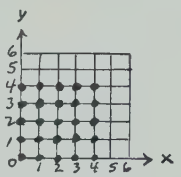


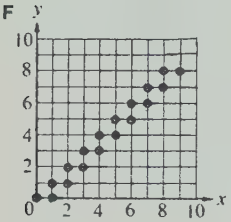
B

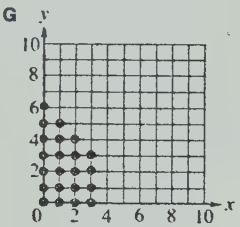


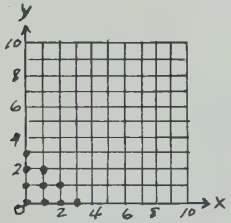
C 

D 

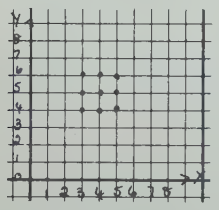
E 

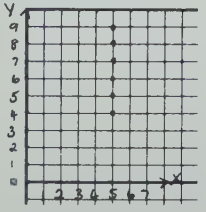
F 

G 

H 

2. A $x + y \leq 1$ **B** $x < 3$

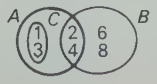
3. 

4. 

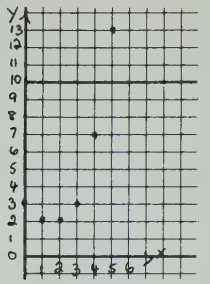
E-58 REVIEWING THE IDEAS

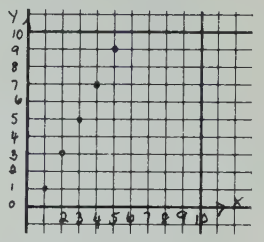
1. A yes **B** yes **C** no **D** yes **2. A** true **B** false **C** true **D** true **E** false **3. A** {1, 2, 3, 4, 6, 8} **B** {2, 4} **C** {1, 2, 3, 4} **D** {1, 3} **E** {1, 2, 3, 4, 6, 8} **F** ∅

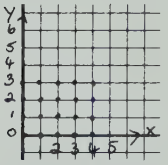
4.



5. 6 **6.** 50 **7. A** {9} **B** {8} **C** {0, 1, 2, 3} **D** {8, 9, 10} **8.** yes **9. A** 7 **B** 3 **10. A** 11 **B** 9 **11. A** 11, 4 **B** 4, 5 **12.** $8x + 9 = 33$ **13.** 3 **14.** 7 **15. A** {6, 7} **B** {0, 1, 10} **C** {0, 1, 2, 3, 4, 6, 7, 8, 9, 10} **D** {3, 4, 5, 6, 7, 8} **16. A** 3 **B** 4 **C** 7 **D** 12

16. 

17. 

18. 

UNIT E: MODULE 3

E-62 DISCUSSING THE IDEAS

? Number	Factors
5	{1, 5}
6	{1, 2, 3, 6}
7	{1, 7}
8	{1, 2, 4, 8}
9	{1, 3, 9}
10	{1, 2, 5, 10}
11	{1, 11}
12	{1, 2, 3, 4, 6, 12}
13	{1, 13}
14	{1, 2, 7, 14}
15	{1, 3, 5, 15}
16	{1, 2, 4, 8, 16}
17	{1, 17}
18	{1, 2, 3, 6, 9, 18}
19	{1, 19}
20	{1, 2, 4, 5, 10, 20}

1. 1 **2.** 12, 18, 20 **3.** 2, 3, 5, 7, 11, 13, 17, 19 **4. A** yes **B** 0 **5. A** yes **B** yes **C** no **6.** yes

E-63 USING THE IDEAS

1. A 84 **B** 42 **C** 28 **D** 21 **E** 14 **F** 12
2. 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84 **3.** 4 **4.** 6
5. A 1, 2, 5, 7, 10, 14, 35, 70 **B** 1, 2, 3, 6, 11, 22, 33, 66
C 1, 2, 3, 4, 6, 8, 12, 16, 24, 48
D 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150 **C** 1, 2, 3, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 140, 210, 420 **6. A** yes **B** no **C** yes **D** yes **E** no **F** yes

7. A 28 **B** 496 **8.** 6, 10, 15, 30, 1 **9. A** 1, 3, 5, 7, 13, 15, 21, 35, 39, 65, 91, 105, 195, 273, 455, 1365 **B** 1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, 30, 33, 35, 42, 55, 66, 70, 77, 105, 110, 154, 165, 210, 231, 330, 385, 462, 770, 1155, 2310

Think no

E-64 DISCUSSING THE IDEAS

1. 22 **2.** no $57 = 19 \cdot 3$ **3. A** 51, 187, 1001 **B** 109, 1009 **4. A** neither **B** neither **5.** yes – 2 **6.** 27, 57, 77, 87

E-65 USING THE IDEAS

2. 4 **3. A** 3 **B** 6 – 3, 2 **C** 9 **D** 5 **4. B** 25

E-66 DISCUSSING THE IDEAS

1. 0 – 100, 25 **2.** 1100 – 1200, 12 **3. A** 691 **B** 31 **C** 853 **D** 241 **E** 409 **F** 1091 **G** 709 **H** 907 **4. A** 83 **B** 128 **C** 26 **D** 111 **E** 194 **F** 99 **G** 37

E-67 USING THE IDEAS

1. A 79 **B** 179 **C** 379 **D** 479 **2.** no; 1, 3, 9, 31, 93, 279 **3. A** 64 **B** 89 **C** 5 **D** 120 **E** 198 **F** 167 **4.** 13, 113 **5.** 101, 401, 601, 701, 1201 **6. A** 3 and 5; 5 and 7; 11 and 13; 17 and 19; 29 and 31; 41 and 43; 59 and 61; 71 and 73 **B** 1019 and 1021; 1031 and 1033; 1049 and 1051; 1061 and 1063; 1091 and 1093 **C** $1217 - 2 = 1215$; $5.243 = 1215$ and $1217 + 2 = 1219$; $25.53 = 1219$ **7.** 3, 5, 7 **8.** 5, 7, 11 **9. A** 17, 23, 31, 41, 53, 67, 83, 101, 121, 143 **B** No, 121 and 143 are not prime **10.** 17, 19, 23, 29, 37, 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257, 289, 323, 359, 397; No, 289 and 323 are not prime numbers.

Think 1, 4, 9, 16, 25, 36, 49; perfect squares

E-68 DISCUSSING THE IDEAS

? A $2 \times 2 \times 2 \times 3 \times 2$ **B** $2 \times 2 \times 2 \times 2 \times 3$ **C** $3 \times 2 \times 2 \times 2 \times 2$ **1. A** yes **B** yes **2.** $2 \times 2 \times 3 \times 3$ **3. A** 7 **B** 2 **C** 13 **D** 2 **E** 3 **F** 5 **G** 7 **H** 3 **I** 11 **4. A** $2 \times 2 \times 3 \times 5$ **B** $2 \times 3 \times 7$ **C** $2 \times 2 \times 5 \times 7$ **D** $2 \times 2 \times 3 \times 5 \times 5$ **E** $5 \times 5 \times 5 \times 5$

E-69 USING THE IDEAS

1. A $5 \times 5 \times 3$ **B** $2 \times 5 \times 2 \times 2$ **C** $5 \times 5 \times 3 \times 3$ **2. A** 3×5 **B** $2 \times 2 \times 3$ **C** 2×11 **D** 3×13 **E** $2 \times 5 \times 5$ **F** $2 \times 5 \times 7$ **G** $3 \times 3 \times 5$ **H** $2 \times 2 \times 2 \times 2$ **I** $3 \times 3 \times 3 \times 3$ **J** $2 \times 2 \times 3 \times 5$ **K** $2 \times 3 \times 11$ **L** $2 \times 2 \times 5 \times 5$ **3. A** $2 \times 7 \times 7$ **B** $3 \times 3 \times 7$ **C** $2 \times 3 \times 2 \times 5$ **D** $2 \times 2 \times 3 \times 13$ **E** $2 \times 2 \times 7 \times 7$ **F** $2 \times 3 \times 5 \times 5$ **G** $11 \times 2 \times 2 \times 3$ **H** $3 \times 3 \times 2 \times 2 \times 2$ **I** $3 \times 5 \times 2 \times 2 \times 2 \times 2$ **J** $3 \times 3 \times 3 \times 7$ **K** $2 \times 2 \times 2 \times 2 \times 3 \times 3$ **L** $2 \times 5 \times 7 \times 7$ **4. A** $2^3 \cdot 3$ **B** $2^3 \cdot 3^2$ **C** $2 \cdot 5^3$ **D** 3^4 **E** $2^3 \cdot 3^3$ **F** 10^3 **G** $2^3 \cdot 3 \cdot 5$ **H** $2^4 \cdot 3$ **I** $2^2 \cdot 3^3$ **J** $2^2 \cdot 11^2$ **K** $2^5 \cdot 3^2$ **L** $3^2 \cdot 5^2$ **M** $3^2 \cdot 7^2$ **N** $2^2 \cdot 3^2 \cdot 17$ **O** $3^2 \cdot 11^2$ **5. A** 8 **B** 8 **C** 9 **D** 12 **E** 16 **F** 8 **6. A** {1, 2, 3, 4, 6, 8, 12, 24} **B** {1, 2, 3, 5, 6, 10, 15, 30} **C** {1, 2, 4, 5, 10, 20, 25, 50, 100} **D** {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72} **E** {1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 120, 125, 250, 500, 1000} **F** {1, 7, 11, 13, 77, 91, 143, 1001}

Think

1. **2.**

E-70 DISCUSSING THE IDEAS

? 28 – cm squares **1. A** 1, 2, 4, 7, 8, 14, 28, 56 **B** 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84 **C** 1, 2, 4, 7, 14, 28 **D** 28 **2.** 6 **3. A** 6 **B** 3 **C** 10 **D** 5 **E** 2

E-71 USING THE IDEAS

1. A {1, 2, 3, 4, 6, 12} **B** {1, 2, 3, 6, 9, 18} **C** {1, 2, 3, 6} **D** 6 **2. A** {1, 2, 3, 4, 6, 9, 12, 18, 36} **B** {1, 2, 3, 6, 9, 18, 27, 54} **C** {1, 2, 3, 6, 9, 18} **D** 18 **3. A** 4 **B** 2 **C** 7 **D** 15 **E** 4 **F** 10 **G** 7 **H** 24 **I** 17 **J** 10 **4. A** 5 **B** 8 **C** 3 **D** 24 **E** 14 **F** 24 **G** 22 **5. A** relatively prime **B** 3 **C** relatively prime **D** relatively prime **E** relatively prime **F** 3 **G** relatively prime **H** relatively prime **I** 3 **J** 13 **6. A** 15 **B** 2 **C** 14 **D** 66 **E** 90 **F** 315

Think 3

E-72 DISCUSSING THE IDEAS

? 23 **1.** 4 **3. B** 28 **4.** 47

E-73 USING THE IDEAS

1. A 14 **B** 23 **C** 23 **D** 27 **E** 35 **F** 64 **G** 16 **H** 37 **I** 113 **J** 19 **K** 56 **L** 213 **3. A** 1 **B** 0 **4.** no, 11 **5. A** 32 **B** 15

Think yes

20

E-74 DISCUSSING THE IDEAS

? 360 **2.** – 3 **3.** 24 **4. A** 30 **B** 60 **C** 36

E-75 USING THE IDEAS

1 A 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66 **B** 20, 40, 60 **C** 60 **D** 60 **2. A** 12 **B** 30 **C** 36 **D** 30 **E** 56 **F** 40 **G** 40 **H** 80 **I** 42 **J** 120 **3. A** 60 **B** 12 **C** 36 **D** 60 **E** 24 **F** 12 **G** 120 **H** 180 **I** 12 **J** 60 **K** 36 **L** 120 **4. A** 28 **B** 100 **C** 210 **D** 72 **E** 165 **F** 180 **G** 108 **H** 432

Think **1.** 20 hrs **2.** 1200 km

E-76 DISCUSSING THE IDEAS

? A yes **B** no **C** no **D** yes **2. a** if it is even **B** if it ends with a 0 or 5 **C** if it ends with 0 **D** if it ends with 25, 50, 75 or 00 **E** if it ends with 00 **3.** divisible by 3 **4.** *C, D*

E-77 USING THE IDEAS

1. A 12, 20, 24, 30, 48, 70 **B** 15, 20, 30, 35, 70 **C** 20, 30, 70 **2. A** true **B** false **C** true **D** true **3. A** 6, yes **B** 5, no **C** 12, yes **D** 18, yes **E** 3, yes **F** 16, no **G** 27, yes **4.** *A, B, C, E, F, G, I* **5.** *A, D, F, G, H, I, J, K, L* **6. A** 3×37 **B** $2 \times 3 \times 19$ **C** 3×89 **D** $3 \times 5 \times 13$ **E** $3 \times 2 \times 5 \times 367$

E-78 DISCUSSING THE IDEAS

? *A, B, D, E* **2. B** yes, no

E-79 USING THE IDEAS

1. *A, B, E, F, G, H, I, J* **2.** *A, B, D, E, F, G, I, J, K, L*
3. A yes **B** yes **C** yes **4.** *A, B, D, E* **5.** *A, D, E, G, I*
6. *C, E, F, G, H*
Think 2520

E-80 DISCUSSING THE IDEAS

? A 16, 19, 22 **B** 27, 35, 44 **C** 37, 50, 65
D 243, 729, 2187 **E** 713, 851, 989 **3. A** 13 **B** 19
C 21 **4.** 16, 21, 26

E-81 USING THE IDEAS

1. A 11, 13, 15 **B** 10 000; 100 000; 1 000 000
C 32, 64, 128 **D** 23, 27, 31 **2. A** 15, 19, 23
B 10, 15, 21 **C** 16, 25, 36 **D** 11, 18, 27
3. 2, 5, 8, 11, 14 **4.** 1, 0, 1, 0, 1 **5. A** $3n$ **B** $n + 1$
C $n(n + 1)$ **D** n^2 **6. A** 34, 55, 89, 144
B $(n - 2)^{th}$ term + $(n - 1)^{th}$ term
Think 2, 4, 8, 15, 26, 64

E-82 SQUARE NUMBERS

1. 16, 25, 36, 49 **2.** $1 + 3 + 5 + 7 = 16$;
$1 + 3 + 5 + 7 + 9 = 25$; $1 + 3 + 5 + 7 + 9 + 11 = 36$
3. A 9 **B** 16 **C** 25 **D** 36 **E** 100 **F** 2500 **4. A** *D*: $1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$; *E*: $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 5^2$; **B** $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 8^2$ **C** $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 10^2$ **5. A** 6^2
B 7^2 **C** 9^2

E-83 TRIANGULAR NUMBERS

1. 10, 15, 21, 28 **2.** 15, 21, 28, 36, 45, 55, 66, 78, 91, 105 **3.** $15 = 1 + 2 + 3 + 4 + 5$; $21 = 1 + 2 + 3 + 4 + 5 + 6$; $28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$; $36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$ **4. A** $10 + 15 = 25$; $15 + 21 = 36$; $21 + 28 = 49$
B $45 + 55 = 100$ **5.** 28; 55; 210; 5050; 500 500

E-84 DISCUSSING THE IDEAS

1. A 46 **B** 58 **C** 288 **2. A** yes **B** no **C** no
D yes **E** yes **3.** yes

E-85 USING THE IDEAS

1. A 22 **B** 76 **C** 148 **D** 244 **E** 313 **F** 2998
2. A yes **B** no **C** yes **D** yes **E** no **F** no **G** no
H no **I** no **J** yes **3. A** {1, 4, 7, 28} **B** {1, 4, 16}
C {1, 10} **D** {1, 34} **E** {1, 7, 10, 70}
F {1, 4, 19, 76} **G** {1, 25} **H** {1, 4, 10, 25, 100}
I {1, 4, 13, 52} **J** {1, 7, 13, 91} **4. A** yes **B** yes
C yes **D** no **E** no **F** yes **5.** 4 **6. A** 4·4 **B** 4·22
C 7·10 **D** 13·4 **E** 7·7 **7.** 10·10; 4·25

E-86 REVIEWING THE IDEAS

1. A {1, 2, 3, 6, 9, 18} **B** {1, 2, 3, 5, 6, 10, 15, 30}
C {1, 2, 11, 22} **D** {1, 2, 5, 10, 25, 50}
E {1, 2, 4, 8, 16} **F** {1, 2, 4, 7, 14, 28}
G {1, 3, 5, 15, 25, 75} **H** {1, 2, 4, 8, 16, 32, 64, 128}
2. 4 **3.** 97 **4.** 69 **5. A** 2×3^3 **B** $2^3 \times 5$ **C** $2^2 \times 5$
D 3^4 **E** $2 \times 3 \times 5 \times 7$ **F** $2^2 \times 5 \times 11$ **G** $2^2 \times 71$
H $2^2 \times 5^3$ **I** $2^2 \times 3^2 \times 5^2$ **6. A** 19 **C** 41 **E** 29 **F** 61
H 73 **7. A** 8 **B** 5 **C** 3 **D** 7 **E** 22 **F** 15 **G** 1
H 13 **I** 24 **8. A** 21 **B** 37 **C** 43 **D** 47 **9. A** 6, 12, 18, 24, 30, 36, 42, 48, 54, 60 **B** 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 **C** 30, 60, 90, 120 **D** 30 **10. A** 24 **B** 40
C 36 **D** 42 **E** 60 **F** 20 **G** 77 **H** 36 **I** $3^2 \cdot 5^2$
J $5^2 \cdot 7^2$ **K** $2^2 \cdot 3 \cdot 5$ **L** $7^2 \cdot 11^2 \cdot 17$ **11. A** 12 **B** 24
C 120 **D** 105 **E** 36 **F** 60 **G** 60 **H** 54 **I** 180
12. A 54, 66, 70, 222 **B** 54, 66, 111, 222 **C** 54, 66, 222
D 65, 70 **13. A** yes **B** no **C** yes **D** no **14.** if sum of digits is divisible by 9 **15.** *B, C, D* **16. A** 26, 31, 36, 41
B 81, 243, 729, 2187 **17.** 5, 8, 11, 14 **18.** triangular; prime; none; square; triangular

UNIT E: MODULE 4

E-91 USING THE IDEAS

1. A −3 **B** −7 **C** −10 **D** −100 **E** 4 **F** 8 **G** 11
H 99 **I** 89, 376 **D** −10 000 **K** 1 000 000
L −10 000 001 **2.** −87 **3.** −86 **4. A** positive
B negative **C** negative **D** negative **E** positive
F negative **G** negative **H** positive **I** negative
J zero **5. A** negative **B** positive **C** negative
D positive **E** zero **6.** −392

E-92 DISCUSSING THE IDEAS

? A 0 **B** 6 **C** −6 **D** −5 **2.** $3 + -5 = -2$ **3. A** 1
B 3 **C** 0 **D** −5 **4.** $-9 + 14 = n$; $n = 5$

E-93 USING THE IDEAS

1. A 9 **B** 0 **C** 3 **D** −30 **E** −25 **2. A** 3 **B** −2
C −70 **D** 20 **E** −600 **3. A** −3 **B** −1 **C** −3
D −3 **E** −2 **F** −7 **G** −1 **H** −6 **I** −8 **J** −3
4. A 18 **B** −7 **C** −33 **D** 37 **E** 4 **F** 2 **G** 0 **H** 0
I 2 **5. A** −10 **B** -10^0 **C** $5.00 **D** 22 km south

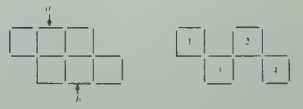
E-94 DISCUSSING THE IDEAS

1. A 11 **B** 0 **C** yes **3.** 0 (opposite); 0 (commutative); 0; −8 (opposite); **4** addition fact; associative; opposite; zero

E-95 USING THE IDEAS

1. A 0, 0 **B** 9, 9 **C** 0, 0 **D** 0, 0 **2. A** 4, 4 **B** 5, 5
C −4, −4 **D** −5, −5 **E** −2, −2 **F** −7, −7 **G** 7, 7
H 9, 9 **I** −10, −10 **J** −5, −5 **K** −8, −8 **L** −8, −8
3. A 1 **B** 2 **C** −12 **D** −3 **E** −1 **F** 5 **G** 3 **H** 4
I −17 **J** −3 **K** −3 **L** −2 **M** −7 **N** −1 **O** 7
P 8 **Q** 11 **R** −8 **S** −13 **T** −36 **U** −26 **V** −200
W 161 **X** −1103 **4. A** 37 **B** −14 **C** 2 **D** 0 **E** 3
F −1 **5. A** −1 **B** −11 **C** −9 **D** −9 **E** −4 **F** 20
G 31 **H** −20 **6. A** 7 **B** −8 **C** 0 **D** −8 **E** 9
F −2 **G** −8 **H** −7 **I** 6

Think



E-96 DISCUSSING THE IDEAS

?1. 3 **2.** −4 **3.** −8 **4.** 8 **5.** 4 **2. A** −1 **B** −3 **C** 3 **D** 4 **3. A** yes **B** yes

E-97 USING THE IDEAS

1. A −4 **B** 5 **C** −9 **D** 7 **E** −9 **F** 4 **G** −6 **H** −10 **I** 6 **J** −10 **K** −9 **L** −5 **M** −11 **N** 5 **O** −8 **P** 12 **Q** −4 **R** −12 **S** 4 **T** 0 **2. A** 5, 6 **B** 5, 7 **C** 10, 11 **D** −6, −4 **E** −6, −3 **F** 8, 11 **3. A** 11 **B** 9 **C** 12 **D** 10 **E** −5 **F** −10 **G** −7 **H** −11 **I** −4 **J** −1 **K** −1 **L** −6 **4. A** 15 **B** −15 **C** 21 **D** −21 **E** 14 **F** −14 **G** 15 **H** −24 **I** −18 **J** −13 **K** −63 **L** 39 **5. A** 10 **B** 16 **C** 10

Think *D*: 30; *E*: 55 **2.** 204

E-98 DISCUSSING THE IDEAS

?1. −4 **2.** −3 **3.** −2 **4.** −1 **5.** 0 **6.** 1 **7.** 2 **8.** 3 **9.** 4 **10.** 5 **3. A** 3 + −7 = −4 **B** 8 + 1 = 9 **C** −9 + 3 = −6 **D** −7 + −6 = −13 **4. A** −4 **B** 4 **C** −3

E-99 USING THE IDEAS

1. A 3 **B** 2 **C** 5 **D** −3 **E** −7 **F** −4 **G** −1 **H** 6 **I** 4 **J** −15 **K** 6 **L** 4 **2. A** 3 **B** −10 **C** 5 **D** −11 **E** 20 **F** −16 **G** −10 **H** −5 **I** 6 **J** 0 **K** −143 **L** −22 **3. A** −5 **B** 5 **C** −6 **D** 2 **E** −4 **F** −4 **G** 8 **H** 1 **I** −16 **J** 3 **K** 2 **L** 12 **4. A** 4 **B** 13 **C** 3 **D** −6 **E** −4 **F** −8 **G** −1 **H** 14 **I** −17

Think Check *AO* bag. If *A* then *AA*, and *OO*, *AO*, *AA*. If *O* then *AO*, *AA*, *OO*.

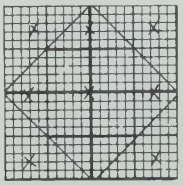
E-100 DISCUSSING THE IDEAS

?A 20 **B** −20 **C** 20 **1. A** 10·−2 **B** −10·−2 **2. A** −20 **B** −10·2

E-101 USING THE IDEAS

1. B 5·−3 = −15 **C** −5·−3 = 15 **D** −10·4 = −40 **E** 10·−4 = −40 **F** −10·−4 = 40 **G** 15·2 = 30 **H** −15·−4 = 60 **I** 12·−5 = −60 **2. A** −12 **B** −16 **C** −8 **D** 10 **E** −24 **F** 16 **G** −70 **H** 18 **I** −49 **J** −90 **K** −40 **L** −10 **3. A** 12 **B** 8 **C** 4 **D** 0 **E** −4 **F** −8 **G** −12 **H** −12 **I** −8 **J** −4 **K** 0 **L** 4 **M** 8 **N** 12

Think



E-102 DISCUSSING THE IDEAS

1. −5 (one); 1 (one) **2.** commutative **3. A** 3 **B** 3 **C** associative **4.** 0, 0, 0, 0, −12, −12 **5.** 0, 0, 0, 0, 0, 12, 12

E-103 USING THE IDEAS

1. A 19, 19 **B** −8, −8 **C** −24, −24 **D** 86, 86 **2. A** −12, −12 **B** −18, −18 **C** −36, −36 **D** −2, −2 **E** −126, −126 **F** −12, −12 **3. A** −16, −16 **B** −3, −3 **C** −84, −84 **D** −27, −27 **E** −384, −384 **F** −7, −7 **G** −48, −48 **H** −360, −360 **4. A** 0, 0, 0, 0, −12 **B** 0, 0, 0, 0, −10 **C** 0, 0, 0, 0, 0 **5. A** −72 **B** −42 **C** 45 **D** 0 **E** −161 **F** −448 **G** 686 **H** 1200 **I** −972 **J** −2436 **K** 2813 **L** 0 **M** −920 **N** −4800 **O** 62000 **P** 10000 **Q** 0 **R** −125 **S** 256 **T** −360 **6. A** negative **B** positive **C** zero

E-104 DISCUSSING THE IDEAS

?A 3 **B** −5 **C** −4 **D** −12 **1. B** −4*n* = 20 **C** 7*n* = −28 **D** −1*n* = 12 **2.** 3 **4.** −6 **5. A** 0 **B** every integer except 0

E-105 USING THE IDEAS

1. A −3 **B** 3 **C** −1 **D** −15 **E** 0 **F** −5 **G** 13 **H** −8 **I** 0 **J** 86 **K** −7 **L** −3 **M** −8 **N** −2 **O** −2 **2. B** 4 **C** −6 **D** −1 **E** 3 **F** −5 **G** −9 **H** −1 **I** 0 **J** −8 **K** −1 **L** −6 **M** −24 **3. A** −3 **B** −6 **C** 2 **D** −4 **E** −7 **F** −7 **G** −4 **H** −4 **I** −3 **J** −1 **4. A** positive **B** negative **C** positive

Think b = 9, a = 10

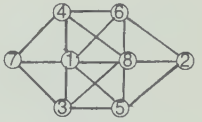
E-106 DISCUSSING THE IDEAS

? −19 −16 −13 −10 −7 −4 −1 2 5 8 11 **1. A** −16 **B** 5 **2. A** −1 **B** 9 **C** 3 **3. A** 18 **B** −12 **C** 27 **D** 12 **E** 6 **F** 96 **4. A** −6 **B** 2 **C** 22

E-107 USING THE IDEAS

1. A −11 **B** 9 **C** −3 **D** 7 **E** 15 **2. A** −10 **B** 6 **C** −28 **D** 8 **E** 0 **3. A** −17 **B** 5 **C** −13 **D** 15 **E** 95 **4. A** 5 **B** 5 **C** 26 **D** 26 **E** 2 **5. A** 8 **B** −1 **C** 64 **D** −27 **E** −2 **6. A** −2 **B** 4 **C** 64 **D** 3 **E** 5 **7. A** 5 **B** −60 **C** 72 **D** −16 **E** 27 **F** −9 **G** 25 **H** −46 **I** −3 **8. A** −5 **B** −92 **C** 6 **D** −4 **E** 14 **F** −1 **G** −4

Think



E-108 DISCUSSING THE IDEAS

?A 7 **B** −3 **C** −8 **D** 0 **E** 6 **F** −7 **G** 2 **H** −3 **I** 7 **1. B** −2 > −5 **C** 6 > −2 **D** −7 = −7 **E** 4 < 10 **F** 8 > 1 **G** −1 < 1 **H** −9 > −12 **I** −10 < −3 **2. A** 3 **B** 2 < 5 **C** 3 **D** −5 < −2 **3. A** left **B** right

E-109 USING THE IDEAS

1. A 2 **B** 4 **C** 7 **D** 98 **2. A** > **B** > **C** > **D** < **E** > **F** < **G** > **H** > **I** < **J** < **K** > **L** > **3. A** −10 **B** 10 **C** −1, −2, . . . , −10 **D** 2, 1, 0, −1, −2 **E** −9, −10 **F** −3, −2, −1 **G** −3 **H** −8, −9, −10, 8, 9, 10 **4. A** less **B** greater **C** less **D** greater **E** less

Think

6	−8	−7	3
−5	1	0	−2
−1	−3	−4	2
−6	4	5	−9

E-110 DISCUSSING THE IDEAS

? **A** $\{-6, -7, -8, -9, -10\}$ **B** $\{5, 6, 7, 8, 9, 10\}$
C $\{-3, -2, -1, 0, 1, 2\}$
D $\{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
E $\{-6, -7, -8, -9, -10\}$ **F** $\{-9, -10\}$
1. A $x > 4$ **B** $\{5, 6, 7, 8, 9, 10\}$
2. A $\{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
B $\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2\}$
C $\{-3, -2, -1, 0, 1, 2\}$ **3. A** $\{-8, -9, -10\}$
B $\{7, 8, 9, 10\}$ **C** $\{-8, -9, -10, 7, 8, 9, 10\}$

E-111 USING THE IDEAS

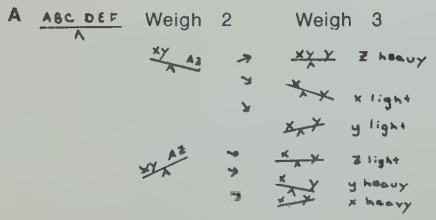
1. A $\{-2, -1, 0, 1, 2, 3, 4\}$ **B** $\{-4, -3, -2, -1, 0, 1\}$
C $\{-2, -1, 0, 1\}$ **D** $\{-4, -3, 4\}$
E $\{-2, -1, 0, 1, 2, 3, 4\}$ **F** $\{-4, -3, -2, -1, 0, 1\}$
2. A $\{-3, -2, -1, 0, 1\}$ **B** $\{-3, -2\}$
C $\{-2, -1, 0, 1, 2, 3, 4, 5\}$ **D** $\{-3, -2, -1\}$
E $\{-3, -2, -1, 0\}$ **F** $\{-3, -2, -1\}$ **G** $\{1, 2, 3, 4, 5\}$
H $\{-3, -2, -1\}$ **I** $\{-1, 0, 1, 2, 3, 4, 5\}$ **J** $\{-2\}$
K $\{0, 1, 2\}$ **L** $\{0, 1\}$ **M** $\{-3, 3, 4, 5\}$
N $\{-3, -2, 2, 3, 4, 5\}$ **O** $\{-3, 3, 4, 5\}$
P $\{0\}$ **Q** $\{-2, -3\}$ **R** $\varnothing$ **3. A** $\{8, 9, 10\}$
B $\{-8, -9, -10\}$ **C** $\{-10\}$
D $\{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
E $\{-10, -9, -8, -7, -6, -5, -4\}$
F $\{-10, -9, -8, -7, -6, -5, -4, -3\}$
G $\{-4, -5, -6, -7, -8, -9, -10\}$
H $\{-10, -9, -8, -7, -6\}$ **I** $\{5, 6, 7, 8, 9, 10\}$
J $\{-2, -1, 0, 1, 2\}$
K $\{-2, -3, -4, -5, -6, -7, -8, -9, -10\}$
L $\{-3, 4, -4, 5, -5, 6, -6, 7, -7, 8, -8, 9, -9, 10, -10\}$
Think $19 \times 3 = 55 + 2$

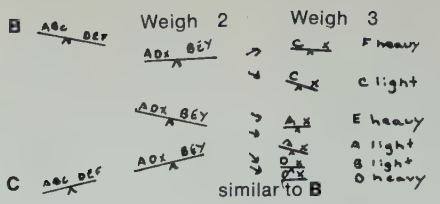
E-112 DISCUSSING THE IDEAS

? **B** 4, 5, 7, 13, 10, 30 **1. A** positive **B** positive
C 0 **2. A** 4 **B** 10 **C** 0 **D** 11 **E** 1 **F** 15 **3. A** 2
B 8 **C** 23 **D** 14 **E** 100 **4. A** 2 **B** 6 **C** 8 **D** 2

E-113 USING THE IDEAS

1. A 8, 2, 20, 53, 3, 16 **B** 4, 2, 2, 3, 4, 6
C 1, 1, 6, 2, 2, 6 **2. A** 5 **B** 5 **C** 18 **D** 2 **E** 47 **F** 4
G 4 **H** 3 **I** 6 **J** 20 **3. A** 5 **B** 6 **C** 5 **D** 5 **E** 8
F 19 **G** 9 **H** 4 **4. A** 3, −3 **B** 8, −8 **C** 4, −4
D 10, −10 **E** 5, −5 **F** 3, −5 **G** 4, −2 **H** 3, −3
5. A −7 **B** 5 **C** −3 **D** 2 **E** −2 **F** −13
Think label coins *A*, *B*, *C*, *D*, *E*, *F*, *x*, *y*, *z*
Weigh 1





E-114 DISCUSSING THE IDEAS

1. A 7 units **B** 7 **C** 20 units **D** 20 **2.** 9, −9
3. A 5 **B** 5 **C** 7 **D** 7 **4. A** 2 **B** −6, 2

E-115 USING THE IDEAS

1. A 8 **B** 12 **C** 8 **D** 12 **E** 15 **F** 2 **G** 49
H 100 **I** 14 **J** 14 **K** 15 **L** 11 **2. A** 0, 12 **B** 0, 8
C 0, −4 **D** 0, −18 **E** 1, 11 **F** 1, 7 **G** −1, −3
H −1, −19 **3. A** = **B** > **C** = **D** > **E** = **F** >
4. A $\{-10, -9, 9, 10\}$ **B** $\{-3, -2, -1, 0, 1, 2, 3\}$
C $\{0, 1, 2, 3, 4\}$
D $\{-10, -9, -8, -7, -6, -5, -4, -3, 9, 10\}$
Think **1.** *B* **2.** *A*: \$8800; *B*: \$8900 **3.** *A*: \$16 400; *B*: \$16 600 **4.** *A*

E-116 DISCUSSING THE IDEAS

? −2

1. $\boxed{\div 2}$

2. 9, −4, −2 **3.** −12, −3

E-117 USING THE IDEAS

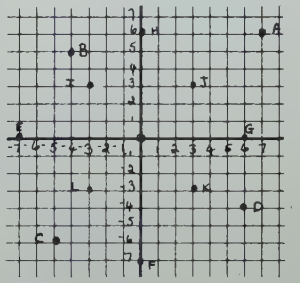
1. A −3 **B** 8 **C** −5 **D** −8 **E** −10 **F** 4
2. A −9, −3 **B** −6, −3 **C** −4, −1 **3. A** 10, 5
B −6, −2 **C** 8, −4 **D** −3, −6 **E** 3, −1 **4. A** 2
B −4 **C** −9 **D** 10 **E** −3 **F** 5 **G** −8 **H** 5 **I** −2
5. B 12, −12, 4, −4 **C** −5, 5, −8, 2
Think INTEGERS; ADDITION

E-118 DISCUSSING THE IDEAS

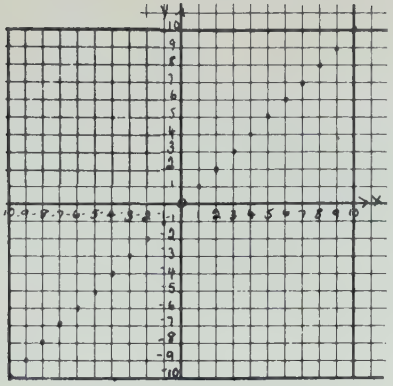
C (4, 4) **D** (−3, −2) **E** (3, −2) **F** (0, 3) **G** (−2, 1)
H (−5, −3) **I** (3, 0) **J** (−4, 0) **K** (−1, −3) **1.** (0, 0)
3. second coordinate is zero **4. A** first **B** second
C fourth **D** third

E-119 USING THE IDEAS

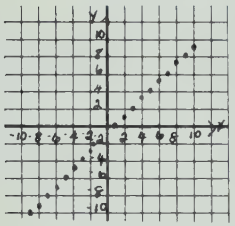
1.



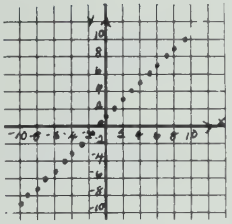
2.



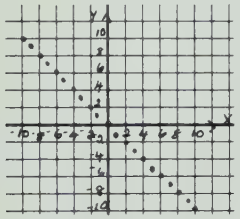
3.



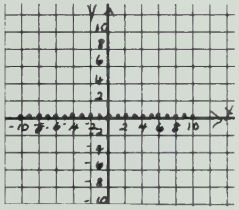
4.



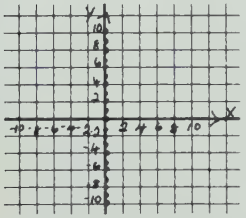
5.



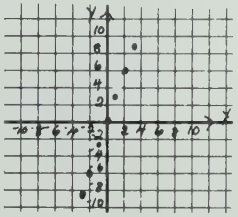
6.



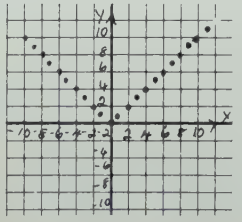
7.



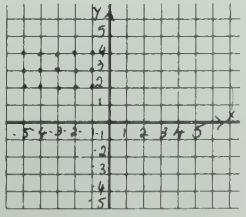
8.



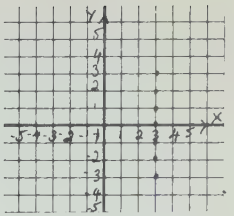
9.



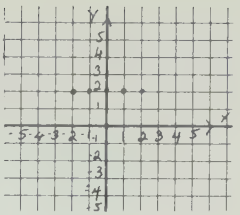
10. A



B

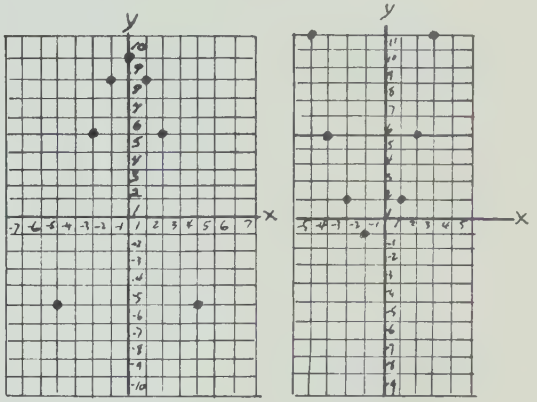


C



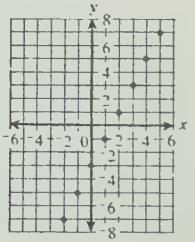
E-120 DISCUSSING THE IDEAS

1. 0 **2.** −4, 4 **3.** 3, −3 **5.** 5, 1, −1, −1, 1, 5, 11

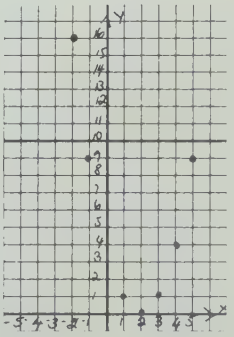


E-121 USING THE IDEAS

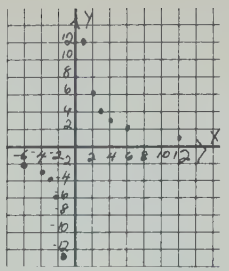
1. A −3, −1, 1, 3, 5, 7



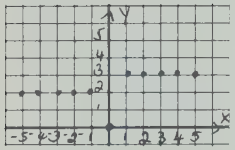
B 4, 1, 0, 1, 4, 9, 16



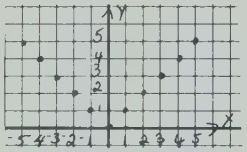
C 2, 3, 4, 6, 12, −12, −6, −4, −3, −2



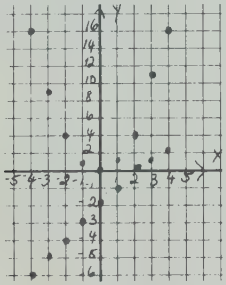
2.



3.



4. A

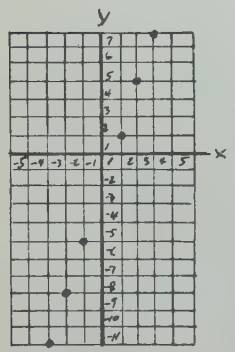


Think			
	256	2	64
	8	32	128
	16	512	4

E-122 DISCUSSING THE IDEAS

3. 4, 7, −5, −8, −11

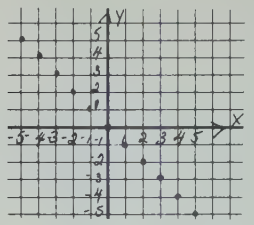
4.



E-123 USING THE IDEAS

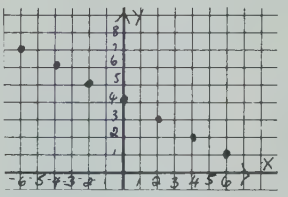
1. −4, 3, −3, 2, 1, −1, 0

2.



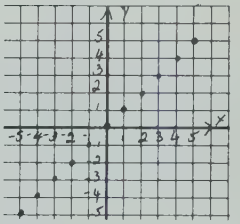
3. 2, 1, 6, 7

4.



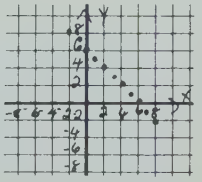
5. Answers will vary.

6.

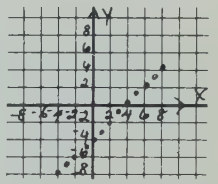


7. $y = x + 2$

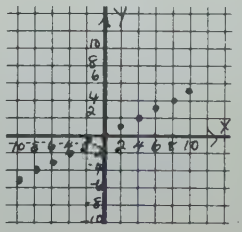
8. A



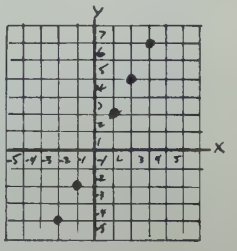
B



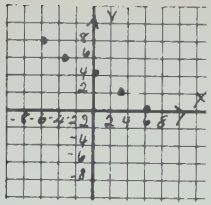
C



D



E



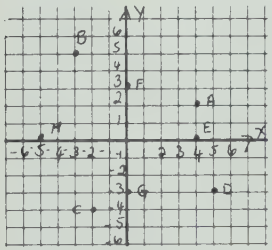
Think

L	I	K	E
L	A	K	E
M	A	K	E
M	A	T	E
M	A	T	H

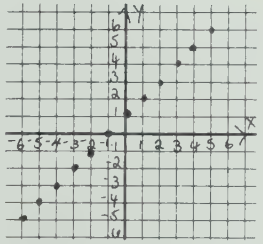
E-124 REVIEWING THE IDEAS

1. A 5 **B** −8 **C** 12 **2.** zero **3. A** 9 **B** −7 **C** −31
4. A 2 **B** −2 **C** 10 **D** 0 **5. A** 16 (zero)
B −9 (one) **C** −8 (opposite) **D** 17 (commutative +)
E 8 (associative +) **F** 2 (distributive)
G −20 (associative ×) **6. A** 42 **B** −42 **C** 42
7. A −9 **B** −6 **C** 8 **D** 0 **8. A** $\{-5, -4, -3, -2\}$
B $\{0, 1, 2, 3, 4, 5\}$ **C** $\{-1, 0, 1, 2\}$ **D** $\{4, 5, -5\}$
9. A 9 **B** 9 **10. A** 13 **B** 27 **C** 6 **D** 3 **E** 3, 1
F 2, 6 **11. A** 7, 2 **B** −6, 2 **C** 3, 5 **D** 1, −4
E −8, −5

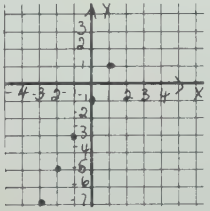
12.



13.



14. A −5, −1, 1

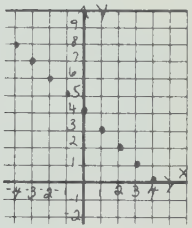


B 0, −3, −4, −3

15. A *b* **B** *a* **C** *b* **D** *b*

E-126 CUMULATIVE REVIEW

1. A MDCCCXCIII **B** 1617 **2.** 7^4 **3. A** 10^6 **B** 2^7
C 10^2 **D** 1 **4. A** 5 004 039 **B** $(7 \cdot 10^7) + (8 \cdot 10^6) +$
$(4 \cdot 10^5) + (6 \cdot 10^4) + (3 \cdot 10^3) + (0 \cdot 10^2) + (9 \cdot 10^1) + 7$
5. A 747 000 **B** 750 000 **C** 700 000 **6.** 32 000
7. 700 **8. A** 1176 **B** 8424 **C** 631 896 **D** 1795, R4
9. A 101 **B** 11001 **C** 1001101 **10. A** 1111 **B** 110
C 1111 **11. A** $\{1, 3, 5, 7, 9\}$ **B** $\{0, 2, 4, 6, 8, 10\}$ **C** *X*
12. *B* **13.** 22 **14. A** $\{4\}$ **B** $\{0, 1, 2, 3\}$ **C** $\{3, 4, 5\}$
D $\{3, 4, 5\}$ **15. A** $7x$ **B** $15x$ **C** $12x$ **D** $10x$
16. $6x - 4 = 26$ **17.** 5 **18. A** $\{15, 16, 17, 18\}$
B $\{3, 5\}$ **C** $\{3, 6, 9\}$ **D** $\{1, 3\}$ **19. A** 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 **B** 4, 6, 8, 9, 10, 12, 14, 15, 16, 18
20. A $2 \times 7 \times 7$ **B** $2 \times 3 \times 17$ **C** $5 \times 5 \times 5 \times 5$
21. A 2 **B** 10 **22. A** 40 **B** 105 **23. A** 3, 5
B 2, 3, 5, 10 **C** 2, 3 **D** 3, 5, 9 **24.** 4, 7, 12, 19, 28
25. A 46 **B** −97 **C** 1000 **26. A** 18 **B** −8
27. A −6 **B** −13 **C** 12 **D** 25 **28. A** −32 **B** 45
C 0 **29. A** −5 **B** −9 **C** 50 **D** −3 **30. A** 3
B −6 **C** −225 **D** −4
31. A −1, −8, 10, 0 **B** −2, 10, 0 or 3, 40 **32. A** >
B < **C** < **D** > **33. A** $\{4, 6\}$
B $\{-4, -2, 0, 2, 4, 6\}$ **C** $\{-4, -2\}$ **D** $\{-4, -2\}$
34. A 4 **B** 12 **C** 68 **D** 68 **E** 0 **F** 1 **35. A** 0, 7
B 1, 5 **36. A** 2, 1, 5, 6, 7, 8
B



UNIT F: MODULE 1

F-2 DISCUSSING THE IDEAS

2. $\frac{1}{3}$ **3.** $\frac{2}{6}, \frac{3}{9}, \frac{4}{12}$ **4. A** $\frac{10}{15}$ **B** $\frac{20}{30}$ **C** $\frac{200}{300}$ **5. A** $\frac{3}{5}$ **B** $\frac{2}{5}$
C $\frac{1}{2}$ **D** $\frac{9}{10}$ **E** $\frac{17}{50}$

F-3 USING THE IDEAS

1. $\frac{2}{5}, \frac{4}{10}, \frac{6}{15}$, or $\frac{8}{20}$ **2. A** $\frac{4}{16}, \frac{5}{20}, \frac{6}{24}$ **B** $\frac{8}{12}, \frac{10}{15}, \frac{12}{18}$ **C** $\frac{12}{8}, \frac{15}{10}, \frac{18}{12}$
D $\frac{16}{20}, \frac{20}{25}, \frac{24}{30}$ **E** $\frac{20}{28}, \frac{25}{35}, \frac{30}{42}$ **F** $\frac{36}{32}, \frac{45}{40}, \frac{54}{48}$ **3. A** $\frac{2}{10}, \frac{3}{15}$ **B** $\frac{8}{6}, \frac{12}{9}$
C. $\frac{10}{16}, \frac{15}{24}$ **D** $\frac{6}{20}, \frac{9}{30}$ **E** $\frac{10}{6}, \frac{15}{9}$ **F** $\frac{2}{16}, \frac{3}{24}$ **G** $\frac{14}{16}, \frac{21}{24}$ **H** $\frac{10}{24}, \frac{15}{36}$
I $\frac{10}{12}, \frac{15}{18}$ **J** $\frac{22}{20}, \frac{33}{30}$ **K** $\frac{14}{32}, \frac{21}{48}$ **L** $\frac{14}{40}, \frac{21}{60}$ **4. A** 6, $\frac{2}{3}$ **B** 4, $\frac{2}{3}$
C 24, 1 **D** 7, $\frac{6}{7}$ **E** 2, $\frac{5}{2}$ **F** 3, $\frac{2}{3}$ **G** 4, $\frac{5}{3}$ **H** 5, $\frac{2}{3}$ **I** 5, $\frac{5}{8}$
J 10, 0 **K** 7, $\frac{9}{10}$ **L** 15, $\frac{3}{4}$ **M** 16, $\frac{3}{4}$ **N** 13, $\frac{3}{4}$ **O** 12, 1
5. A $\frac{16}{24}$ **B** $\frac{12}{24}$ **C** $\frac{10}{24}$ **D** $\frac{9}{24}$ **E** $\frac{6}{24}$ **F** $\frac{20}{24}$ **G** $\frac{21}{24}$
6. A improper **B** proper **C** proper **D** improper
E proper **F** improper
Think 622

F-4 DISCUSSING THE IDEAS

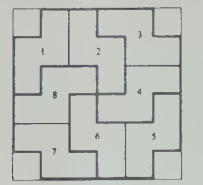
2. A no **B** yes **C** no **D** yes **E** yes **3. A** 5
B 10 **C** 3 **D** 60 **E** 30

F-5 USING THE IDEAS

1. A $\frac{3}{10}$ **B** $\frac{1}{2}$ **C** $\frac{9}{10}$ **D** $\frac{2}{5}$ **E** $\frac{4}{5}$ **2. A** 45, 45, yes
B 24, 24, yes **C** 120, 150, no **3. A** yes **B** no **C** yes

D yes **E** yes **F** no **G** yes **H** yes **4. A** 3 **B** 15
C 45 **D** 8 **E** 12 **F** 1 **G** 25 **H** 60 **I** 2 **J** 66
K 135 **L** 125

Think



F-6 DISCUSSING THE IDEAS

2. A $-\frac{1}{2}$ **B** $-\frac{7}{8}$ **C** -1 **D** $-\frac{4}{3}$ **E** $-\frac{13}{8}$ **F** -2
3. A $-\frac{2}{3}$ **B** 0

F-7 USING THE IDEAS

1. A $-\frac{3}{8}$ **B** $\frac{5}{6}$ **C** $\frac{1}{4}$ **D** $-\frac{11}{5}$ **C** $\frac{37}{45}$ **2.** 0 **3. A** $\frac{6}{5}$
B $-\frac{7}{5}$ **C** $\frac{4}{5}$ **D** $-\frac{4}{5}$ **E** $-\frac{2}{5}$ **F** $\frac{1}{5}$ **G** $-\frac{9}{5}$ **4. A** $-\frac{3}{4}$ **B** $\frac{5}{8}$
C $-\frac{11}{4}$ **D** $\frac{7}{8}$ **E** $\frac{11}{6}$ **F** $-\frac{8}{3}$ **G** -2 **H** $-1\frac{3}{4}$ **I** $\frac{2}{3}$ **J** 0
K $\frac{5}{4}$ **L** 0 **M** 4 **N** 0 **O** $-\frac{5}{2}$ **5.** $-\frac{4}{8}, -\frac{5}{10}, -\frac{6}{12} \ldots$
6. A negative **B** positive **C** zero
Think 19 units minimum

F-8 DISCUSSING THE IDEAS

1. 0 **2.** 0 **3.** $-\frac{2}{3}$ **5.** 0 **6. A** positive **B** positive
C positive **D** negative **E** negative **F** positive

F-9 USING THE IDEAS

1. A $-\frac{4}{3}$ **B** $\frac{5}{6}$ **C** $-\frac{3}{5}$ **D** $\frac{6}{10}$ **E** $\frac{9}{7}$ **2. A** T **B** F **C** F
D T **E** T **3. A** *P* **B** *Q* **C** *P* **D** *Q* **E** *Q* **F** *Q*
4. A *S* **B** *R* **C** *U* **D** *S* **E** *S* **F** *R* **G** *T* **H** *T* **I** *R*
J *S* **K** *S* **L** *U*
Think 3

F-10 DISCUSSING THE IDEAS

4. A $-\frac{6}{12}$ **B** $-\frac{1}{8}$ **C** $-\frac{1}{7}$ **6.** $a + c$

F-11 USING THE IDEAS

1. A $\frac{5}{6}$ **B** $\frac{1}{10}$ **C** $-\frac{1}{8}$ **D** $-\frac{5}{4}$ **E** 1 **F** $\frac{2}{10}$ **G** $-\frac{2}{100}$
H $-\frac{12}{10}$ **I** $-\frac{4}{8}$ **J** $\frac{4}{5}$ **K** $\frac{10}{5}$ **L** $-\frac{10}{9}$ **2. A** $-\frac{1}{7}$ **B** $\frac{2}{4}$
C $-\frac{7}{10}$ **D** $-\frac{8}{8}$ **E** $\frac{4}{6}$ **F** $\frac{8}{8}$ **G** $-\frac{2}{12}$ **H** $-\frac{6}{4}$ **I** $-\frac{9}{10}$ **J** 0
K $\frac{19}{100}$ **L** $-\frac{6}{9}$ **3. A** $\frac{7}{4}$ **B** $\frac{3}{8}$ **C** $\frac{1}{10}$ **D** 0 **E** $-\frac{1}{2}$ **F** $-\frac{7}{6}$
G $-\frac{2}{10}$ **H** $\frac{1}{3}$ **I** 1 **4. A** 1 **B** -1 **C** $-\frac{1}{3}$ **D** $-\frac{8}{8}$ **E** 1
F $-\frac{9}{10}$ **G** $-\frac{19}{20}$ **H** -1 **I** $\frac{4}{15}$ **J** 1 **K** $-\frac{1}{6}$ **L** $\frac{6}{4}$ **M** $-\frac{4}{6}$
N $\frac{3}{2}$ **O** -1 **P** $\frac{13}{100}$ **Q** $-\frac{3}{10}$ **R** $\frac{34}{100}$
Think **A** 4 **B** 5 **C** 5 **D** 5

F-12 DISCUSSING THE IDEAS

? 1. *C* **2.** *D* **3.** *A* **4.** *E* **5.** *B* **2. A** 12 **B** 6
C 20 **D** 10 **E** 100 **3. A** 15 **B** -4 **C** 20 **D** -9
E -9 **4.** $-\frac{5}{12}$

F-13 USING THE IDEAS

1. A 3, 4, 7 **B** 4, 9, 13 **C** -30, 7, -23
D -9, -20, -29 **2. A** $\frac{4}{7}$ **B** $-\frac{1}{2}$ **C** $-\frac{2}{3}$ **D** $-\frac{3}{4}$ **E** $\frac{1}{6}$
F $-\frac{3}{8}$ **G** $-\frac{1}{2}$ **H** $-\frac{4}{3}$ **I** $\frac{9}{8}$ **J** $\frac{20}{20}$ **K** $-\frac{1}{6}$ **L** $\frac{1}{12}$ **M** $-\frac{19}{24}$
N $\frac{17}{12}$ **O** $\frac{1}{3}$ **P** $-\frac{18}{40}$ **3. A** $101\frac{5}{6}$ **B** $21\frac{13}{16}$ **C** 46 **D** $53\frac{11}{12}$
E $68\frac{3}{8}$ **F** 733 **G** $184\frac{57}{100}$ **H** $1197\frac{33}{100}$ **4. A** $27\frac{7}{8}$ **B** $18\frac{4}{5}$
C $12\frac{1}{2}$ **D** $15\frac{3}{10}$

Think

7	-7	-6	4
-4	2	1	-1
0	-2	-3	3
-5	5	6	-8

F-14 DISCUSSING THE IDEAS

1. A 8 **B** -7 **C** 4 **2. A** $\frac{4}{5}$ **B** $-\frac{7}{10}$ **C** $\frac{2}{5}$ **3.** $a - c$
4. A $\frac{17}{12}$ **B** -1 **C** $-\frac{1}{8}$

F-15 USING THE IDEAS

1. A 1 **B** $-\frac{4}{5}$ **C** $2\frac{2}{5}$ **D** -1 **E** 1 **F** $\frac{1}{2}$ **G** 1 **H** -1
I $-\frac{2}{5}$ **J** $1\frac{1}{4}$ **K** $-1\frac{1}{2}$ **L** $\frac{1}{2}$ **M** $\frac{4}{25}$ **N** $-1\frac{133}{1000}$ **O** $-\frac{2}{5}$
2. A $\frac{3}{8}$ **B** $\frac{5}{12}$ **C** $\frac{7}{12}$ **D** $\frac{1}{5}$ **E** $\frac{1}{5}$ **F** $\frac{1}{12}$ **G** $\frac{3}{20}$ **H** $\frac{1}{24}$ **I** $\frac{13}{100}$
J $\frac{357}{1000}$ **3. A** $1\frac{1}{8}$ **B** -1 **C** $-\frac{1}{3}$ **D** $\frac{5}{6}$ **E** $1\frac{3}{8}$ **F** $1\frac{13}{30}$
G $-1\frac{3}{10}$ **H** $\frac{7}{20}$ **I** $-\frac{1}{2}$ **J** $-\frac{7}{8}$ **K** $\frac{13}{20}$ **L** $\frac{1}{10}$ **M** $1\frac{1}{6}$ **N** $\frac{1}{6}$
O $-\frac{1}{6}$ **4. A** $5\frac{7}{12}$ **B** $24\frac{3}{10}$ **C** $22\frac{3}{10}$ **D** $12\frac{5}{6}$ **E** $46\frac{7}{8}$
5. A $5\frac{2}{5}$ **B** $3\frac{3}{20}$ **C** $2\frac{4}{5}$

Think

1.

-1	1	-1
1	1	1
-1	1	-1

2.

1	-1	1
-1	-1	-1
1	-1	1

F-16 DISCUSSING THE IDEAS

1. A $\frac{1}{3}$ **B** $\frac{1}{3}$ **C** $\frac{1}{10}$ **2. A** $\frac{1}{6}$ **B** $\frac{3}{5}$ **3. A** $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$
B $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$ **C** $5 \cdot \frac{1}{6} = \frac{5}{6}$ **D** $5 \cdot \frac{1}{3} = \frac{5}{3}$

F-17 USING THE IDEAS

1. A $\frac{1}{12}$ **B** $\frac{1}{16}$ **C** $\frac{1}{3}$ **D** $\frac{1}{3}$ **E** $\frac{1}{42}$ **F** $\frac{1}{10}$ **G** $\frac{1}{10}$ **H** $\frac{1}{2}$
2. A $\frac{4}{3}$ **B** $\frac{3}{4}$ **C** 0 **D** 1 **E** 4 **3.** 1 **4. A** one **B** one
C commutative × **D** associative ×
E commutative × **F** distributive **5. A** $-\frac{1}{4}$ **B** $-\frac{3}{10}$
C $-\frac{2}{3}$ **D** $-\frac{3}{8}$ **E** $-\frac{5}{6}$ **F** $-\frac{17}{100}$ **6.** additive; inverse
7. A $\frac{1}{3}$ **B** $\frac{3}{8}$ **C** $\frac{9}{10}$ **D** $\frac{3}{4}$ **8. A** $-\frac{1}{4}$ **B** $-\frac{1}{10}$ **C** $\frac{7}{8}$ **D** -1
9. A $-\frac{1}{6}$ **B** $-\frac{1}{24}$ **C** $\frac{1}{18}$ **D** $\frac{1}{10}$ **E** $-\frac{1}{1000}$ **F** $\frac{1}{100}$

F-18 DISCUSSING THE IDEAS

2. A $\frac{9}{40}$ **3. A** $-\frac{8}{15}$ **B** $\frac{21}{40}$

F-19 USING THE IDEAS

1. A $\frac{8}{15}$ **B** $\frac{5}{21}$ **C** $\frac{3}{2}$ **D** $\frac{9}{32}$ **E** $\frac{1}{2}$ **F** $\frac{5}{3}$ **G** 1 **H** $1\frac{1}{2}$
I $\frac{2}{3}$ **J** $\frac{1}{4}$ **K** 0 **I** $2\frac{2}{3}$ **J** $3\frac{1}{3}$ **K** $\frac{9}{10}$ **L** $\frac{27}{100}$ **2. A** $-\frac{9}{16}$
B $-\frac{5}{21}$ **C** $\frac{1}{4}$ **D** $-\frac{4}{15}$ **E** $\frac{1}{12}$ **F** $\frac{1}{2}$ **G** -1 **H** -2 **I** $\frac{1}{4}$
J $\frac{4}{9}$ **K** $-\frac{5}{2}$ **L** $\frac{20}{9}$ **M** -6 **N** $-\frac{21}{10}$ **O** $-\frac{15}{2}$ **P** $-\frac{21}{5}$
Q 45 **R** 1 **S** 1 **T** 63 **3. A** positive **B** positive
C negative **4. A** 14 **B** -14 **C** 12 **D** -25 **E** 9
F 16 **5. A** $\frac{32}{15}$ **B** $\frac{21}{16}$ **C** -2 **D** $-\frac{35}{9}$ **E** $\frac{15}{4}$ **F** 40
G -25 **H** 25 **I** 20
Think Start at 5 (49 lots)

F-20 DISCUSSING THE IDEAS

? A $\frac{1}{3}$ **B** $\frac{4}{3}$ **C** $-\frac{2}{5}$ **D** $-\frac{1}{10}$ **1. A** $\frac{10}{11}$ **B** -2 **C** $-\frac{1}{3}$
D $\frac{5}{11}$ **E** 1 **2. B** whole number **3.** negative **4.** 1, -1
5. $\frac{10}{37}$

F-21 USING THE IDEAS

1. A $\frac{5}{2}$ **B** -3 **C** $\frac{8}{9}$ **D** $\frac{15}{16}$ **E** $-\frac{1}{4}$ **F** $\frac{7}{3}$ **G** 1 **H** $-\frac{6}{11}$
I $\frac{3}{10}$ **J** $\frac{5}{37}$ **K** $\frac{1}{10}$ **L** -100 **2. A** $\frac{8}{3}$ **B** $\frac{4}{5}$ **C** $\frac{7}{8}$ **D** $\frac{5}{6}$
E $\frac{7}{12}$ **F** $\frac{9}{4}$ **G** $\frac{1}{10}$ **H** 6 **I** $\frac{2}{3}$ **J** $\frac{8}{41}$ **K** $\frac{3}{30}$ **L** 1 **3. A** $\frac{1}{15}$
B $-\frac{1}{28}$ **C** 8 **D** $-2\frac{1}{2}$ **E** $-2\frac{2}{3}$ **F** $4\frac{2}{7}$ **4. A** $\frac{3}{2}$ **B** $\frac{9}{10}$
C 2 **D** $-\frac{3}{8}$ **E** $\frac{7}{5}$ **F** $\frac{19}{7}$

Think **1.** yes **2.** yes for A and B but not for C

F-22 ELECTRICAL ENERGY

1. 50 **2.** $1\frac{7}{20}$ **3.** $19\frac{1}{6}$ **4.** $4\frac{1}{2}$ **5.** $\frac{1}{3}$ **6.** 21 **7.** 10 000
8. $1\frac{28}{75}$ **9. A** \$11.70 **B** $1\frac{1}{2}$¢ **C** \$65.70 **D** 50.4¢

F-23 IMPROVING COMPUTATIONAL SKILLS

1. A $45\frac{4}{5}$ **B** $20\frac{1}{2}$ **C** $220\frac{3}{5}$ **D** 100 **E** $199\frac{1}{30}$ **F** $1\frac{1}{30}$
G $1\frac{47}{1000}$ **H** $1\frac{13}{60}$ **I** $3\frac{73}{100}$ **J** $\frac{111}{200}$ **2. A** $\frac{14}{25}$ **B** $\frac{12}{25}$ **C** $\frac{1}{15}$
D $-6\frac{4}{25}$ **E** $\frac{22}{90}$ **F** $8\frac{1}{3}$ **G** $8\frac{3}{4}$ **H** $7\frac{1}{10}$ **I** $45\frac{3}{5}$ **J** $91\frac{1}{8}$
3. A $\frac{7}{25}$ **B** $\frac{1}{6}$ **C** $\frac{5}{16}$ **D** $17\frac{1}{3}$ **E** 20 **F** $15\frac{1}{4}$ **G** 13
H $183\frac{3}{4}$ **I** 100 **J** $88\frac{2}{3}$

Think **A** $\frac{1}{5}$ **B** 0 **C** $\frac{1}{2}$ **D** 4

F-24 DISCUSSING THE IDEAS

1. $\frac{28}{15}$ **2.** c/d **3. A** $-\frac{2}{5}$ **B** $\frac{15}{16}$ **C** $-1\frac{1}{3}$ **D** $-\frac{1}{2}$
4. the multiplicative inverse

F-25 USING THE IDEAS

1. A $\frac{3}{2}$ **B** $\frac{3}{2}$ **C** $\frac{2}{3}$ **D** $\frac{3}{2}$ **E** -2 **F** $-\frac{2}{3}$ **G** $\frac{15}{8}$ **H** $\frac{9}{2}$
I $-\frac{12}{5}$ **2. A** $\frac{2}{3}$ **B** $\frac{5}{2}$ **C** $\frac{3}{2}$ **D** $\frac{7}{6}$ **E** $-\frac{3}{2}$ **F** $-\frac{3}{4}$ **G** -1
H 4 **I** $\frac{3}{2}$ **J** $\frac{2}{3}$ **K** $\frac{2}{3}$ **L** $-\frac{3}{10}$ **3. A** $3\frac{1}{3}$ **B** $1\frac{4}{5}$ **C** $1\frac{11}{14}$
D $2\frac{1}{4}$ **E** $-\frac{1}{4}$ **F** -3 **G** 2 **H** $2\frac{2}{3}$ **4. A** $\frac{5}{6}$ **B** 25 **C** 3
D $-\frac{3}{4}$ **E** $1\frac{1}{4}$ **F** $-\frac{2}{3}$ **G** $-\frac{4}{9}$ **H** $-2\frac{1}{2}$ **5. A** $1\frac{7}{8}$ **B** $1\frac{7}{15}$
C $1\frac{7}{13}$ **D** $1\frac{7}{8}$ **E** $1\frac{1}{7}$ **F** $4\frac{2}{3}$

Think 17

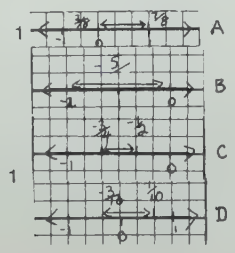
F-26 DISCUSSING THE IDEAS

? A -1 and -2 **B** -2 and -3 **C** -3 and -4 **A** $-\frac{7}{3}$
B $-\frac{3}{2}$ **C** $-\frac{5}{6}$ **D** $-\frac{1}{4}$ **E** $-\frac{1}{3}$ **F** $\frac{1}{2}$ **G** $-5/-6$ **H** $\frac{7}{4}$ **I** $\frac{7}{3}$
1. $-\frac{7}{3}$ **2.** $\frac{7}{3}$ **3. A** $>$ **B** $>$ **C** $<$ **D** $=$ **4. A** $<$ **B** $<$
C $<$ **D** $<$

F-27 USING THE IDEAS

1. A 2 **B** 12 **C** -8 **D** 5 **E** -15 **F** -25 **G** 7
H 77 **I** 2 **J** 7 **2. A** 4, 5; $<$ **B** 10, 9; $>$
C -24, -25; $>$ **D** 28, 33; $<$ **E** 140, 120; $>$
F 15, -20; $>$ **3. A** $<$ **B** $<$ **C** $>$ **D** $>$ **E** $>$ **F** $<$
G $>$ **H** $=$ **I** $<$ **J** $>$ **K** $<$ **L** $=$ **M** $<$ **N** $<$
O $>$ **P** $>$ **4. A** $-\frac{3}{2} < x < \frac{3}{4}$ **B** $-\frac{15}{4} < x < 0$
C $-\frac{7}{3} < x < \frac{9}{10}$

5.



F-28 DISCUSSING THE IDEAS

? \$72 **1.** reduced **3.** \$72 **4. A** \$32 **B** \$64
5 \$31.96

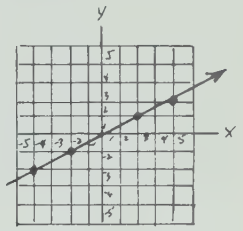
F-29 USING THE IDEAS

1. \$72.67 **2.** \$135 **3.** \$9 **4. A** \$4.80 **B** \$4.72
5. 16 **6.** 60 km/h **7.** 1000 **8.** 9 **9.** $\frac{1}{4}$

Think **A** \$16 **B** *A*: \$2; *B*: \$2; *C*: \$2

F-30 DISCUSSING THE IDEAS

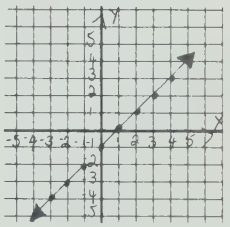
?

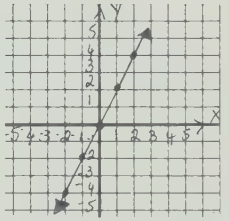


3. segment

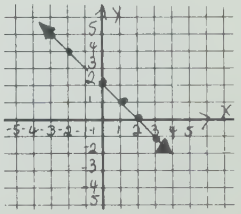
F-31 USING THE IDEAS

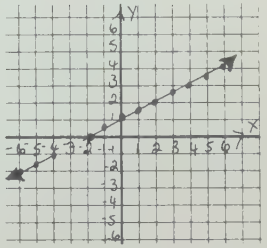
1. -2, $-2\frac{1}{2}$ **2.** -1, 0, 3



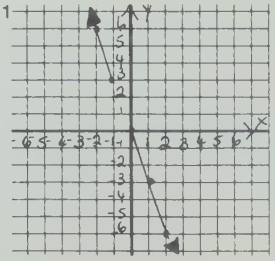


3. 3, 4, $2\frac{1}{2}$ **4.** 2, $\frac{5}{2}$, $\frac{1}{2}$

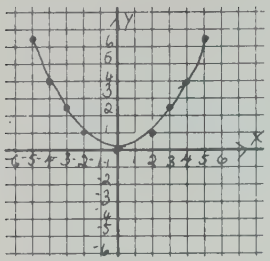




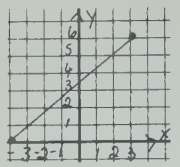
5. 0, -1, $-\frac{3}{2}$, 3, 1



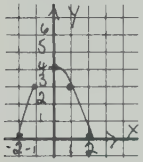
6. 0, $\frac{1}{4}$, 1, $\frac{9}{4}$, $\frac{1}{4}$



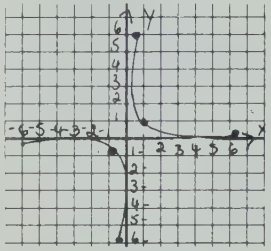
7. 1, 2, $\frac{5}{2}$, 3, 4



8. 3, 0, $1\frac{3}{4}$, $1\frac{3}{4}$



9. 4, $\frac{1}{2}$, -4, $-\frac{1}{2}$

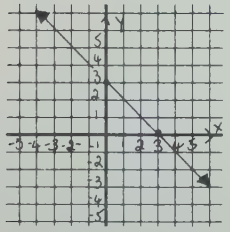


F-32 DISCUSSING THE IDEAS

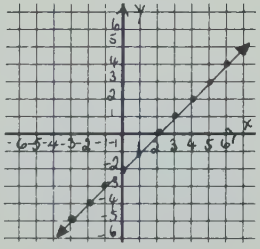
2. A no **B** no **3.** yes **4.** $3\frac{9}{10}$ **5.** yes

F-33 USING THE IDEAS

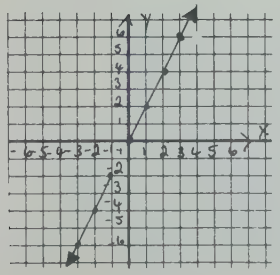
1. 1, 5



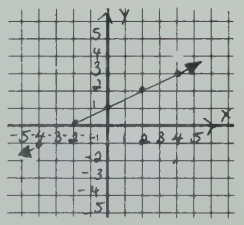
2. -4



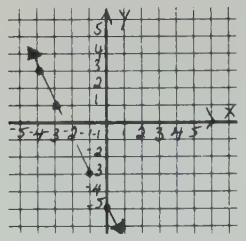
3. 8



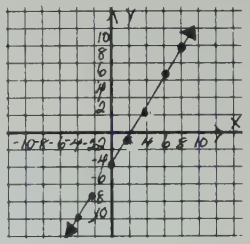
4. 1, 3, -1



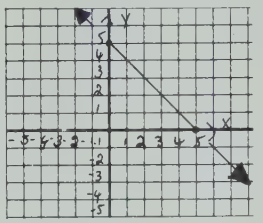
5. -5, 5, -7



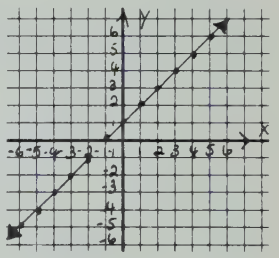
6. -1, -4, -10



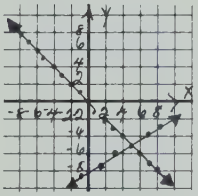
7. A



B

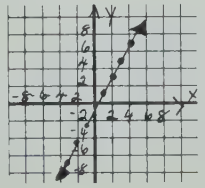


C



8. (3, 1)

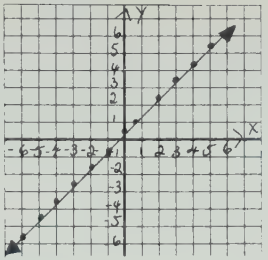
9. (5, -5)



F-34 REVIEWING THE IDEAS

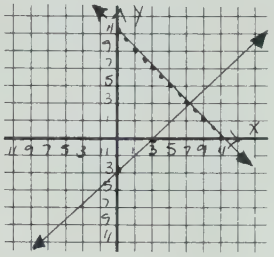
1. A $-\frac{3}{7}$ **B** $\frac{2}{3}$ **C** $\frac{5}{7}$ **D** 0 **2. A** 0 **B** $\frac{3}{7}$ **C** $-\frac{5}{6}$ **D** $-\frac{1}{6}$
E $\frac{5}{8}$ **F** $-\frac{3}{8}$ **3. A** $-\frac{1}{7}$ **B** -1 **C** $\frac{6}{7}$ **D** $1\frac{2}{3}$ **E** $-\frac{1}{6}$ **F** $\frac{2}{3}$
4. A $\frac{4}{5}$ **B** $-\frac{2}{5}$ **C** $\frac{5}{9}$ **D** $-\frac{1}{6}$ **E** $3\frac{1}{3}$ **F** $-1\frac{1}{2}$ **G** 1
H $-1\frac{1}{4}$ **I** -9 **5. A** $1\frac{1}{3}$ **B** $-1\frac{1}{2}$ **C** -6 **D** $-2\frac{1}{3}$ **E** $\frac{1}{4}$
F $-\frac{1}{5}$ **G** $-\frac{3}{4}$ **H** $\frac{5}{7}$ **6. A** $\frac{3}{5}$ **B** $-\frac{1}{2}$ **C** $1\frac{1}{3}$ **D** $1\frac{2}{5}$ **E** $2\frac{1}{2}$
F $-\frac{1}{3}$ **G** -4 **H** $-1\frac{7}{9}$ **I** $-\frac{5}{8}$ **7. A** 16 **B** $\frac{11}{10}$ **C** $\frac{3}{7}$ **D** $\frac{1}{2}$
E $\frac{5}{9}$ **F** 31 **G** $\frac{1}{5}$ **H** $\frac{1}{2}$ **8. A** $\frac{1}{6}$ **B** $\frac{3}{5}$ **C** $-\frac{3}{10}$ **D** $-\frac{1}{2}$
E $-1\frac{1}{2}$ **F** $-1\frac{1}{3}$ **9.** $-\frac{3}{4} < x < \frac{1}{2}$ **10.** \$99 **11.** $\frac{3}{4}$, $\frac{1}{4}$, 0, $-\frac{1}{4}$

12.



13. Answers may vary. Some pairs are: (−1, 13), (2, 11), (1, 9), (2, 7), . . .

14.



UNIT F: MODULE 2

F-38 DISCUSSING THE IDEAS

? $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$, $\frac{1}{1000000}$ **1. A** $\frac{1}{10^6}$ **B** 10^{-8} **2. A** $1/7^2$
B 2^{-3} **3. A** $\frac{23}{100}$ **B** $\frac{625}{1000}$ **C** $32\frac{7}{10}$ **D** $500\frac{1}{20}$

F-39 USING THE IDEAS

1. A $1/10^2$ **B** $1/10^4$ **C** $1/10^5$ **D** $1/10^{14}$ **E** $1/6^2$
F $1/5^3$ **G** $\frac{1}{28}$ **H** $\frac{1}{34}$ **I** $\frac{1}{73}$ **J** $1/8^5$ **2. A** 10^{-3} **B** 10^{-4}
C 7^{-5} **D** 3^{-3} **E** 10^{-1} **F** 10^{-7} **G** 10^{-3} **H** 10^{-4}
I 2^{-1} **J** 3^{-5} **3. A** 10^3 **B** 10^{-3} **C** 10^{-2} **D** 10^{2}
E 10^4 **F** 10^{-1} **G** 10^{-5} **H** 10^{-4} **I** 10^6 **J** 10^0
4. A $\frac{18}{25}$ **B** $\frac{703}{1000}$ **C** $3\frac{1}{2}$ **D** $1\frac{39}{100}$ **E** $807\frac{2}{5}$ **F** $\frac{5631}{10000}$
G $12\frac{37}{100}$ **H** $310\frac{3}{50}$ **5.** $7\frac{3}{4}$ **6.** *C*

Think

F-40 DISCUSSING THE IDEAS

? A 4925.0 **B** 49.250 **C** 492.50 **1. A** 0.001
B 0.0001 **C** 0.2 **D** 0.03 **E** 0.27 **F** 0.179
2. A hundredths thousandths ten thousandths
B −2 −3 −4 **3. A** $7 \cdot 10^1 + 5 \cdot 10^0 + 4 \cdot 10^{-1}$
B $2 \cdot 10^0 + 7 \cdot 10^{-1} + 8 \cdot 10^{-3}$
C $1 \cdot 10^2 + 2 \cdot 10^1 + 9 \cdot 10^0 + 3 \cdot 10^{-1} + 5 \cdot 10^{-2}$
D $3 \cdot 10^0 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 1 \cdot 10^{-3} + 6 \cdot 10^{-4}$
4. A 0.9, 0.09, 0.0095) **B** 0.6, 0.06, 0.006
C 0.1, 0.01, 0.001 **D** 0.8, 0.08, 0.008

F-41 USING THE IDEAS

1. A hundreds, 3×10^2 **B** hundredths, 6×10^{-2}
C thousandths, 7×10^{-3} **2. B** 0.7 **C** 0.19 **D** 0.239
E 0.9 **F** 0.03 **G** 0.743 **H** 0.6197 **I** 0.39 **J** 0.007
K 0.0291 **L** 0.009 **3. B** 7.142 **C** 56.375 **D** 475.3
E 25.07 **F** 27.046 **G** 62.0539 **H** 3.0007 **4. A** $3 \cdot 10^0 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 7 \cdot 10^{-3}$ **B** $2 \cdot 10^2 + 9 \cdot 10^1 + 8 \cdot 10^0 + 4 \cdot 10^{-1} + 2 \cdot 10^{-2}$ **C** $6 \cdot 10^{-1} + 1 \cdot 10^{-2} + 2 \cdot 10^{-3} + 9 \cdot 10^{-4}$
D. $6 \cdot 10^{-3} + 7 \cdot 10^{-4}$ **5. A** 23.1 **B** 8.03 **C** 965.81
D 700.03 **6. A** 0.7 **B** 0.70 **C** 0.700. **7.** 0.750; 0.7500
8. A 72.435 **B** 8.020 **C** 0.074
Think **A** 5 **B** 3 **C** 10 **D** 7 **E** 5 **F** 3

F-42 DISCUSSING THE IDEAS

? A 0.25 km **B** 2.6 km **C** 0.65 km **2.** 0.053

F-43 USING THE IDEAS

1. A 111.01 **B** 1.541 **C** 1661.62 **D** 58.966
E 0.222 **2. A** 14.5 **B** 5.09 **C** 0.038 **D** 22.8
E 6.217 **3. A** 7.985 **B** 10.802 **C** 21.222 **D** 20.276
4. A 13.96 **B** 46.166 **C** 24.103 **D** 23.064
5. A 2.558 **B** 12.483 **C** 3.85 **D** 99.1224 **6. A** 5.6
B −1.191 **C** −15.9 **D** 0.0656 **E** −0.26 **F** −0.94
G 2.17 **H** −4.25

F-44 CELSIUS TEMPERATURES

1. A 1.3°C **B** 2.2°C **2.** 13.8 **3.** 3.6 **4.** 105.6
5. 146.1 **6.** 47.8 **7.** −273.16°C **8.** 194.66 **9.** no
10. 11.8

F-45 SPEED RECORDS

1. 789.657 **2.** 39.83 **3.** 225.773 **4.** 0.97 **5.** 5.29
6. 1.77 **7.** 0.24

F-46 DISCUSSING THE IDEAS

? 0.06 **3. A** 0.0048 **B** 3.5 **C** 0.09585 **4.** right, left

F-47 USING THE IDEAS

1. A hundredths **B** thousandths **C** hundred thousandths **2. A** 127.8 **B** 3.78 **C** 22.26 **D** 1.625
E 5250 **F** 14.744 **G** 27.82 **H** 4.7124 **I** 77.89012
3. 90¢ **4.** 1.2 cm **5.** 3.01 s **6.** $21 **7.** 1008
8. 300 000 km/s **9. A** −1.68 **B** −0.24 **C** 2.24
D −0.793 **E** −2.56 **F** 0.504 **G** −3.1714 **H** −0.0684
I 0.36519
Think 1

F-48 DISCUSSING THE IDEAS

? A *J* **B** *N* **C** *L* **D** *I* **E** *K* **F** *H* **G** *M*
1. $3 \cdot 2 \times 10^4$ **2.** *A, B, D, E* **3. A** $5 \cdot 8 \times 10^3$
B $1 \cdot 7 \times 10^{-2}$ **C** $1 \cdot 0 \times 10^4$ **D** $3.76 \times 10^{\circ}$ **E** $8 \cdot 0 \times 10^{-5}$

F-49 USING THE IDEAS

1. A 1 **B** 2 **C** −1 **D** −2 **E** 3 **F** 6 **G** 5 **H** −8
I 0 **2. A** 2.3×10^2 **B** 1.28×10^3 **C** 6.19×10^3
D 3.029×10^3 km **E** 4.0×10^4 **F** 4.44×10^6
G 8.7×10^9 **H** 3.0×10^{22} **3. A** 7000 **B** 0.062
C 0.00000545 **D** 90 300 000 **E** 6.43
F 0.000000000005 **G** 11 000 000 000 **H** 0.0000000337
4. A 0.02 **B** 0.35 **C** 0.000064 **D** 0.000053
E 0.000045 **F** 0.00000025 **G** 0.0000000106
H 0.000000000000000003 **5. A** 3.0×10^7 **B** 2.0×10^{23}
C 2.0×10^{-4} **D** 5.6×10^{-2} **E** 1.0×10^8 **F** 1.0×10^2
G 4.2×10^{13} **H** 6.0×10^{-7} **I** 1.0×10^{-6}

F-50 DISCUSSING THE IDEAS

? 3.85×10^5 km **1. A** yes **B** yes **C** no **D** yes
2. A 3 **B** 4 **3. A** 4 **B** 3 **C** 4 **D** 1 **4.** 3
5. A 5×10^2 **B** 5×10^1

F-51 USING THE IDEAS

1. A 2 **B** 3 **C** 2 **D** 3 **E** 4 **F** 5 **2. A** 500 cm
B 0.00005 mm **C** 50 000 000 yr **D** 50 000 kg
C 0.00000005 cm **F** 5 ml **G** 5000 m
H 50 000 000 000 km **I** 0.005 cm **3. A** $\frac{1}{130}$ **B** $\frac{1}{144}$
C $\frac{1}{60}$ **D** $\frac{1}{210}$ **4. A** $(3.2 \pm .05) \times 10^{-2}$ **B** 2.14×10^{-3}
C 4.0×10^3

F-52 DISCUSSING THE IDEAS

? A 0.543 **B** .047 **C** .025 **3. A** 3.62 **B** 0.00243
C 0.023 **4. A** 42.1 **B** 0.032 **C** 0.15 **D** 7.3

F-53 USING THE IDEAS

1. A 3.8 **B** 0.57 **C** 0.063 **D** 3.56 **E** 0.0023
F 0.00015 **G** 5.294 **H** 0.74 **2. A** 35.7 **B** 38 **C** 6.4
D 0.062 **E** 0.41 **F** 7.7 **G** 1.7 **H** 0.036 **3.** 125
4. 40 **5.** 1.31 **6.** 192 **7.** 64 s **8.** 80 **9. B** -2
B -0.5 **C** -700 **D** 0.2 **E** 0.05 **F** 30
Think **1.** 3 **2.** 7 **3.** 3 **4.** 3

F-54 DISCUSSING THE IDEAS

? A 6.93 **B** 6.93 **C** 6.93 **D** 693 **E** 0.0693
F 0.00693 **2. A** 2 **B** 3 **C** 6.93 **3. A** 110.4 . . .
B 510.5 . . . **C** 0.01892 . . . **4.** 20

F-55 USING THE IDEAS

1. A 3.6×10^7 **B** 6×10^4 **C** 4.4×10^1 **D** 9×10^{-1}
E 1.6×10^{-4} **F** 1.6×10^{-5} **G** 1.4×10^{-4}
H 3.2×10^{-2} **I** 2.8×10^{-9} **2. A** 2.3 **B** 4.7 **C** 3.14
D 0.4 **E** 3.258 **F** 0.0037 **3. A** 10^3 **B** 10^4 **C** 10^{-2}
D 6×10^3 **E** 3×10^8 **F** 2×10^0 **G** 4×10^{-3}
H 2×10^{-4} **I** 2×10^2 **4. A** 2×10^2 **B** 2×10^3
C 3×10^1 **D** 3×10^{-3} **E** 2×10^3 **F** 5×10^{-8}
G 1×10^5 **H** 2 **I** 3×10^2 **5. A** 0.002 **B** 500 **C** 14
D 0.27 **6. A** 8×10^{-9} **B** 8.1×10^{13} **C** 6.4×10^7
D 1.6×10^{-15} **E** 3.2×10^{-9} **F** 10^{30} **7.** 3×10^6

F-56 DISCUSSING THE IDEAS

? 0.573 0.5732 **1.** *A* **2. A** *D* **B** 0.5732 **3. A** 0.3
B 0.33 **C** 0.333 **D** 0.3333 **4. A** 4.74 **B** 3.66
C 7.96 **D** 15.46 **E** 0.81

F-57 USING THE IDEAS

1. A 0.4; 0.6; 1.5 **B** 0.43; 0.62; 1.52 **C** 0.435; 0.622; 1.516 **2. A** 0.83 **B** 0.81 **C** 0.43 **D** 0.73 **E** 0.55
3. A 0.500 **B** 0.778 **C** 0.583 **D** 0.657 **E** 0.146
4. A 0.607 **B** 0.583 **C** 0.582 **D** 0.632 **5. A** *D* **B** *C*
6. A 0.750 **B** 0.688 **C** 0.500 **D** 0.438 **E** 0.375

F-58 DISCUSSING THE IDEAS

2. A 20, 18, 6, 2, $0.6\frac{2}{3}$ **B** 200, 198, 66, 2, $0.66\frac{2}{3}$
C 2000, 1998, 666, 2, $0.666\frac{2}{3}$ **6.** $\frac{4}{11}$, $\frac{7}{11}$ **7. A** $0.1\frac{2}{3}$
B $0.12\frac{1}{2}$ **C** $0.142\frac{6}{7}$

F-59 USING THE IDEAS

1. A $0.06\frac{1}{4}\%$ **B** $0.01\frac{1}{2}$ **C** $0.34\frac{3}{10}$ **2. A** $0.5\frac{2}{5}$, $0.58\frac{1}{3}$,
$0.583\frac{1}{3}$ **B** $0.8\frac{1}{3}$, $0.83\frac{1}{3}$, $0.833\frac{1}{3}$ **C** $1.3\frac{1}{3}$, $1.33\frac{1}{3}$, $1.333\frac{1}{3}$
D $0.9\frac{3}{8}$, $0.93\frac{3}{4}$, $0.937\frac{1}{2}$ **E** $0.2\frac{6}{27}$, $0.21\frac{23}{37}$, $0.216\frac{8}{37}$ **F** $0.1\frac{23}{27}$,
$0.18\frac{14}{27}$, $0.185\frac{5}{27}$ **3. A** $\frac{1}{8}$ **B** $\frac{1}{8}$ **C** $\frac{1}{12}$ **D** $\frac{3}{8}$ **E** $\frac{5}{8}$ **F** $\frac{5}{12}$
4. A $0.2\frac{11}{32}$, $0.23\frac{7}{16}$, $0.234\frac{3}{8}$, $0.2343\frac{3}{4}$, $0.23437\frac{1}{2}$, 0.234375
B $0.3\frac{73}{125}$, $0.35\frac{21}{25}$, $0.358\frac{2}{5}$, 0.3584 **C** $0.1\frac{7}{8}$, $0.18\frac{3}{4}$, $0.187\frac{1}{2}$,
0.1875 **D** $0.6\frac{4}{25}$, $0.61\frac{3}{5}$, 0.616 **E** $0.9\frac{3}{4}$, $0.97\frac{1}{2}$, 0.975 **F** $0.4\frac{7}{8}$,
$0.48\frac{3}{4}$, $0.487\frac{1}{2}$, 0.4875
Think 15

F-60 DISCUSSING THE IDEAS

? A 0.8333 . . . **B** 0.3600 . . . **C** 0.555 . . .
D 0.428571428571 . . . **1.** $0.\overline{185}$ $1.2\overline{5}$

F-61 USING THE IDEAS

1. A $0.\overline{3}$ **B** $0.\overline{7}$ **C** $0.1\overline{5}$ **D** $0.3\overline{24}$ **E** $0.5\overline{6}$ **F** $0.\overline{9}$
G $0.0\overline{24}$ **H** $0.3\overline{715}$ **I** $0.0\overline{426}$ **J** $0.5\overline{07}$ **K** $0.6\overline{2}$ **L** $0.1\overline{}$
2. A $5.2\overline{63}$ **B** $0.7\overline{4}$ **C** $3.2\overline{8}$ **D** $526.1\overline{4}$ **E** $73.4\overline{6}$
F $77.77\overline{13}$ **G** $56.66\overline{15}$ **H** $0.04\overline{321}$ **I** $0.7\overline{6243}$
3. A $0.\overline{3}$ **B** $0.\overline{6}$ **C** $0.\overline{1}$ **D** $0.\overline{3}$ **E** $0.\overline{8}$ **F** $0.\overline{09}$
G $0.\overline{27}$ **H** $0.\overline{81}$ **I** $0.\overline{16}$ **J** $0.\overline{01}$ **K** $0.\overline{02}$ **L** $0.\overline{71}$
M $0.\overline{142857}$ **N** $0.\overline{571428}$ **4. A** 1 **B** 1 **C** 1 **D** 1
E 1 **F** 2 **G** 2 **H** 2 **I** 1 **J** 2 **K** 2 **L** 2 **M** 6
N 6 **5. A** $0.\overline{2}$ **B** $\frac{7}{5}$ **C** $0.\overline{09}$ **D** $\frac{10}{11}$ **E** $0.\overline{43}$ **F** $0.2\overline{85}$
G $0.\overline{24}$ **H** 9 **I** $1.\overline{561}$ **J** 2 **6. A** $\frac{1}{3}$ **B** $\frac{71}{99}$ **C** $\frac{28}{99}$ **D** $\frac{17}{18}$
C $\frac{277}{333}$ **F** $\frac{127}{198}$ **G** $\frac{1741}{2475}$ **H** $\frac{23}{73}$

F-62 COMPUTING WITH REPEATING DECIMALS

1. A 0.666 . . . **B** 0.8787 . . . ; 1.902993 **2. A** 0.444 . . .
B 0.4949 . . . **C** 0.0101 . . . **3. A** 0.666 . . .
B 0.3232 . . . **C** 1.332 **4.** Since the divisor is 17 and if it repeats, it should do so after 17 places. **5. A** 0.9999 . . .
B 0.9999 . . .
Think $\frac{2}{27} = 0.\overline{074}$

F-63 SPECIFIC GRAVITY

1. A 19 300 **B** 35.2 **C** 798 **D** 78.6 **E** 311.65
F 9000 **G** 90 000 **H** 1000 **2. A** 1.3 g **B** 1300 g
3. 499.2 kg **4.** 13.55 **5.** 51.8

F-64 REVIEWING THE IDEAS

1. A 10^4 **B** 10^{-1} **C** 10^{-3} **2.** 8400.56 **3.** $5.10^2 + 6.10^1 + 3 \times 10^0 + 8.10^{-2} + 1.10^{-3}$ **4. A** hundreds
B hundredths **5. A** 25.98 **B** 173.876 **C** 0.1665
6. A 3.72 **B** 4.71 **C** 5.47 **D** 96.921 **7.** 9.405
8. A 8.75 **B** 4080 **C** 9.715 **D** 0.0215 **E** 0.156
F 199.71 **9. A** 900 000 **B** 6300 **C** 0.053
D 0.00000388 **10. A** 2.5×10^3 **B** 4.5×10^4
C 5.9×10^{-2} **D** 3.1×10^{-4} **11.** 3.9×10^5 **12.** 5×10^2
13. A 0.14 **B** 14 **C** 14 **D** 0.00014 **E** 0.14
F 140 000 **14. A** 0.25 **B** 1.25 **C** 560 **D** -9.1
15. A 4.8×10^5 **B** 2.1 **C** 4.8×10^7 **D** 4×10
16. A $0.\overline{313}$ **B** 0.267 **C** 0.889 **17.** $0.63\frac{4}{11}$ **18. A** $0.\overline{75}$
B $0.\overline{304}$ **19. A** $0.8\overline{3}$ **B** $0.\overline{5}$ **C** $0.\overline{285714}$

UNIT F: MODULE 3

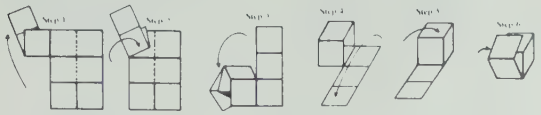
F-68 DISCUSSING THE IDEAS

3. pizzas: $\frac{1}{6}$ pop: $\frac{3}{8}$ doughnuts: $\frac{5}{3}$ chips: $\frac{2}{10}$
4. A $\frac{6}{4}$, $\frac{2}{6}$, $\frac{12}{8}$ **B** $\frac{10}{6}$, $\frac{15}{9}$, $\frac{20}{12}$ **5. A** 3:4 **B** 5:4 **C** $\frac{2}{5}$ **D** 1:3

F-69 USING THE IDEAS

1. A 4 to 8; 5 to 10; 6 to 12 **B** 12 : 8; 15 : 10; 18 : 12
C 12 to 16; 15 to 20; 18 to 24 **D** $\frac{20}{32}$, $\frac{25}{40}$, $\frac{30}{48}$ **E** 8 : 20; 10 : 25; 12 : 30 **F** $\frac{20}{24}$; $\frac{25}{30}$; $\frac{30}{36}$ **2. A** 6 **B** 15 **C** 20 **D** 9 **E** 50
F 21 **G** 15 **H** 10 **I** 60 **J** 18 **3. A** 1 : 2 **B** 5 : 4
C 7 : 4 **D** 14 : 17 **E** 2 : 13 **4. A** $\frac{3}{4}$ **B** $\frac{2}{5}$ **C** $\frac{9}{10}$ **D** $\frac{6}{7}$
E $\frac{9}{20}$ **5. A** 0.64 **B** 0.65 **C** 0.29 **D** 1.29 **E** 0.22
6. 39

Think



F-70 DISCUSSING THE IDEAS

? *EF*:*AE* *GD*:*AD* *HC*:*AC* *IB*:*AB*
1. A 4 cm, 12 cm **B** 4:12 **C** 1:3 **4.** 12

F-71 USING THE IDEAS

1. A $3 \cdot 3 = 1 \cdot 9$; $9 = 9$ **B** $5 \cdot 12 = 3 \cdot 20$; $60 = 60$
C $6 \cdot 10 = 4 \cdot 15$; $60 = 60$ **D** $3 \cdot 28 = 7 \cdot 12$; $84 = 84$
E $5 \cdot 60 = 3 \cdot 100$; $300 = 300$ **F** $90 \cdot 27 = 30 \cdot 81$;
$2430 = 2430$ **2. A** 18 **B** 30 **C** 16 **D** 10 **E** 9 **F** 5
G 10 **H** 1 **I** 7 **J** 7 **K** 24 **L** 21 **3. A** $1.17 **B** 20
C 175 km **D** 750 **4.** 26 km **5. A** $\frac{9}{8}$ **B** $3\frac{1}{9}$ **C** $4\frac{2}{3}$
D $7\frac{4}{3}$ **6. A** 24 **B** 12 **C** 6 **D** 15

F-72 RATIO AND UNITS OF MEASURE

1. A 700 **B** 7000 **C** 19 000 **D** 3 **E** 2 **F** 3
G 3.2 **H** 3 **I** 20 000 **J** 4 **K** 200 **L** 4.5 **M** 900
N 30 **O** 300 **P** 0.6 **Q** 200 **R** 25 200 **2. A** 60, 2
B 7, 200 **C** 60, 50 **3. A** $\frac{1}{30}$ **B** $\frac{5}{12}$ **C** $\frac{1}{200\,000}$ **D** $\frac{2}{5}$ **E** $\frac{17}{50}$

F-73 SPORTS RATIOS

1. A $\frac{729}{64}$ **B** 0.482 **2.** 0.625 **3.** 0.544 **4.** 1.95
5. 2.20 **6. A** 0.841 **B** 0.159 **7.** 0.574 **8.** 0.730
9. 0.318 **10. A** $\frac{11}{26}$ **B** $\frac{31}{78}$ **C** $\frac{7}{39}$

F-74 DISCUSSING THE IDEAS

? $10\frac{4}{5}$ cm **1. A** $x/3 = \frac{36}{10}$ **2. A** yes **B** $x/40 = \frac{3}{5}$
C 24 **4.** yes

F-75 USING THE IDEAS

1. 24 **2.** 12 **3.** 4.25 **4.** 3.3 **5.** 24 **6.** 0.8
7. A 25 mm **B** 2.5 mm **C** 100 mm **D** 30 mm **8.** 3 times

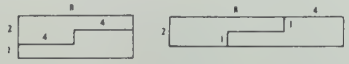
F-76 DISCUSSING THE IDEAS

? 140 metres 190 metres 270 metres 320 metres
1. 1:5000 **3.** 4:1 **4.** 6 mm **5. A** 1:8 000 000 **B** 224 km

F-77 USING THE IDEAS

1. 3.12 m **2.** 4.92 m **3.** about 47 m **4. A** 0.013 mm
B 0.006 mm **5.** 1 : 25 000 000

Think



F-78 RATIOS AND ELECTRICITY

1. 3.6 **2. A** 1.6 **B** 6.25 **C** 5.0 **D** 1.40 **3.** 14.4
4. 0.17 **5.** 31.5

F-79 EXPANSION OF METALS

1. 5.6 **2.** 0.0156 **3.** 0.66 **4.** 0.021 **5.** 20.75

F-80 DISCUSSING THE IDEAS

? A $\frac{2}{100}$ **B** $\frac{74}{100}$ **C** $\frac{40}{100}$ **D** $33\frac{1}{3}/100$ **E** $\frac{35}{100}$ **F** $1\,37\frac{1}{2}/100$

F-81 USING THE IDEAS

1. A 19% **B** 26% **C** 4% **D** 150% **E** 0.5%
F 100% **G** 65% **H** 24% **2. A** 5% **B** 4% **C** $2\frac{1}{2}$%
D $3\frac{1}{8}$% **E** $3\frac{1}{2}$% **F** $2\frac{5}{6}$% **G** 2.3% **H** 37.1% **I** $\frac{1}{2}$%
J 0.75% **3. A** 40, $13\frac{1}{3}$ **B** 200, $66\frac{2}{3}$ **C** 500, $41\frac{2}{3}$
D 100, $12\frac{1}{2}$ **4. A** 58% **B** 72% **C** 83% **D** 75%
E 69% **F** 6% **G** 154% **H** 88% **I** 57% **J** 96%
5. A 58.3% **B** 7.8% **C** 24.1% **D** 362% **E** 23.9%
F 0.7% **6.** 5% **7. A** $\frac{30}{41}$ is 73% not 75% **B** 4

Think 23421314 or 41312432

F-82 DISCUSSING THE IDEAS

1. A .05 **B** 0.44 **C** 1.25 **D** 0.164 **E** 0.01 **2. A** $\frac{1}{5}$
B $\frac{1}{25}$ **C** $\frac{1}{8}$ **D** $\frac{1}{1000}$ **E** 10 **3. A** $\frac{1}{10}$ **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** $\frac{3}{10}$
E $\frac{9}{10}$ **F** $\frac{2}{5}$ **G** 1 **H** $\frac{4}{5}$ **I** $\frac{3}{4}$ **J** $\frac{1}{3}$ **4.** 48% **5. A** 22%
B 36% **C** 214% **D** 1%

F-83 USING THE IDEAS

1. 0.735; 0.015 **2.** 0.87 **3.** 32% **4.** 71% **5.** 32.9%
6. 4% **7.** 2% **8.** 53% **9.** 95% **10.** 81% **11.** 0.085
12. 2.2%

F-84 DISCUSSING THE IDEAS

? A 50% **B** 37.5% **C** 117% **D** 27.3% **E** 20%
F 4% **3.** *A*, *B*, *E*, *F*, *G* **4. A** 20% **B** 45%

F-85 USING THE IDEAS

1. 90% **2.** 80% **3.** 40% **4.** 75% **5.** 25% **6.** 75%
7. 10% **8.** 25% **9.** 80% **10.** $66\frac{2}{3}$% **11.** $58\frac{1}{3}$%
12. 14% **13. A** 20% **B** $33\frac{1}{3}$% **C** $6\frac{2}{3}$%

Think 89, 98 & 99

F-86 DISCUSSING THE IDEAS

? dress, watch **2.** $45 **4.** no
5. $72, $18, $54, $3.24, $57.24;
$69.95, $17.49, $52.46, $3.14, $55.60;
$54.00, $13.50, $40.50, $2.43, $42.93

F-87 USING THE IDEAS

1. A $4.50 **B** $13.50 **2. A** $5.50 **B** $5.78
3. A yes **B** no **4.** $112.63 **5.** *B* **6.** Erton's: $12;
Sampson's: $12 **7.** $129.60 **8. A** $126 **B** $3.60 less

F-88 DISCUSSING THE IDEAS

1. A $\frac{1}{3}$ **B** $\frac{1}{3}$% **2. A** $4.20 **B** $175 **C** $10 **3.** 250
4. A 3.0 **B** 1.2 **C** 10.0 **D** 1.05

F-89 USING THE IDEAS

1. A 0.005 **B** 2.74 **C** 0.003 **D** 8 **E** 0.007
F 0.0025 **G** 0.0001 **H** 2.1 **I** 3.5 **J** 0.002 **K** 0.0005
L 0.00125 **2.** 0.007 m **3.** 0.6 cm **4.** 1.6 **5.** $12,000
6. 95 400 **7.** $1308 **8.** 0.0001%

F-91 USING THE IDEAS

1. A 6, $\frac{6}{84}$, 7% **B** 8, $\frac{8}{78}$, 10.3% **C** 9, $\frac{9}{80}$, 11.3% **D** 5, $\frac{5}{70}$,
7% **2. A** 6 **B** 4.3% **3. A** 32 **B** 22.5% **4.** 6.6%
5. 58.3% **6.** 16%

F-92 REVIEWING THE IDEAS

1. 4 : 10, 6 : 15, 8 : 20 **2. A** 1 : 4 **B** 3 to 8 **C** 2 to 3
3. 16, 9 **4.** 3 : 4 = 9 : 12 **5.** product of the extremes
6. A 8 **B** 5 **C** 16 **D** 25 **E** 8 **F** 6 **7.** 60 km
8. 0.09 mm **9.** 4 cm **10.** *C* **11.** 5 **12. A** 11%
B 8% **C** 12% **D** 4% **13. A** 4% **B** 8%
C 20% **D** $\frac{1}{2}$% **14. A** 60% **B** 17% **C** $66\frac{2}{3}$%
D 125% **E** 0.2% **F** 2.25% **15.** 85% **16.** $44
17. $1.00 **18.** 25% **19.** $960 **20.** 97%

F-94 CUMULATIVE REVIEW

1. 37.5% or $\frac{3}{8}$ **2.** $\frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \frac{25}{30}, \frac{30}{36}$ **3. A** $\frac{2}{3}$ **B** $-\frac{5}{4}$ **4. A** $\frac{7}{6}$
B $\frac{1}{2}$ **C** $\frac{22}{69}$ **5. A** 2 **B** 9 **C** 81 **D** 64 **E. A** < **B** =
C < **D** > **7. A** $14\frac{11}{15}$ **B** $1\frac{1}{15}$ **C** $5\frac{25}{28}$ **D** $\frac{55}{72}$
8. A commutative + **B** commutative + **C** one
D distributive **9. A** $\frac{7}{24}$ **B** $2\frac{2}{27}$ **C** 1 **D** $\frac{1}{5}$ **10. A** $1\frac{1}{2}$
B $\frac{1}{4}$ **C** −9 **D** no **11. A** $\frac{1}{12}$ **B** $-\frac{5}{8}$ **C** $1\frac{2}{3}$ **D** $-\frac{37}{40}$
12. A $-\frac{1}{4}$ **B** $\frac{3}{10}$ **C** $\frac{45}{49}$ **D** $-\frac{4}{11}$ **13.** $3.32 **14.** $5200
15. A 0.47 **B** 0.32 **C** 0.4 **D** 0.375 **E** $0.16\frac{2}{3}$
F 0.2857142 **16. A** $\frac{2}{25}$ **B** $\frac{3}{10}$ **C** $\frac{33}{100}$ **D** $2\frac{2}{5}$ **E** $16\frac{11}{100}$
F $8\frac{7}{20}$ **17. A** 77.906 **B** 43.821 **18. A** 7.3×10^{-3}
B 5.8×10^{4} **C** 8.2×10^{-5} **D** 9.8×10^{10} **19. A** 1.632
B 2.8038 **C.** 6.4 **D** 3.9

F-95

20. A 71 **B** 8.2 **C** 2160 **D** 0.217 **E** 6.39
F 0.0554 **21. A** 86.8 **B** 5.53 **C** 20.978 **D** 0.0216
22. A 8.14 **B** 62.8 **C** 9600 **D** 0.0283 **23. A** 19 230
B 2610.41 **C** 27 **D** 0.0719 **24. A** 28 **B** 1.8
C 1200 **D** 27 000 **25. A** 4 **B** 6 **C** 500 **26.** 3:2
27. A 16 **B** 18 **C** 7 **D** 64 **E** 8 **F** 9 **28.** 3 to 9
29. A 75% **B** 70% **C** 33% **D** 53% **E** 110%
F 16% **30.** 84% **31.** 2% **32.** $9.80 **33. A** $4.56
B $80.51 **34.** 7.1% **35.** $500 **36.** 0.06

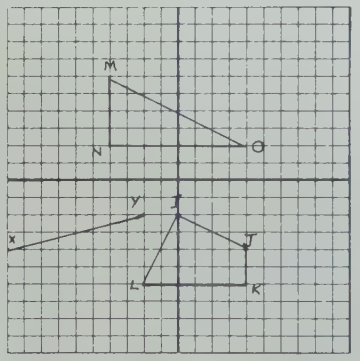
UNIT G: MODULE 1

G-2 DISCUSSING THE IDEAS

1. A B^1, C^1, D^1

G-3 USING THE IDEAS

1.



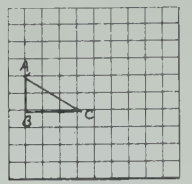
2. A triangle **B** segment **C** quadrilateral
3. A *DEF* **B** *D*, *E*, *F* **4. A** *xyz* **B** *x*, *y*, *z*
5. Same except for its opposite direction.

G-4 DISCUSSING THE IDEAS

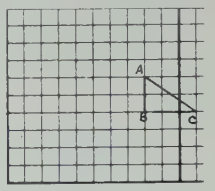
1. right 2, down 4 **2. B** right 6, down 1
3. A right 1, up 4 **B** right 4, down 2 **C** right 5, up 2

G-5 USING THE IDEAS

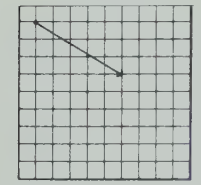
1. A



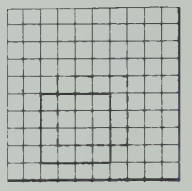
B



C



2.



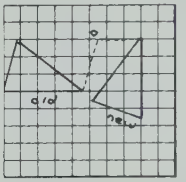
4. yes **5. A** right 1, down 1 **B** left 4, up 5 **C** left 2, down 7 **D** right 4, up 4 **E** down 3 **F** right 2 **6. C** no

G-6 DISCUSSING THE IDEAS

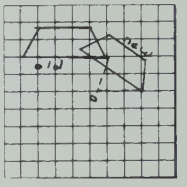
? *GHI* **2. A** *J* **B** *K* and *B*, *L* and *C* **3.** B^1

G-7 USING THE IDEAS

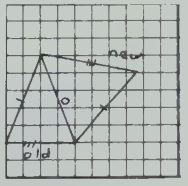
1. A yes **B** no **C** yes **D** yes **E** yes **F** yes
2. A *F*, *G*, *H* **B** 45° clockwise
3. A



B



C



4. *D*, *E*, *A*, *B*

G-8 DISCUSSING THE IDEAS

3. A *XZY* **B** *X*, *Z*, *Y*

G-9 USING THE IDEAS

1. *A-H*, *S-Q*, *T-L*, *M-E* **4. C** no **D** yes **5. A** 4 **B** 2
6. A 3 **B** 2 **C** 0 **D** uncountable

G-12 DISCUSSING THE IDEAS

1. A translation **B** no **2. A** $\angle JKL \cong \angle RST$; $\angle KLJ \cong \angle STR$; $\overline{LK} \cong \overline{TS}$; $\overline{LJ} \cong \overline{TR}$

G-13 USING THE IDEAS

1. A $\overline{DF}$ **B** $\overline{FE}$ **C** $\overline{DE}$ **D** $\angle D$ **E** $\angle F$ **F** $\angle E$
2. A $\overline{HI} \cong \overline{MN}$; $\overline{IJ} \cong \overline{NO}$; $\overline{JK} \cong \overline{OL}$; $\overline{KH} \cong \overline{LM}$ **B** $\angle I \cong \angle N$; $\angle J \cong \angle O$; $\angle K \cong \angle L$; $\angle H \cong \angle M$ **3. A** I' **B** $\angle A'AC$ **C** yes **D** $\angle A''A'B' \cong \angle A'AB$ **4. A** *RSTU* & *EFGD* **B** *RSTU* & *MLKJ* **C** *RSTU* & *YXWZ*
5. A $\overline{RS} \cong \overline{EF}$, $\overline{ST} \cong \overline{FG}$; $\overline{TU} \cong \overline{GD}$, $\overline{UR} \cong \overline{DE}$; $\angle R \cong \angle E$, $\angle S \cong \angle F$; $\angle T \cong \angle G$, $\angle U \cong \angle D$; $RSTU \cong EFGD$ **B** $\overline{RS} \cong \overline{ML}$, $\overline{ST} \cong \overline{LK}$; $\overline{TU} \cong \overline{KJ}$, $\overline{UR} \cong \overline{JM}$; $\angle R \cong \angle M$, $\angle S \cong \angle L$; $\angle T \cong \angle K$, $\angle U \cong \angle J$; $RSTU \cong MLKJ$ **C** $\overline{RS} \cong \overline{YX}$, $\overline{ST} \cong \overline{XW}$; $\overline{TU} \cong \overline{WZ}$, $\overline{UR} \cong \overline{ZY}$; $\angle R \cong \angle Y$, $\angle S \cong \angle X$; $\angle T \cong \angle W$, $\angle U \cong \angle Z$; $RSTU \cong YXWZ$

G-14 DISCUSSING THE IDEAS

? $\triangle JBF$ and $\triangle NMF$; $\triangle CBH$ and $\triangle GFA$
1. $\triangle JBF \cong \triangle NMF$; $\overline{JB} \cong \overline{NH}$, $\overline{BF} \cong \overline{MF}$, $\overline{JF} \cong \overline{NF}$, $\angle J \cong \angle N$, $\angle B \cong \angle M$, $\angle F \cong \angle F$ $\triangle CBH \cong \triangle GFA$; $\overline{CB} \cong \overline{GF}$, $\overline{BH} \cong \overline{FA}$, $\overline{CH} \cong \overline{GA}$, $\angle C \cong \angle G$, $\angle B \cong \angle F$, $\angle H \cong \angle A$ **2. A.** $\overline{KL} \cong \overline{YZ}$, $\overline{JL} \cong \overline{XZ}$ **B** $\angle K \cong \angle Y$, $\angle L \cong \angle Z$ **3. A** $\overline{RT} \cong \overline{UV}$; $\overline{RS} \cong \overline{UW}$; $\overline{ST} \cong \overline{WV}$; **B** $\angle R \cong \angle U$; $\angle S \cong \angle W$; $\angle T \cong \angle V$ **C** $\triangle RST \cong \triangle UVW$

G-15 USING THE IDEAS

1. A $\overline{D'E'}$ **B** $\overline{F'E'}$ **C** $\overline{D'F'}$ **D** $\angle D'$ **E** $\angle E'$ **F** $\angle F'$
2. A $\overline{MK}$ **B** $\overline{KL}$ **C** $\overline{ML}$ **D** $\angle M$ **E** $\angle K$ **F** $\angle L$
G $\triangle MKL$ **3.** $\overline{RS} \cong \overline{XW}$; $\overline{RT} \cong \overline{XY}$; $\overline{ST} \cong \overline{WY}$; $\angle R \cong \angle X$; $\angle S \cong \angle W$; $\angle T \cong \angle Y$ **4. A** $\triangle ABD \cong \triangle CBD$ **B** $\triangle EFH \cong \triangle GFH$ **C** $\triangle JKL \cong \triangle LMJ$ **D** $\triangle WXY \cong \triangle VZY$ **E** $\triangle POS \cong \triangle QOR$ **F** $\triangle DGH \cong \triangle HED$
Think 27

G-16 DISCUSSING THE IDEAS

2. A $\angle R \cong \angle V$; $\angle T \cong \angle W$; $\angle S \cong \angle U$ **B** $\angle R \cong \angle K$; $\angle P \cong \angle J$; $\angle Q \cong \angle L$ **C** $\angle A \cong \angle C$; $\angle BDA \cong \angle BDC$; $\angle ABD \cong \angle CBD$ **D** $\angle X \cong \angle W$; $\angle Y \cong \angle Z$; $\angle XRY \cong \angle WRZ$

G-17 USING THE IDEAS

1. A *DEF*, *IHG* **B** *I* **C** *H* **D** *G* **2. A** *UVW*, *YXZ* **B** $\angle U \cong \angle Y$; $\angle V \cong \angle X$; $\angle W \cong \angle Z$ **3. A** $\overline{SQ}$; $\overline{SQ} \cong \overline{SQ}$, $\overline{SP} \cong \overline{SR}$; $\overline{PQ} \cong \overline{RQ}$; $\triangle SPQ \cong \triangle SRQ$ **B** $\overline{MQ}$; $\overline{MQ} \cong \overline{MQ}$, $\overline{MP} \cong \overline{MN}$; $\overline{PQ} \cong \overline{NQ}$; $\triangle PMQ \cong \triangle NMQ$ **C** $\overline{SU}$; $\overline{SU} \cong \overline{SU}$, $\overline{SR} \cong \overline{ST}$; $\overline{RU} \cong \overline{TU}$; $\triangle SRU \cong \triangle STU$ **D** $\overline{DB}$; $\overline{DB} \cong \overline{DB}$, $\overline{AD} \cong \overline{CD}$; $\overline{AB} \cong \overline{CD}$, $\overline{AB} \cong \overline{BC}$; $\overline{AD} \cong \overline{BC}$; $\triangle ADB \cong \triangle CDB$ **4.** $\overline{AD} \cong \overline{AD}$; $\overline{AB} \cong \overline{DC}$; $\overline{BD} \cong \overline{CA}$; $\triangle ABD \cong \triangle DCA$ **5. A** yes, *SSS* **B** yes

G-18 DISCUSSING THE IDEAS

? yes **1. A** yes **B** yes **2.** $\overline{DE}$ and $\overline{EF}$ **3. A** *SAS* **B** *SSS* **C** *SAS* **D** *SAS*

G-19 USING THE IDEAS

1. A $\triangle ABC \cong \triangle FDE$ **B** *SAS* **C** $\overline{FD}$, $\angle F$, $\angle D$
2. A $\overline{CD}$ **B** $\overline{AC}$ **C** $\angle 2$ **D** *CAB*; *SAS* **3. A** *SAS* **B** *SSS* **C** *SSS* **D** *SAS*
Think
$\triangle ACD \cong \triangle AED$; $\triangle ABG \cong \triangle AFG$; $\triangle BGC \cong \triangle FGE$; $\triangle GCD \cong \triangle GED$; $\triangle BCE \cong \triangle FEC$; $\triangle ACG \cong \triangle AEG$; $\triangle ACF \cong \triangle AEB$

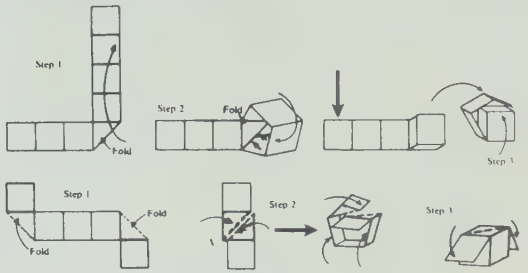
G-20 DISCUSSING THE IDEAS

2. A *WU*, *XY* **B** *XZ*, *YZ* **C** $\angle Z$ **3. A** *ASA* **B** *SAS* **C** *ASA*

G-21 USING THE IDEAS

1. A *SUT*, *ASA* **B** $\angle S$ **C** $\overline{TS}$, $\overline{SU}$ **2. A** *IHJ* **B** $\overline{ZX}$, $\overline{YX}$, $\angle I$ **3. A** *SSS* **B** *ASA* **C** *SAS* **D** *ASA*
4. A translation **B** reflection **C** rotation **D** translation

Think



G-22 DISCUSSING THE IDEAS

1. yes **2.** intersecting **3.** same **4.** 90^0 **5.** parallel
6. A yes **B** yes **C** $\angle 4 = \angle 8$; $\angle 3 = \angle 7$; $\angle 1 = \angle 5$; $\angle 2 = \angle 6$; $\angle 4 = \angle 2$; $\angle 8 = \angle 6$; $\angle 5 = \angle 7$; $\angle 3 = \angle 5$; $\angle 2 = \angle 8$

G-23 USING THE IDEAS

3. yes **4.** $M \perp T$ **5.** $S \parallel T$ **6.** $A \parallel C$
Think the second car

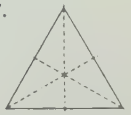
G-25 USING THE IDEAS

1. A 6 **B** *CEGA* **C** yes **2. A** yes **B** rectangle
3. A *AED* and *BFC* **B** *AEFB* and *DEFC*
4. A triangular **B** hexagonal

G-26 REVIEWING THE IDEAS

1. A *Z* **B** $\overline{XY}$ **2.** Answers will vary, however, the final image should be "left 2 and down 2" from the original position. **3. A** clockwise **B** *E* **C** $\triangle EDF$
4.

5. *RST* **6.** 2 **7.**



8. true **9. A** $\triangle MNP \cong \triangle QRS$ (*SAS*)
B $\triangle HIJ \cong \triangle GFE$ (*ASA*) **C** $\triangle ABD \cong \triangle CDB$ (*SSS*)
10. $P \parallel M$ **11. A** $\angle 1$, $\angle 8$, $\angle 5$ **B** $\angle 4$, $\angle 6$, $\angle 7$
12. A *ABC* and *DEF* **B** rectangles

UNIT G: MODULE 2

G-30 DISCUSSING THE IDEAS

1. 9 units **4.** $2\frac{1}{2}$

G-31 USING THE IDEAS

1. A $l(\overline{AB}) = 4$; $l(\overline{CD}) = 2$; $l(\overline{EF}) = 2$ **B** $l(\overline{AB}) = 4$; $l(\overline{CD}) = 2\frac{1}{2}$; $l(\overline{EF}) = 1\frac{1}{2}$ **C** 8 **2. A** $l(\overline{GH}) = 0.8$; $l(\overline{IJ}) = 0.7$; $l(\overline{KL}) = 0.4$ **B** 1.9 **3. A** 26 **B** 3.6 **C** 5.4 **D** 19.54
4. A 4 by 2 **B** 4 by $1\frac{1}{2}$ **C** 4.2 by 1.7 **5. A** 12 **B** 11 **C** 11.8

G-32 DISCUSSING THE IDEAS

1. A 0.01 **B** 100 **C** 10 **D** 1000 **2. A** meters **B** km **C** mm **D** cm **3.** 1.9 cm

G-33 USING THE IDEAS

1. A $l(\overline{AB}) = 9$; $l(\overline{CD}) = 7$; $l(\overline{EF}) = 3$ **B** 9, 7.5, 3.5 **C** 88, 74, 33 **2. A** 5 **B** 3 **C** 1.2 **D** 2000 **E** 1600 **F** 2.27 **G** 470 **H** 1500 **I** 35.8 **J** 0.25 **K** 1 **L** 1 **3. A** 1.52 **B** 5.48 **C** 27.6 **D** 3.66 **E** 3.4 **4. A** 250 **B** 38 **C** 300 **D** 5 **5. A** 10.8 **B** 91 **C** 0.086

G-34 DISCUSSING THE IDEAS

2. A 8 **B** 7 **C** 3.5 **D** 1.75 **3.** $l \times w$ **5. B** $b \times h$ **D** 4.07

G-35 USING THE IDEAS

1. A 345 **B** 6.12 **C** 0.0567 **2.** 20.68 **3.** $247.75 **4. A** 322 **B** 23.04 **C** 8.51 **5. A** 23.4 **B** 40.32 **6. A** 423.36 **B** 1747.36 **C** 1324

Think 8

G-36 DISCUSSING THE IDEAS

3. A $b \cdot h$ **B** $\triangle VWS \cong \triangle STV$; $\triangle VRW \cong \triangle RVU$ **4. A** a **B** b **C** $\frac{1}{2} \times$ (product of two sides)

G-37 USING THE IDEAS

1. A 60 **B** 12 **C** 81 **2. A** 6.9 **B** 49.8 **C** 7.9625 **3. A** 28 **B** 20 **C** 48 **4.** 532.5 **5. A** 96 **B** 1182 **C** 71.04

Think 16, yes

G-38 DISCUSSING THE IDEAS

1. 3 **2. A** 45° **B** 60° **C** 10° **3. A** $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{1}{6}$ **D** $\frac{1}{10}$ **E** $\frac{1}{60}$ **F** $\frac{1}{120}$ **G** $\frac{1}{240}$

G-39 USING THE IDEAS

1. A 153°7′ **B** 26°53′ **C** 74°10′ **2.** 19°41′ **3.** 54°21′45″ **4. A** 30° **B** 180° **C** 150° **D** 150° **E** yes **5.** 132°40′ **6. A** 57°13′ **B** 56°53′ **7. A** 110°55′50″ **B** 147°12′4″ **C** 180° **D** 126°40′13″ **8. A** 59°14′9″ **B** 51°42′20″ **C** 88°48′49″ **D** 35°45′8″

G-40 DISCUSSING THE IDEAS

1. sum = 180° **2. A** yes **B** yes **C** 180° **3.** 180° **4.** 36° **5. A** 180° **B** 180° **C** 360° (2 × 180°) **6.** 540° (3 × 180°)

G-41 USING THE IDEAS

1. A 50° **B** 23° **C** 145° **D** 45° **2. A** 90° **B** 80° **C** 74° **D** 77° **E** 93°29′ **F** 66°45′ **G** 95°4′ **H** 83°55′30″ **3.** 60° **4.** 42° **5.** 144° **6.** 123° **7. A** 540° **B** 108° **8. A** 720° **B** 120° **9. A** 1440° **B** 144° **10.** $M\angle H = 87°30'$; $M\angle G = 43°45'$; $M\angle I = 48°45'$

Think 180°

G-42 DISCUSSING THE IDEAS

1. 25, 4, 13, 17; 13, 9, 4, 13; 13, 9, 10, 19

2. $a^2 + b^2 = c^2$

G-43 USING THE IDEAS

1. A 9, 9, 18 **B** 16, 1, 17 **2.** 45 **3.** 2 **4.** 41 **5.** 25 **6.** 1 **7.** 200 **8.** 6.25 **9.** 71.41 **10.** 223.49 **11.** 9,2; 6, 7

G-44 DISCUSSING THE IDEAS

2. 5 **3.** 25, 5

G-45 USING THE IDEAS

1. A 8.5 **B** 13 **C** 17 **2. A** 4 **B** 12 **C** 12 **3.** 10 cm **4.** 24 **5.** 500 **6. A** 24 **B** 20 **C** 36 **7. A** 20 **B** 0.5 **C** 25 **D** 9 **E** 16 **F** 10

G-46 DISCUSSING THE IDEAS

1. A 9 **B** 5 **C** 45 **D** 8 **E** 360 **3. A** $l \cdot w$ **4. A** 6 **B** 6

G-47 USING THE IDEAS

1. A 15 **B** 7.5 **C** 15 **2. A** 337.5 **B** 279 **C** 27.225 **D** 0.875 **E** 3.5 **F** 93.75 **G** 840 **3. A** 1620 **B** 1.62 **4.** 24 **5.** 81

Think 6

G-48 DISCUSSING THE IDEAS

1. $\frac{1}{1000}$ **2. A** $\frac{1}{2}$ **B** $\frac{1}{100}$ **3.** 1 gram **4. A** $1920c^2$ **B** 1920 ml **C** 1.92 ℓ **D** about 1000 grams

G-49 USING THE IDEAS

1. A 560 **B** 24 **C** 600 **2. A** 8 **B** 1.1529 **C** 4.35132 **3. A** 560 g **B** 24 g **C** 600 g **4. A** 8 **B** 1.1529 **C** 4.35132 **5.** 1000 **6.** C **7. A** 80 cm^3 **B** 32 cm^3

Think mm

G-50 DISCUSSING THE IDEAS

1. A yes **B** yes **C** yes **3.** Yes, the heights and bases appear the same.

G-51 USING THE IDEAS

1. A 84 **B** $38\frac{1}{2}$ **C** 89.46 **D** 95.472 **2. A** 20 **B** 190 **C** 18 **3.** 2 592 276 m^3 **4. A** 49.5 **B** 113 850 **5.** 2500 cm^3 **6. A** 320 **B** 1408

G-52 REVIEWING THE IDEAS

1. A 3.7 **B** 0.37 **C** 0.037 **2. A** 55 **B** 80 **3. A** 75 **B** 53.94 **C** 11.3 **D** 20 **4.** 166.6667 m **5. A** 49.4 **B** 0.8075 **6. A** 72°29′36″ **B** 71°37′42″ **7.** 52°45′ **8.** 23° **9.** 360° **10.** 13 **11.** 5 m **12.** 3000 **13. A** 0.2 kg **B** 1 ℓ **14.** 14.4 **15. A** 480 **B** 480 **C** 160

UNIT G: MODULE 3

G-56 DISCUSSING THE IDEAS

2. A $\angle 1 = \angle 2$ **B** yes **C** $\triangle ABD \cong \triangle BCE$; ASA **D** yes **E** parallelogram **F** yes

G-57 USING THE IDEAS

1. A yes **B** yes **2. A** 7 **B** 6 **C** 9 **3. A** 6 **B** 6 **C** 6 **4. A** 4 **B** 4.5 **C** 5 **D** 8 **E** 7.5 **F** 4.5 **5. A** 3 **B** 6 **C** 3.75 **D** 7.5

G-58 DISCUSSING THE IDEAS

2. B $\angle T \cong \angle W$; $\angle R \cong \angle U$; $\angle S \cong \angle V$ **C** 3:1 3:1 **3. A** 62° **B** 5 to 6

G-59 USING THE IDEAS

1. $l(\overline{DE}) = 6$; $l(\overline{EF}) = 15$; $l(\overline{DF}) = 12$ **2. A** yes **B** yes

C yes **3.** yes **4.** 60° **5.** $m\angle P = 27°$; $m\angle Q = 90°$; $m\angle R = 63°$ **6.** 141 × 96 **7.** 60 **8.** 3 to 4

Think

1. **2.** **3.**

G-60 DISCUSSING THE IDEAS

1. yes **2.** yes **3. A** yes **B** yes **4.** 12

G-61 USING THE IDEAS

1. $m\angle R = 35°$; $m\angle S = 120°$; $m\angle T = 25°$ **2.** 6 **3.** 5 **4.** 7.5 **5.** $1\frac{1}{7}$ **6.** $5\frac{2}{3}$ **7.** 1.32 **8.** $\frac{2}{3}$ **9.** 3.57 **10.** $1\frac{49}{51}$ **11.** 15 m **12.** 192 m **13.** 0.9

G-62 DISCUSSING THE IDEAS

? $\triangle ABC \sim \triangle ONM$; $\triangle GHJ \sim \triangle JLK$; $\triangle STW \sim \triangle DEF$ **3.** $a2/c2$ **5.** $a2/c2$ **6. A** yes **B** QPR

G-63 USING THE IDEAS

1. AMN **2.** w_2/y_2 **3.** $b/f = c/g = d/h$ **4. A** $\frac{1}{2}$ **B** $\frac{1}{2}$ **5. A** $\frac{4}{5}$ **B** $\frac{4}{5}$ **6.** $\frac{1}{2}$ **7.** $l(a) = 20$; $l(b) = 15$; $l(c) = 10$; $l(d) = 5$ **8.** 14.86 m

G-64 DISCUSSING THE IDEAS

1. 12.27 **2. A** 1.28

G-65 USING THE IDEAS

1. A 0.35 **B** 0.34 **C** 0.95 **2. A** 0.5 **B** 60° **3. A** 1 **B** 45° **4. A** 0.95 **B** 0.35 **C** 2.7 **5.** 5 **6.** 4.55 **7.** 3.8 **8. A** 70° **B** 4.1 m

G-66 DISCUSSING THE IDEAS

2. A sin **B** tan **3.** 17.2

G-67 USING THE IDEAS

1. A 8.55 **B** 12.3 **2. A** 18.75 **B** 35.29 **3. A** 0.42 **B** 0.38 **C** 0.92 **4. A** 0.225 **B** 0.220 **5. A** 5 **B** 0.6 **C** 0.8 **D** 0.75

G-69 USING THE TRIGONOMETRIC TABLE

1. A 407 **B** 1.036 **C** 0.574 **D** 0.602 **E** 6.314 **F** 0.454 **G** 0.035 **H** 0.574 **2. A** 75° **B** 27° **C** 70° **D** 4° **3.** 90° **4. A** 4.07 **B** 17.42 **C** 5.32 **D** 9.96 **5. A** 37° **B** 67° **C** 44° **D** 56° **6.** 3° **7.** 2355.6 km

G-70 REVIEWING THE IDEAS

1. 8 **2.** 7 **3. A** $\angle J \cong \angle P$; $\angle K \cong \angle Q$; $\angle L \cong \angle R$ **B** $\overline{JK}$ and $\overline{PQ}$; $\overline{KL}$ and $\overline{QR}$; $\overline{JL}$ and $\overline{PR}$ **4.** $4\frac{3}{8}$ **5.** $\angle S$ **6.** $\frac{9}{7}$ **7.** A and E, C and D, B and F **8. A** sin **B** cos **C** tan **9.** 37° **10. A** 0.454 **B** 0.788 **C** 0.344 **D** 0.326 **E** 1.111 **F** 0.993 **11. A** 70° **B** 9° **C** 30° **D** 60° **12.** $x = 23.8$; $y = 49.1$

UNIT G: MODULE 4

G-75 USING THE IDEAS

1. A 1 cm **B** chord **C** $\overset{\frown}{ACD}$ **2. A** interior **B** on **C** exterior **3.** 10, 15, 21, 28, 36, 45 **4. A** OB **B** no **C** yes

G-76 DISCUSSING THE IDEAS

1. B yes **2. A** z **3.** cuts along a diameter **4. B** circle

G-77 USING THE IDEAS

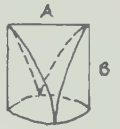
1. A circle **B** radius **C** Anywhere on the sphere **2.** no **3.** interior to sphere **4. A** hemisphere **B** spherical cap **5. A** $\overline{OB}$ **B** no **C** yes **6. A** 90° **B** sum = 180°

G-79 USING THE IDEAS

1. A 28 cm **B** 22 cm **2. A** A 6 cm; B 6 cm; C 6 cm; D 6 cm **B** A **C** D **3. A** D **B** F **C** E

4. A **B**

Think To see the circular shape, view from A. To see the triangular shape, view from B.



G-80 DISCUSSING THE IDEAS

2. A triangle **B** octagon **C** yes **3. A** decrease **B** increase

Think

G-81 USING THE IDEAS

1. to **4.** Constructions left to the student. **5.** 3.14

Think

G-82 DISCUSSING THE IDEAS

2. A 3.14 **B** 3.142 **3.** 31.4 cm **4.** 25.12 cm

G-83 USING THE IDEAS

1. A 25.12 **B** 47.1 **C** 19.468 **D** 376.8 **E** 2.23 **F** 4.08 **G** 3.14 dm **H** 7.54 **2.** 26.5 **3. A** 31.4 **B** 42.7 **C** 232.36 **D** 5.02 **4.** 40 000 **5. A** 37.68 **B** 12 **6. A** 490 **B** 245 **7.** 3.140868

G-84 DISCUSSING THE IDEAS

1. yes **4. A** 50.24

G-85 USING THE IDEAS

1. A 153.86 **B** 314 **C** 4.52 **D** 0.015 **2.** 5 **3. A** 16, 50.24, 200.96 **B** 2.5, 15.7, 19.625 **C** 2, 4, 12.56 **D** 10, 62.8, 314 **E** 2.25, 4.5, 15.90 **F** 0.24, 0.75, 0.045 **4. A** 1017.36 **B** 278.64 **5.** 169.18

G-86 CIRCUMFERENCE AND AREA PROBLEMS

1. A 11.304 **B** 5.652 **C** 5.652 **D** 10.1736 **E** 5.0868 **2.** 571.48 g **3.** 9.8 **4.** 400 **5.** 9555.3 **6.** 14.6 **7. A** 3.5325, 6.28, 9.81

G-88 DISCUSSING THE IDEAS

1. A less **2. A** less **3. D** 5 **4.** B

? 15 black, 28 orange **2.** no **3. A** 350 in 1000 **B** 0.35 **4. A** 65 **B** 35 **5. A** 1 in 4 **B** $\frac{1}{4}$ **C** 50

G-89 USING THE IDEAS

1. A 113.04, 1130.4, 376.8 **B** 452.16, 2712.96, 452.16 **C** 1256, 75 360, 7536 **D** 0.785, 0.196, 0.79 **E** 2.96, 27.5, 148.7 **2.** 5.6 **3. A** 10, 113.04, 37.68, 301.44, 188.4 **B** 2, 4.52, 7.54, 2.41, 7.54 **C** 12, 78.5, 31.4, 314, 204.1 **4.** 1.96 m^3 **5. A** 616 **B** same

G-91 USING THE IDEAS

1. A 113.04, 113.04 **B** 904.32, 452.16 **C** 6202.7, 1632.3 **D** 2143.6, 803.8 **E** 4.19, 12.56 **2. A** 4082 g **3. A** 514 457 600 **B** 363 207 060 **C** 1 097 509 500 000 **4. A** 37 503 959 **B** 21 602 280 000 km^3 **5.** 4.19 **6. A** 7234.56 cm^3 **B** 6589.44 cm^3 **C** 1808.6 cm^2 **D** 1647.36 cm^2 **7.** $(\frac{4}{3}\cdot\pi\cdot r^3/\pi\cdot r^2\cdot h) = (\frac{4}{3}/2) = \frac{2}{3}$ **8.** $8x$, $4x$ **9.** $27x$, $9x$

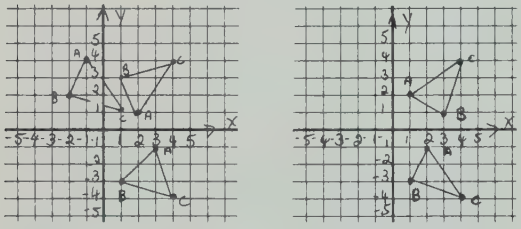
G-92 REVIEWING THE IDEAS

1. A AB **B** OC **C** O **D** $\overset{\frown}{BCD}$, or $\overset{\frown}{ABC}$ **2.** great circle **3.** B **4.** 3.14 **5.** $C = \pi d$ or $2\pi r$ **6.** 77.24 **7.** 2.8 **8.** πr^2 **9.** 117.6 **10.** 125.6 **11.** 784 **12.** 4.46 **13. A** 6 cm **B** 28.26 **C** 4.5 **14.** 5.19 **15.** 1.73 **16.** 305.3 **17. A** 38.8 **B** 55.4

G-94-95 CUMULATIVE REVIEW

1. $A'(-1, 4)$, $B'(-2, 2)$, $C'(2, 1)$ **2.** $A''(2, 1)$, $B''(1, 3)$, $C''(5, 4)$

3.



4. A $\angle Y$ **B** $\overline{XZ}$ **5.** C and D (ASA); B and F (SSS); A and E (SAS) **6.** $\angle 3$, $\angle 5$, $\angle 7$ **7. A** $A = 9.87$; $P = 13.6$ **B** $A = 24$; $P = 24$ **C** $A = 10.6$; $P = 15$ **D** $A = 30.59$; $P = 28.8$ **8.** $m\angle 1 = 114°$; $m\angle 3 = 24°$; $m\angle 4 = 66°$; $m\angle 5 = 90°$; $m\angle 6 = 66°$ **9. A** 110° **B** 70° **C** 50° **10. A** 17 **B** 16 **C** 10 **D** 0.5 **11. A** 1134 **B** 105 **12.** 1380 **13.** 4 **14.** $g = 6$; $h = 5.1$ **15. A** $\frac{5}{13}$ **B** $\frac{12}{13}$ **C** $\frac{5}{12}$ **16.** 16.97 **17.** 3.14 to 1 **18.** 31.4 **19.** 113.04 **20. A** 785 **B** 314 **C** 471 **21. A** 96 **B** 75.36 **22. A** 33.5 **B** 50.24

UNIT H: MODULE 1

H-3 USING THE IDEAS

1. A 50 black; 30 orange; 20 white **B** 500, 0.5; 300, 0.3 200, 0.2 **2. A** 50 **B** 0.5, 0.5 **3. A** 6 **B** $\frac{1}{6}$ **4. A** 3 **B** $\frac{1}{3}$ **C** 20 **D** 100

Think

17	113	47
89	59	29
71	5	101

H-4 DISCUSSING THE IDEAS

1. yes **2. A** $A + F$, $C + E$, $B + D$ **B** no **3. A** $\frac{3}{100}$, $\frac{10}{100}$, $\frac{5}{100}$, $\frac{32}{100}$ **B** no **4. A** yes **B** yes

H-5 USING THE IDEAS

1. A $\frac{1}{2}$ **B** $\frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{6}$ **C** $\frac{1}{4}, \frac{1}{2}$ **D** $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ **2.** 1 **3. A** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ **B** yes **C** $\frac{1}{12}$ **4. A** 25 **B** A: $\frac{18}{25}$; B: $\frac{6}{25}$; C: $\frac{1}{25}$ **C** A: 72; B: 24; C: 4

H-6 DISCUSSING THE IDEAS

? 60 **1.** $\frac{3}{5}$ **2. A** $\{2, 4\}$ **B** $\frac{2}{5}$ **3.** yes; $\frac{2}{5}$ **4.** yes **5.** 0

H-7 USING THE IDEAS

1. A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\{2, 3\}$, $\frac{1}{2}$ **D** $\{1, 2, 3, 4\}$, 1 **E** $\varnothing$, 0
2. A $\frac{1}{2}$ **B** $\frac{1}{3}$ **C** $\frac{1}{6}$ **D** 1 **E** $\frac{2}{3}$ **3. A** $\frac{1}{3}$ **B** $\frac{2}{3}$ **C** $\frac{5}{6}$ **D** $\frac{1}{3}$
E $\frac{1}{2}$ **F** $\frac{2}{3}$ **4. B** $P(\{2, 4, 6, 8\}) = \frac{4}{8}$ **C** $P(\{4, 8\}) = \frac{2}{8}$
D $P(\{1, 2, 4\}) = \frac{3}{8}$ **E** $P(\{1, 4, 9\}) = \frac{3}{8}$
F $P(\{2, 3, 5, 7\}) = \frac{4}{8}$ **G** $P(\{1, 2, 3, 6\}) = \frac{4}{8}$
H $P(\{8, 9\}) = \frac{2}{8}$ **I** $P(\{7, 8, 9\}) = \frac{3}{8}$
J $P(\{1, 2, 3, 4, 5, 6\}) = \frac{6}{8}$ **K** $P(\{1, 2, 3, 4, 5, 6, 7\}) = \frac{7}{8}$
L $P(\{1, 2, 3, 4, 5, 6, 8\}) = \frac{7}{8}$ **M** $P(\{2, 4, 6\}) = \frac{3}{8}$
N $P(\{1, 2, 4\}) = \frac{3}{8}$

H-8 DISCUSSING THE IDEAS

? 250, 500, 250 **2. A** $P(H, T) = \frac{1}{4}$; $P(T, H) = \frac{1}{4}$; $P(H, H) = \frac{1}{4}$; $P(T, T) = \frac{1}{4}$ **C** $\frac{1}{2}$ **3.** 12

H-9 USING THE IDEAS

1. A $\{(H, H), (H, T), (T, H), (T, T)\}$ **B** $\frac{1}{4}$ **2. A** $\{(H, T)\}$
B $\{(H, H), (H, T)\}$ **C** $\{(H, T), (T, T)\}$
D $\{(H, H), (H, T), (T, H)\}$ **E** $\{(H, T), (T, H)\}$
F $\{(H, H), (T, T)\}$ **3. A** $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$ **E** $\frac{1}{2}$ **F** $\frac{1}{2}$
4. A $(D, 3)$ **B** $\{(A, 1), (A, 2), (A, 3), (B, 1), (B, 2), (B, 3), (C, 1), (C, 2), (C, 3), (D, 1), (D, 2), (D, 3)\}$ − 12 **C** $\frac{1}{12}$
5. A $\frac{1}{36}$ **B** $\frac{1}{9}$ **C** $\frac{1}{12}$ **D** $\frac{5}{6}$ **E** $\frac{5}{36}$ **F** $\frac{1}{6}$ **G** $\frac{1}{36}$ **H** $\frac{15}{36}$

H-10 DISCUSSING THE IDEAS

1. A 8 **B** (H, H, H) **2. A** F **B** $\frac{1}{8}$ **3. B** 16 **C** $\frac{1}{16}$
4. A 32 **B** $\frac{1}{32}$

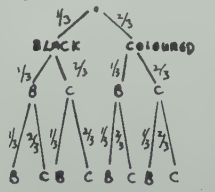
H-11 USING THE IDEAS

1. A $\frac{1}{4}$ **B** $\frac{1}{8}$ **C** $\frac{1}{8}$ **D** $\frac{1}{8}$ **E** $\frac{1}{16}$ **F** $\frac{1}{16}$ **G** $\frac{1}{8}$ **H** $\frac{1}{16}$ **I** $\frac{1}{16}$

2.

3. A $\frac{1}{4}$ **B** $\frac{1}{8}$ **C** $\frac{1}{8}$ **D** $\frac{1}{8}$ **E** $\frac{3}{8}$ **F** $\frac{1}{8}$

4.



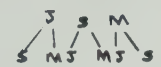
5. A $\frac{2}{9}$ **B** $\frac{1}{9}$ **C** $\frac{4}{9}$ **D** $\frac{4}{27}$ **E** $\frac{4}{27}$ **F** $\frac{8}{27}$ **G** $\frac{1}{27}$ **H** $\frac{4}{27}$

H-12 DISCUSSING THE IDEAS

? 24 **1. A** 6 **2.** 24 (total number)

H-13 USING THE IDEAS

1. A



B 6 **2.** 6 **3.** 24 **4.** 6 **5.** 120 **6.** 720
Think 10:16¢, 31¢, 36¢, 40¢, 56¢, 61¢, 65¢, 76¢, 80¢, 85¢

H-14 DISCUSSING THE IDEAS

? 24 − 5 **2. A** yes **B** 120 **3. A** 720 **B** 5040
4. A 6 **B** 120 **C** 3 628 800 **D** 144 **E** 5

H-15 USING THE IDEAS

1. 5! = 120; 6! = 720; 7! = 5040; 8! = 40 320;
9! = 362 880; 10! = 3 628 800; 11! = 39 916 800;
12! = 479 001 600 **2.** 6 **3.** 362 880 **4.** 5040 **5.** 120
6. 720 **7. A** 10! = 3 628 800 **B** 7! = 5040
C 11! = 39 916 800 **D** 5 **E** 20 **F** 6 **G** 90 **H** 60
I 5 **J** 7! = 5040 **K** 4! = 24 **L** 6! = 720 **M** 72 **N** 5
Think *ETTF*

H-16 DISCUSSING THE IDEAS

? 12 **2. A** 4 **B** 3 **C** $4 \times 3 = 12$ **3.** 120 **4.** 20

H-17 USING THE IDEAS

1. A 5 **B** 4 **C** 20 **2.** 60 **3.** 116 280 **4.** 8 **5. A** 9
B 10 **C** 90 **7.** 900 **8.** 15 600

H-18 DISCUSSING THE IDEAS

? 10 **5. A** 6 **B** 5 **C** 20 **D** 7

H-19 USING THE IDEAS

1. 6 **2.** 4 **3. A** 4 **B** 1 **4.** 21 **5.** 45 **6. A** 3
B 15 **C** 5 **D** 120 **7.** 15 **8.** 22 100
Think 6

H-20 DISCUSSING THE IDEAS

? 33 **2. A** 6 **B** $\frac{1}{3}$ **3.** $\frac{2}{3}$

H-21 USING THE IDEAS

1. A (V, S), (V, B), (S, B) **B** 2 **C** $\frac{2}{3}$ **D** $\frac{1}{3}$ **2. A** 20
B 2 **C** $\frac{1}{10}$ **3. A** 10 **B** $\frac{1}{10}$ **4. A** 45 **B** 10 **C** $\frac{2}{9}$ **D** $\frac{2}{9}$
E $\frac{5}{9}$

H-22 DISCUSSING THE IDEAS

? $\frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{4}{5}$ **1.** 1 **3.** $P(EUF) = P(E) + P(F) - P(E \cap F)$
4. $\frac{4}{5}$

H-23 USING THE IDEAS

1. A $E: \{2, 4, 6\}, P(E) = \frac{1}{2}$; $F: \{1, 2, 3\}, P(F) = \frac{1}{2}$
B $E \cap F = \{2\}$; $P(E \cap F) = \frac{1}{6}$ **C** $E \cup F = \{1, 2, 3, 4, 6\}$;
$P(E \cup F) = \frac{5}{6}$ **D** $\frac{5}{6}$ **2. A** $\frac{3}{10}$ **B** $\frac{1}{2}$ **C** $\frac{1}{10}$ **D** $\frac{7}{10}$ **3. A** $\frac{1}{12}, \frac{1}{9}, \frac{5}{36}, \frac{1}{6}, \frac{5}{36}$ **B** $\frac{23}{36}$ **C** $\frac{1}{36}, \frac{1}{18}, \frac{1}{12}, \frac{1}{9}$ **D** $\frac{19}{36} = \frac{5}{18}$
E $\{2, 3, 4, 5, 6, 7, 8\}$ **F** $\frac{13}{18}$ **G** $\{4, 5\}$ **H** $\frac{7}{36}$ **I** $\frac{13}{18}$

H-24 REVIEWING THE IDEAS

1. 75 **2. A** $\frac{1}{4}$ **B** about 10 white; about 30 red
3. A $\{4, 5, 6\}$ **B** $\{1, 2, 3, 4\}$ **C** $\{2, 4, 6\}$ **D** $\{2, 3, 5\}$
4. A $\frac{1}{2}$ **B** $\frac{2}{3}$ **C** $\frac{1}{2}$ **D** $\frac{1}{2}$

5.

6. A $\frac{1}{4}$ **B** $\frac{1}{6}$ **C** $\frac{1}{18}$ **D** $\frac{1}{9}$ **E** $\frac{1}{12}$ **F** $\frac{1}{36}$ **7.** 24 **8.** 72
9. 6 **B** 20 **C** 5 **10.** 10 **11. A** 24 **B** 6 **C** 36
D 120 **12. A** $E = \{(B, 1), (B, 2), (B, 3)\}$, $P(E) = \frac{1}{4}$
B $F = \{(A, 2), (B, 2), (C, 2), (D, 2)\}$, $P(F) = \frac{1}{3}$ **C** $\frac{1}{12}$ **D** $\frac{1}{2}$

UNIT H: MODULE 2

H-28 DISCUSSING THE IDEAS

4. 25 cm

H-29 USING THE IDEAS

1. *B* **2. A** 56.25 **B** 400 **C** 36 **D** 9 **E** 625 **F** 1
3. A 10 s **B** 25 m **4.** 9 m **5. A** 25 cm **B** 75 **C** 37

H-30 DISCUSSING THE IDEAS

? Eva B **2. B** 59.3 **3.** 33 kg

H-31 USING THE IDEAS

1. A 5 140 000 000 km **B** 1 700 888 800 km **C** Saturn
2. A 412, 442 **B** 137.3, 147.3; Mary **C** Linda 47;
Mary 40 **3. A** 30 **B** 81.4 **4. A** 19.3 **B** 47.4 **5.** 92

H-32 DISCUSSING THE IDEAS

3. A 13 **B** 15 and 16

H-33 USING THE IDEAS

1. 77 **2. A** 40, 59, 65, 65, 65, 67, 68, 72, 76, 80, 100
B 67 **3. A** 12 **B** 12 **4. A** 7.4 **B** 7.96 **C** mean by
0.56 **5.** 4230 **6. A** 23 **B** 31.75
Think yes

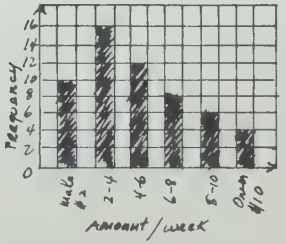
H-34 DISCUSSING THE IDEAS

? 8 **1. A** 8 **B** 7 **C** 17 **2. A** yes 8 **B** no
3. 0 and 3

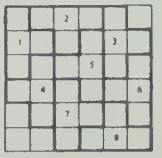
H-35 USING THE IDEAS

1. 13 **2.** 38 **3. A** 2 and 5 **B** 2 and 5 **4. A** $2–4

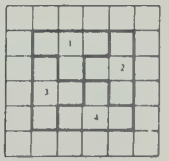
B



Think **1.**



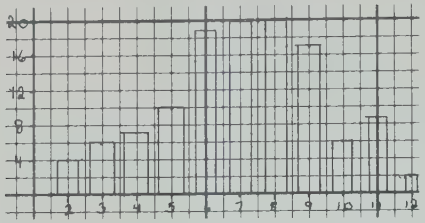
2.



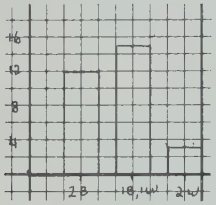
H-37 USING THE IDEAS

1. A 45 **B** 1, 4, 7, 10, 9, 6, 5, 3 **C** 22.2% **D** $6.6\overline{6}$

2.

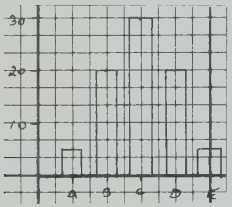


3.



4. A $A: \frac{1}{16}; B: \frac{1}{4}; C: \frac{3}{8}; D: \frac{1}{4}; E: \frac{1}{16}$ **B** $A: 5; B: 20; C: 30; D: 20; E: 5$

C

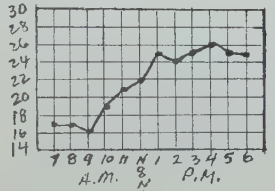


H-38 DISCUSSING THE IDEAS

? about 78 cm **2. A** 3 **B** 5 **3. A** 84

H-39 USING THE IDEAS

1. A $A: 26; B: 22$ **B** $A: 27; B: 24.5$ **C** $A: 25.5; B: 22$ **D** $A: 1.5; B: 2.5$ **2. A** 1600: 0.5 billion; 1700: 0.8 billion; 1800: 1 billion; 1900: 1.5 billion; 1950: 3 billion **B** 4.5 billion **C** 10 billion **3. A** −10°C **B** 11°C **C** 21°C **D** 17 **4.**

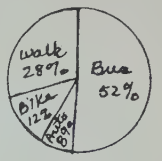


H-40 DISCUSSING THE IDEAS

2. pie 72° malt 84° fries 84° **3.** 96°
5. eating 12.5%, 45° school 25%, 90° other 12.5%, 45° recreation 8.3%, 30° homework 4.2%, 15° sleeping 37.5%, 135°

H-41 USING THE IDEAS

1. A basketball, $\frac{6}{15}$; other, $\frac{2}{15}$; tennis, $\frac{7}{12}$; golf, $\frac{1}{10}$; baseball, $\frac{3}{20}$; football, $\frac{1}{8}$ **B** basketball, 40%; other, 13.3%; tennis, 8.3%; golf, 3.3%; baseball, 15%; football, 20%
2. A $3.00 **B** $2.40 **C** 10%; $1.20 **3.** 437 by bus; 235 walk; 100 ride bicycles; 68 by car. **4.**



Think A–B

H-43 USING THE IDEAS

1. A 13.8% **B** 15.884 **C** 10 801 **2. A** bias **B** unbias **C** bias **D** bias **E** bias **F** unbias
3. A yes **B** undecided votes **D** A: 5 960 000, 4 760 000; B: 5 000 000, 3 800 000; C: 1 440 000, 240 000

H-44 DISCUSSING THE IDEAS

3. 0.851 **4.** $\frac{23}{25}, \frac{45}{49}$

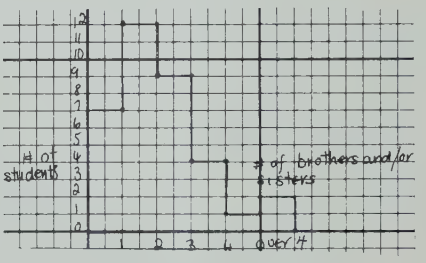
H-45 USING THE IDEAS

1. A 180 **B** 64 **C** 6 **D** 1 **E** no **2.** 0.2882634
3. A 73% **B** 876 **C** 324

H-48 REVIEWING THE IDEAS

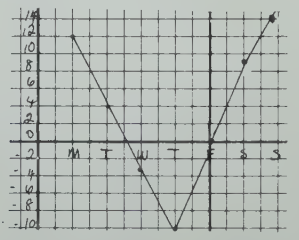
1. $13 **2. A** 18 **B** 87.4 **C** 88.5 **3. A** 1 **B** 1

4.



5. M: 820; T: 827; W: 830; T: 825; F: 837

6.



7. A 15% **B** 72° **8.** under 20: 12 000; 20–39: 7500; 40–60: 6000; over 60: 4500 · **9. A** no **B** no
10. 625 000 **11.** 59

UNIT H: MODULE 3

H-52 DISCUSSING THE IDEAS

? 1.41 **1.** less **2.** more **3.** yes **5. A** yes **B** $-\sqrt{2}$

H-53 USING THE IDEAS

1. A $\sqrt{3}$ **B** $-\sqrt{3}$ **C** $\sqrt{17}$ **D** $-\sqrt{17}$ **E** positive, 6 **F** negative, 6 **G** positive, 7 **H** $-\sqrt{47}$ **I** *n* **2. A** 2 **B** 5 **C** 6 **D** 15 **E** 100 **F** 225 **3. A** $\sqrt{9}$ **B** $\sqrt{7}$ **C** $\sqrt{11}$ **D** $\sqrt{16}$ **E** $\sqrt{17}$ **F** $\sqrt{19}$ **G** $\sqrt{63}$ **H** $\sqrt{64}$ **4.** $7 = \sqrt{49}$; $8 = \sqrt{64}$; $9 = \sqrt{81}$; $10 = \sqrt{100}$

H-54 DISCUSSING THE IDEAS

? no **1.** *C* **2.** no **3. B** $\sqrt{5}$ **C** $\sqrt{5}$, $-\sqrt{5}$

H-55 USING THE IDEAS

1. $10, \sqrt{10}, \sqrt{10}, -\sqrt{10}$ **2.** $17, \sqrt{17}, \sqrt{17}, -\sqrt{17}$ **3.** $26, \sqrt{26}, \sqrt{26}, -\sqrt{26}$ **4.** $3, \sqrt{3}, \sqrt{3}, -\sqrt{3}$ **5.** $6, \sqrt{6}, \sqrt{6}, -\sqrt{6}$

H-56 DISCUSSING THE IDEAS

1. A incorrect **B** correct **2. A** incorrect **B** incorrect **C** correct **3.** 3 **4.** 3.2

H-57 USING THE IDEAS

1. A 3, 4 **B** 5, 6 **C** 12, 13 **2. A** *F* **B** *J* **C** *C* **D** *A* **E** *L* **F** *E* **G** *H* **H** *B* **I** *K* **J** *D* **K** *Q* **L** *G* **M** *M* **N** *P* **O** *O* **P** *N* **Q** *R* **R** *I* **3. A** 2.8; 2.9 **B** 7.2; 7.3 **C** 10.6; 10.7 **4. A** 2.8 **B** 7.2 **C** 10.6 **5.** 1, 2; 1.7, 1.8; 1.73, 1.74; 1.732, 1.733 **6.** 2, 3; 2.2, 2.3; 2.23, 2.24; 2.236, 2.237 **7.** 5, 6; 5.3, 5.4; 5.38, 5.39; 5.385, 5.386 **8.** 9, 10; 9.3, 9.4; 9.32, 9.33; 9.327, 9.328

H-58 DISCUSSING THE IDEAS

? yes **1. A** 4.3 **B** 4.3

H-59 USING THE IDEAS

1. A 26 **B** 79 **C** 3.5 **D** 0.18 **E** 6.7 **F** 5.3 **G** 57 **H** 11.8 **2. A** 4.1 **B** 9.4 **C** 12.8 **D** 4.6 **E** 13.1 **F** 20.9 **G** 8.3 **H** 14.1 **I** 0.4 **J** 11.8 **3. A** 9.27 **B** 24.96 **C** 3.61 **D** 4.36 **E** 5.65 **F** 7.27 **G** 2.81 **H** 54.77 **I** 0.63 **J** 1.02 **4.** *B* **5. A** 2 **B** 10 **C** 4 **D** 5 **E** −1 **6. A** 67; 667; 6667; 66667

Think 1. 3 : 4 **2.** 7 : 6 **3.** engine 8 cm; dining car 12 cm; caboose 6 cm

H-60 DISCUSSING THE IDEAS

? A 196 **B** 4.123 **C** 5.477 **2. A** 2.236 **B** 2.828 **C** 3.606 **D** 4.472 **3. A** 5.477 **B** 9.487 **C** 10.954 **D** 13.784 **4. A** 18 **B** 13 **C** 16 **D** 19 **5.** 4.183

H-61 USING THE IDEAS

1. A 2.449 **B** 7.746 **C** 3.606 **D** 11.402 **E** 4.243 **F** 13.416 **G** 3.317 **H** 10.488 **I** 14 **J** 13 **2. A** 2.828 **B** 4.472 **C** 3.464 **3. B** 11 **C** 150 **D** 90 **E** 15 **F** 15 **G** 3.162 **H** 11 **4. A** 2.121 **B** 1.225 **C** 3.240 **D** 4.183 **E** 2.915 **F** 4.278 **G** 4.405 **H** 4.025 **I** 3.688 **J** 3.847 **5.** 5.39 **6.** 4.47

H-62 DISCUSSING THE IDEAS

? 3.6 **2.** 8.94 **3. B** 12

H-63 USING THE IDEAS

1. A 3.16 **B** 4.24 **C** 5.385 **2.** 9.5 **3.** 10.4 **4.** yes **5.** 1.97 **6.** 6 km

H-65 USING THE IDEAS

1. A $0.\overline{6}$, 1 **B** $0.\overline{2857142}$, 7 **C** $0.6250\overline{0}$, 1 **D** $0.\overline{384615}$, 6 **E** $0.8\overline{3}$, 1 **F** $0.0\overline{37}$, 3 **G** $0.8\overline{1}$, 2 **2. B** $4.\overline{4}$ **C** $3.\overline{83}$ **D** $7.0\overline{7}$ **E** $842.\overline{842}$ **3. A** 4.444 . . . **B** 4 **C** $\frac{4}{9}$ **4. A** 27.2727 . . . **B** 27 **C** $\frac{3}{11}$ **5. A** 74.074074 . . . **B** 74 **C** $\frac{74}{999}$ **6. A** $\frac{8}{9}$ **B** $\frac{37}{99}$ **C** $2\frac{3}{11}$ **D** $\frac{5}{333}$ **E** $\frac{41}{333}$ **F** $\frac{11}{990}$ **G** $\frac{1}{18}$ **H** $4\frac{3}{110}$ **I** $3\frac{2}{9}$ **J** $\frac{1}{999}$ **7. A** $0.\overline{1}$ **B** $0.\overline{2}$ **C** $0.0\overline{1}$ **D** $0.0\overline{2}$ **E** $0.0\overline{01}$ **F** $0.0\overline{02}$ **8. A** $\frac{1}{9}, \frac{2}{9}$ **B** $\frac{1}{9}, \frac{1}{3}$ **C** $\frac{1}{9}, \frac{4}{9}$ **D** $\frac{1}{99}, \frac{2}{99}$ **E** $\frac{1}{99}, \frac{1}{11}$ **F** $\frac{1}{99}, \frac{10}{99}$ **G** $\frac{1}{99}, \frac{1}{9}$ **H** $\frac{1}{999}, \frac{4}{333}$ **I** $\frac{1}{999}, \frac{23}{999}$ **J** $\frac{1}{99}, \frac{68}{99}$ **K** $\frac{1}{999}, \frac{2}{999}$ **L** $\frac{11}{9999}, \frac{1}{111}$ **M** $\frac{1}{9999}, \frac{23}{9999}$ **N** $\frac{1}{9999}, \frac{467}{9999}$

H-66 DISCUSSING THE IDEAS

? A 0.23323323323 . . . **B** 0.05005000500005 . . . **C** 0.123456789101112131 . . . **D** 0.142857142857142857 . . . **E** 0.246810121416182 . . . **F** 6.26262626262 . . . **G** 0.100100200300400 . . . **H** 1.41425314253142531 42 . . . **I** *a, d, f, h* **2.** *b, c, e, g* **4.** non-repeating **5.** irrational **6. A** yes **B** yes

H-67 USING THE IDEAS

1. A rational **B** rational **C** irrational **D** rational **E** irrational **F** rational **G** rational **H** irrational **2. A** 4.414 **B** 2.828 **C** 0.414 **D** 2.828 **E** 0.2828 **F** $0.902\overline{3}$ **G** 0.7072 **H** 0.707 **3. A** 31.4 **B** 12.56 **C** 28.26 **D** 0.32 **E** 0.02 **F** 9.86 **G** 17.58 **H** 19.63 **4.** 1.62

Think A $\frac{355}{113}$ **B** 3.1415928

H-68 DISCUSSING THE IDEAS

1. A yes, rational **B** irrational **2. A** 2 **B** $\sqrt{3}$ **3. A** 6 **B** 20 **C** 15 **D** 56

H-69 USING THE IDEAS

1. A rational **B** irrational **C** irrational **D** rational **2. A** 6 **B** 18 **C** 21 **D** 24 **E** 12 **F** 105 **3. A** $2 \cdot 3 = 6$ **B** $3 \cdot 5 = 15$ **C** $2 \cdot 7 = 14$ **D** $7 \cdot 5 = 35$ **4.** $\sqrt{6}$ **5.** 2.449 **B** 2.449 **6. A** $\sqrt{15}$ **B** $\sqrt{14}$ **C** $\sqrt{35}$ **D** $\sqrt{21}$ **E** $\sqrt{55}$ **F** $\sqrt{143}$ **G** $\sqrt{187}$ **H** $\sqrt{51}$ **I** 13 **7.** 6

H-70 REVIEWING THE IDEAS

1. A $\sqrt{5}$ **B** $-\sqrt{5}$ **C** positive, 13 **D** negative, 13 **2.** $\sqrt{10}$ **3.** −6 and −7 **4. A** 3, 4 **B** 5, 6 **C** 18, 19 **5. A** 3 **B** 5 **C** 18 **D** 4 **E** 16 **F** 13 **6. A** 3.6 **B** 7.28 **7.** 4.3 **8.** 12.77 **9. A** 3 **B** 10 **C** 52 **D** 7 **10.** 8.1 **11.** $0.\overline{857142}$ **12.** $\frac{23}{69}$ **13. B** irrational **14.** 14 **15.** 6.71 **16.** 33 **17.** 4.863 **18.** 16.97

UNIT H: MODULE 4

H-74 DISCUSSING THE IDEAS

3. A *d* **4. A** true **B** true **C** true **D** true **E** true **F** false

H-75 USING THE IDEAS

1. A 3 **B** 2 **C** 2 **D** 0 **E** 3 **F** 3 **G** 0 **H** 1 **I** 2 **J** 1 **2. A** 8 **B** −24 **C** 9 **D** −28 **E** −40 **F** 30

G −44 **H** −60 **3. A** true **B** true **C** false **D** true **E** true **F** false **4. A** 1 **B** 2 **C** 0 **D** 3 **E** 1 **F** 0 **G** 1 **H** 0 **I** 3 **5. A** 0 **B** 1 **C** 2 **D** 3 **6. A** 3 **B** 0 **C** 3 **D** 2 **E** 0 **F** 3 **G** 2 **H** 0

H-76 DISCUSSING THE IDEAS

? 1. 2 **2.** 3 **3.** 0 **4.** 1 **5.** 2 **6.** 0 **7.** 0 **8.** 2 **9.** 1 **10.** 3 **1. A** 1 **B** 1 **C** 2 **D** 2 **2. A** 1, 2 **B** 3 **3.** 3 **4. B** 1 **C** 2

H-77 USING THE IDEAS

1.

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. A 3 **B** 0 **C** 1 **D** 2 **E** 0 **F** 3 **3.** yes **4.** yes **5. A** 0 **B** 3 **C** 2 **D** 1 **6. A** 2 **B** 0 **7. A** 3 **B** 3 **C** 2 **D** 0 **E** 3 **8. A** $1 \oplus_4 2 = 3$ **B** $2 \oplus_4 1 = 3$ **C** $3 \oplus_4 3 = 2$ **D** $2 \oplus_4 2 = 0$ **E** $3 \oplus_4 0 = 3$

H-78 DISCUSSING THE IDEAS

? 1. 2 **2.** 0 **3.** 3 **4.** 0 **5.** 0 **6.** 1 **7.** 0 **8.** 1 **1. A** 2 **B** 3 **C** 2 **3.** 2 **4.** 0

H-79 USING THE IDEAS

1.

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2. A 2 **B** 0 **C** 3 **D** 3 **E** 0 **F** 2 **3.** yes **4.** 1, 3 **5.** 1 **6. A** no **B** 1 and 1; 3 and 3 **7. A** 2 **B** 2 **C** 1 **D** 1 **E** 2 **F** 2

Think The ratio of the areas.

H-80 DISCUSSING THE IDEAS

2. A *L* **B** *I* **C** *U* **D** *I* **E** *U* **F** *U*

3.

☆	*I*	*R*	*U*	*L*
I	*I*	*R*	*U*	*L*
R	*R*	*U*	*L*	*I*
U	*U*	*L*	*I*	*R*
L	*L*	*I*	*R*	*U*

5. A $U, L \rightarrow R$ **B** $U, I \rightarrow U$

H-81 USING THE IDEAS

1 and **2. A** $L \star L = U$ **B** $U \star R = L$ **C** $R \star R = U$ **D** $L \star R = I$ **3. A** *U* **B** *R* **C** *L* **D** *U* **E** *I* **F** *R* **G** *U* **H** *R* **I** *U* **4. A** *U* **B** *U* **C** *U* **D** *U* **E** *R* **F** *R* **G** *L* **H** *L* **5.** associativity **6.** *I* times any element is that element. **7. A** *I* **B** *L* **C** *U* **D** *R* **8.** yes, *I* and *I*, *L* and *R*, *R* and *L*, *U* and *U*

H-82 DISCUSSING THE IDEAS

1. Horizontal and Vertical Flip **2.** no **3. A** *R* **B** *I* **C** *L* **D** *U* **E** *L* **F** *R* **4. A** *L* **B** *V*

H-83 USING THE IDEAS

1. A H ☆ H = I **B** ⃠ ☆ ∅ = U **C** U ☆ ⃠ = ∅ **D** ∅ ☆ U = ∅

3.

☆	I	R	U	L	H	V	⃠	∅
I	I	R	U	L	H	V	⃠	∅
R	R	U	L	I	∅	⃠	H	V
U	U	L	I	R	V	H	∅	⃠
L	L	I	R	U	⃠	∅	V	H
H	H	⃠	V	∅	I	U	R	L
V	V	∅	H	⃠	U	I	L	R
⃠	⃠	V	∅	H	L	R	I	U
∅	∅	H	⃠	V	R	L	U	I

4. I **5.** not commutative in flip section of octic system; R ☆ H = ∅ while H ☆ R = ⃠ **6.** I ☆ I = I, U ☆ U = I, L ☆ R = I, H ☆ H = I, V ☆ V = I, ∅ ☆ ∅ = I **7. A** I **B** I **C** V **D** V **E** R **F** R **8.** yes **9.** U

H-84 DISCUSSING THE IDEAS

1. (−3, −2), (3, −4), (1, 4) **2.** (−3, 4), (−1, −4) **3. A** 5 **B** $\sqrt{13}$ **C** $\sqrt{29}$ **4.** (−2, 1) **5.** 0

H-85 USING THE IDEAS

1. A (−3, 1) **B** (−1, −1) **C** (0, 3) **D** (4, 5) **E** (3, −4) **F** (4, 2) **2. A** $\sqrt{10}$ **B** $\sqrt{2}$ **C** 3 **D** $\sqrt{41}$ **E** 5 **F** $\sqrt{20}$ **4. A** $\sqrt{17}$ **B** $\sqrt{2}$ **C** 4 **D** 5 **E** 2 **F** $\sqrt{29}$ **G** $\sqrt{18}$ **H** 5 **5. A** *b* **B** *d* **C** *e* **D** *g*

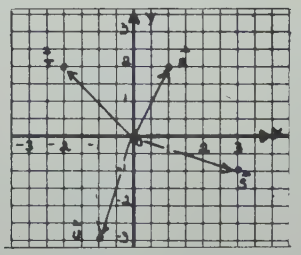
Think 24

H-86 DISCUSSING THE IDEAS

1. (2, 3) **3. A** (5, 2) **B** (−1, −5)

H-87 USING THE IDEAS

1.



2. A (5, 1) **B** (1, 1) **C** (−3, −1) **D** (2, −4) **E** (1, −1) **F** (0, 4) **3.** (−7, 3) **4.** (0, 0) **5. A** (0, 0) **B** (0, 0) **C** yes **6. A** (−6, 5) **B** yes **7. A** (−1, 4) **B** (1, −5)

C (3, 1) **D** (3, 1) **8.** yes **9.** (−8, −11)
Think $65 = 1^2 + 8^2 = 4^2 + 7^2$; $85 = 2^2 + 9^2 = 6^2 + 7^2$; $125 = 2^2 + 11^2 = 5^2 + 10^2$

H-88 DISCUSSING THE IDEAS

1. A no **B** no **C** yes **2. A** $0 \times 1 = 0$ **B** $1 \times 0 = 0$ **C** $1 \times 1 = 1$

H-89 USING THE IDEAS

1.

⊗	0	1
0	0	0
1	0	1

2. A 0 **B** 0 **C** 0 **D** 1 **3. A** closed **B** $1 \otimes 1 = 1$
4. A open **B** $0 \otimes 1 = 0$ **5. A** no **B** open **C** 0
6. A 0 **B** 0 **C** 0 **D** 1 **7.** $0 \otimes 1 = 0$, $1 \otimes 0 = 0$
8. yes

H-90 DISCUSSING THE IDEAS

1. A closed **B** closed **C** open **D** closed
2. B $1 \oplus 0 = 1$ **C** $0 \oplus 0 = 0$ **D** $1 \oplus 1 = 1$
3.

⊕	0	1
0	0	1
1	1	1

H-91 USING THE IDEAS

1. A 1 **B** 0 **C** 1 **D** 1 **E** 0 **F** 0
2. A $(1 \otimes 1) \oplus 0 = 1$ **B** $(0 \otimes 1) \oplus 0 = 0$
C $(0 \otimes 0) \oplus 1 = 1$ **D** $(1 \otimes 1) \oplus 1 = 1$
E $(0 \otimes 1) \oplus 0 = 0$ **F** $(0 \otimes 0) \oplus 0 = 0$
4. A 2 (eg. F and E) **6.** 7 (eg. D, E, J)

H-92 REVIEWING THE IDEAS

1.

$\oplus_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. A 0 **B** 1 **C** 1 **D** 0
3.

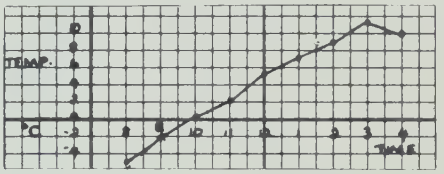
$\otimes_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

4. A *R* **B** *L* **C** *L* **D** *R* **E** *U* **F** *L* **5. A** D **B** D **C** *U* **D** *U* **E** *I* **F** *U* **6. A** (3, 4) **B** 5 **C** (−3, −4)
7. (8, 2) **8.** (5, 2) **9.** (−6, 2) **10.** (4, 4)
11. A parallel **B** series **12.** yes **13. A** 0 **B** 1 **C** 1 **D** 1 **E** 0 **F** 1 **G** 1 **H** 1

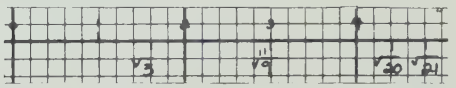
H-94-95 CUMULATIVE REVIEW

1. $\frac{2}{3}$ **2.** $\frac{1}{6}$ **3.** $\frac{3}{8}$ **4. A** $\frac{1}{4}$ **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$
5.

6. 24 **7.** 20 **8.** 21 **9. A** $\frac{2}{3}$ **B** $\frac{1}{3}$ **C** 0 **D** 1
10. A 39 **B** 163 **11.** 88 **12. A** 4 **B** 6 **C** 0
13. A 2H, 1T **B** 3H **C** 24
14.



15. A 40% **B** 72° **16. A** 10 600 **B** 9400
17. $\sqrt{2}$, $-\sqrt{2}$ **18.** 1.4
19.



20. A 3, 4 **B** 6, 7 **C** 9, 10 **D** 24, 25 **21. A** 6.9 **B** 30.5 **C** 75 **22.** $0.\overline{5}$ **23. A** $\frac{8}{25}$ **B** $\frac{711}{1000}$ **C** $\frac{7}{9}$
24. A repeating **B** non-repeating **25. A** 15 **B** 14 **C** 105 **D** 6 **26. A** 3.65 **B** 3.16 **C** 0.82 **D** 1.58
27. A 0 **B** 1 **28. A** 0 **B** 0 **29.** U ☆ L = R **30.** U
31. (9, 6) **32.** (2, −6) **33.** C

SUPPLEMENTARY EXERCISES

S-2 SET 1

1. A 5^2 **B** 8^3 **C** 2^5 **D** 95^4 **E** 10^6 **F** 3^9 **2. A** 10^3 **B** 10^4 **C** 10^5 **D** 10^5 **E** 10^6 **F** 10^7 **G** 10^5 **H** 10^9
3. A 10^2 **B** 10^1 **C** 10^3 **D** 10^5 **E** 10^6 **F** 10^2 **G** 10^6 **H** 1 **4. A** $(3 \cdot 10) + 8$ **B** $(6 \cdot 10^2) + (2 \cdot 10) + 1$ **C** $(8 \cdot 10^2) + 2$ **D** $(1 \cdot 10^3) + (2 \cdot 10^2) + (5 \cdot 10) + 9$ **E** $(2 \cdot 10^3) + (3 \cdot 10) + 5$ **F** $(8 \cdot 10^3) + 1$ **G** $(1 \cdot 10^4) + (4 \cdot 10^3) + (3 \cdot 10) + 5$ **H** $(8 \cdot 10^4) + (1 \cdot 10^3) + (2 \cdot 10^2) + (5 \cdot 10) + 3$ **I** $(6 \cdot 10^5) + (8 \cdot 10^4) + (7 \cdot 10^3) + (5 \cdot 10^2) + (2 \cdot 10) + 4$ **J** $(3 \cdot 10^5) + (5 \cdot 10)$

S-2 SET 2

1. A 70 **B** 90 **C** 130 **D** 640 **E** 1250 **F** 7510 **G** 15 630 **H** 48 110 **I** 200 010 **J** 555 560 **2. A** 400 **B** 600 **C** 1300 **D** 6800 **E** 9100 **F** 17 900

G 24 000 **H** 105 100 **I** 843 700 **J** 503 500 **3. A** 1000
B 3000 **C** 6000 **D** 16 000 **E** 21 000 **F** 135 000
G 458 000 **H** 756 000 **4. A** 5 607 380 **B** 5 607 400
C 5 607 000 **D** 5 610 000 **E** 5 600 000 **F** 6 000 000

S-3 SET 3

1. A 400 **B** 1300 **C** 30 **D** 400 **E** 500 **F** 6000
G 42 000 **2. A** 800 **B** 500 **C** 5400 **D** 30 **E** 18 000
F 9800 **G** 1900 **H** 4 **I** 600 000 **J** 23 000
K 450 000 **L** 6 000 000

S-3 SET 4

1. A 89 **B** 92 **C** 207 **D** 844 **E** 2204 **F** 18 932
G 204 **H** 208 **I** 1500 **J** 1516 **K** 9814 **L** 88 087
2. A 196 **B** 262 **C** 247 **D** 3885 **E** 1266 **F** 700
G 1565 **H** 6862 **I** 367 **J** 19 461 **K** 41 070 **L** 6950
M 11 097 **N** 79 594 **O** 77 274 **P** 896 218
Q 1 590 423 **R** 13 090 851

S-4 SET 5

1. A 33 **B** 33 **C** 47 **D** 162 **E** 686 **F** 449
G 1229 **H** 991 **I** 59 **J** 555 **K** 3194 **L** 9409
2. A 66 **B** 81 **C** 178 **D** 284 **E** 625 **F** 628 **G** 99
H 6961 **I** 8890 **J** 898 **K** 49 908 **L** 28 587
M 27 000 **N** 66 005 **O** 96 081 **P** 90 000 **Q** 99 000
R 99 900 **S** 318 207 **T** 689

S-4 SET 6

1. A 312 **B** 584 **C** 783 **D** 516 **E** 3192 **F** 4515
G 3152 **H** 27 654 **I** 72 720 **J** 37 533 **K** 185 295
L 636 321 **2. A** 972 **B** 1742 **C** 1012 **D** 8855
E 11 000 **F** 45 980 **G** 82 800 **H** 429 786 **I** 349 932
J 270 000 **K** 853 576 **L** 4 507 125 **3. A** 35 979
B 204 240 **C** 292 521 **D** 431 210 **E** 175 910
F 858 945 **G** 369 342 **H** 1 780 731 **I** 4 269 687
J 5 252 065 **K** 5 771 500 **L** 4 066 540

S-5 SET 7

1. A 14, R4 **B** 15 **C** 15, R5 **D** 64 **E** 37, R1
F 63, R3 **G** 167 **H** 83, R4 **I** 137 **J** 253, R4
K 476, R1 **L** 4999, R1 **M** 4287, R4 **N** 5706, R1
O 8785, R4 **2. A** 4, R15 **B** 8, R12 **C** 4, R28
D 29, R2 **E** 83, R4 **F** 34, R11 **G** 28, R3 **H** 122, R50
I 98, R62 **J** 2700 **K** 876, R22 **L** 823, R88
3. A 22, R2 **B** 8, R31 **C** 9, R1 **D** 13, R30
E 9, R270 **F** 9, R821 **G** 46, R5 **H** 68, R11
I 83, R39 **J** 143, R4 **K** 112, R268 **L** 295, R285

S-5 SET 8

1. 1, 3, 5, 7 **2. A** 12 **B** 15 **C** 18 **D** 7 **E** 12
F 17 **G** 10 **H** 6 **I** 10 **J** 100 **3. A** 4 **B** 1 **C** 0
D 6 **E** 3

S-6 SET 9

1. A 7 **B** 4 **C** 9 **D** 9 **E** 6 **F** 4 **G** 5 **H** 9 **I** 7
J 2 **K** 0 **L** 5 **2. A** $\{0, 1, 2, 3, 4\}$ **B** $\{10\}$
C $\{0, 1, 2, 3, 4, 5, 6\}$ **D** $\{8, 9, 10\}$
E $\{3, 4, 5, 6, 7, 8, 9, 10\}$ **F** $\{0, 1, 2, 3, 4, 5\}$ **G** $\{0, 1, 2\}$
H $\{5, 6, 7, 8, 9, 10\}$ **I** $\{0, 1, 2, 3, 4\}$
J $\{0, 1, 2, 3, 4, 5\}$ **K** $\{0, 1, 2, 3, 4, 5, 6\}$ **L** $\{0, 1, 2, 3, 4\}$

S-6 SET 10

1. A 6 **B** 4 **C** 49 **D** 5 **E** 14 **F** 36 **G** 23 **H** 5
I 3 **J** 6 **K** 1 **L** 3 **M** 0 **N** 7 **O** 2 **2. A** 6, 2
B 12, 6 **C** 12, 3 **D** 8. 4 **E** 10, 2 **F** 21, 3 **G** 20, 5
H 0, 0

S-7 SET 11

1. A 0, 1 **B** 1, 2 **C** 0, 0 **D** 5, 9 **E** 2, 7 **F** 8, 3
2. A 2 **B** 3 **C** 3 **D** 4 **E** 2 **F** 3 **3. A** 5 **B** 5
C 6 **D** 3 **E** 4 **F** 1 **G** 2 **H** 7 **I** 9

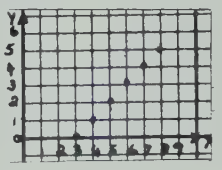
S-7 SET 12

1. 9 **2.** 6 **3.** 3 **4.** 86 **5.** 19 **6.** 29 **7. A** 23
B 10, 13

S-8 SET 13

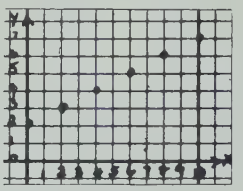
1. 4, 3, 2, 1. 0

2.

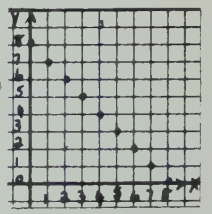


3. $2y$: 8, 10, 12. 14; x: 2, 4, 6, 8, 10

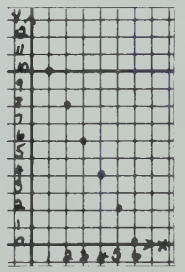
4.



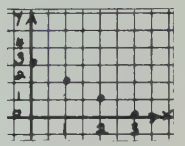
5. A



B



C



S-8 SET 14

1. A $2 \times 3 \times 5$ **B** $2 \times 2 \times 2 \times 7$ **C** $2 \times 2 \times 2 \times 3$
D $2 \times 3 \times 3$ **E** $2 \times 2 \times 2 \times 3 \times 3$ **F** $2 \times 2 \times 11$
G $3 \times 3 \times 3$ **H** $2 \times 3 \times 7$ **I** 3×17 **J** $3 \times 5 \times 7$
K 103 **L** $2 \times 5 \times 11$ **M** $5 \times 5 \times 5$ **N** $2 \times 2 \times 2 \times 17$
O $2 \times 3 \times 5 \times 5$ **P** 5×61 **Q** 3×163 **R** 29
2. A 3×5 **B** 2×3^2 **C** 3^3 **D** $2^2 \times 3^2$ **E** 2×3^3
F $2^2 \times 3 \times 5$ **G** 5^3 **H** $2^2 \times 5 \times 7$ **I** $2^4 \times 3 \times 7$
J $2^2 \times 3^2 \times 7$ **K** $2^2 \times 5^3$ **L** 2^9 **M** $2^2 \times 7 \times 11$
N $2^2 \times 5 \times 41$ **O** $2^6 \times 11$ **P** $2^2 \times 3^2 \times 17$ **Q** $2^3 \times 79$
R $2 \times 3^2 \times 11$ **S** $2 \times 3^4 \times 5$ **T** $2 \times 5^2 \times 7$
U $2 \times 3 \times 5^3$ **V** $2^2 \times 3^2 \times 5^2$ **W** $2^6 \times 3 \times 5$
X $2^2 \times 5 \times 7^2$

S-9 SET 15

1. A 6 **B** 1 **C** 5 **D** 6 **E** 5 **F** 12 **G** 7 **H** 6 **I** 20 **J** 7 **K** 14 **L** 10 **M** 4 **N** 25 **O** 24 **P** 30 **Q** 1 **R** 1 **S** 30 **T** 90 **2. A** 3 **B** 10 **C** 11 **D** 5 **E** 4 **F** 2 **G** 9 **H** 1 **I** 9 **J** 15 **K** 25 **L** 30 **M** 75 **N** 5 **O** 3 **P** 80 **Q** 10 **R** 25

S-9 SET 16

1. A 10 **B** 18 **C** 20 **D** 14 **E** 24 **F** 20 **G** 60 **H** 100 **I** 60 **J** 34 **K** 30 **L** 99 **M** 60 **N** 48 **O** 105 **P** 120 **Q** 130 **R** 90 **S** 48 **T** 180 **2. A** 30 **B** 30 **C** 60 **D** 60 **E** 30 **F** 12 **G** 28 **H** 36 **I** 30 **J** 120 **K** 70 **L** 30 **M** 24 **N** 60 **O** 48

S-10 SET 17

1. A −1 **B** −2 **C** −3 **D** 0 **E** −7 **F** −7 **G** 2 **H** 4 **I** −23 **J** −42 **K** 20 **L** 65 **M** −19 **N** −100 **O** 16 **P** 1 **Q** −198 **R** −48 **S** −212 **T** −935 **2. A** 3 **B** −5 **C** 9 **D** −3 **E** −9 **F** 9 **G** 11 **H** −2 **I** −3 **J** −97 **K** 60 **L** 0 **M** 0 **N** −60 **O** −120

S-10 SET 18

1. A −3 **B** −2 **C** 3 **D** 4 **E** 2 **F** −4 **G** 8 **H** −8 **I** −4 **J** 4 **K** −4 **L** −8 **M** −6 **N** 13 **O** −15 **P** 10 **Q** 20 **R** 2 **S** −15 **T** −9 **2. A** 9 **B** −8 **C** −19 **D** −21 **E** −10 **F** 3 **G** 5 **H** 10 **I** 45 **J** −35 **K** 3 **L** 17 **M** −3 **N** −32 **O** 53 **P** −65 **Q** 43 **R** 28 **S** 96 **T** −108 **U** −224 **V** 0 **W** 560 **X** −396

S-11 SET 19

1. A −20 **B** 24 **C** −60 **D** 48 **E** −150 **F** 72 **G** 0 **H** 53 **I** 90 **J** 0 **K** −280 **L** 120 **M** −43 **N** 1500 **O** 34 **P** −87 **Q** 87 **R** −870 **S** 870 **T** −870 **U** −1600 **V** 0 **W** 140 **X** −1232 **Y** 1350 **2. A** 27 **B** −27 **C** 27 **D** −27 **E** −27 **F** −20 **G** 90 **H** −800 **I** −180 **J** 144 **K** 400 **L** 400 **M** 400 **N** −400 **O** 400

S-11 SET 20

1. A 9 **B** −3 **C** −6 **D** −19 **E** 11 **F** −7 **G** 9 **H** 73 **I** 9 **J** 1 **K** −3 **L** −1 **M** −10 **N** 100 **O** 5 **P** 0 **2. A** −5 **B** 9 **C** 7 **D** −6 **E** −1 **F** −5 **G** −3 **H** 13 **I** 33 **J** 3 **K** −5 **L** 59 **M** 40 **N** 100 **O** −50 **P** −2 **Q** 3 **R** −2 **S** 2 **T** 6 **U** $-\frac{1}{2}$

S-12 SET 21

1. A 3 **B** 3 **C** 5 **D** 8 **E** 42 **F** 0 **G** 6 **H** 2 **I** 2 **J** 6 **K** 2 **L** 6 **2. A** 8 **B** 3 **C** 2 **D** 3 **E** 0 **F** 11 **G** 7 **H** 7 **I** 15 **J** 3, 7 **K** −36, 24 **L** −11, 29 **3. A** < **B** < **C** = **D** = **E** = **F** <

S-12 SET 22

1. A −4, −1 **B** −3, −1 **C** 16, 8 **2. A** −9, −3 **B** 20, 10 **C** −8, −2 **D** 3, 1 **E** 10, −4 **3. A** −5 **B** −9 **C** −39 **D** −78 **E** −10 **F** −2 **G** −3 **H** −9 **I** −9 **J** −5 **K** −9 **L** −6

S-13 SET 23

1. Many are possible.—Sample answers: **A** $\frac{2}{8}, \frac{4}{16}$ **B** $\frac{6}{8}, \frac{9}{12}$ **C** $\frac{2}{12}, \frac{3}{18}$ **D** $\frac{4}{6}, \frac{6}{9}$ **E** $\frac{6}{16}, \frac{9}{24}$ **F** $\frac{14}{20}, \frac{21}{30}$ **G** $\frac{4}{10}, \frac{6}{15}$ **H** $\frac{4}{4}, \frac{9}{6}$ **I** $\frac{10}{16}, \frac{15}{24}$ **J** $\frac{10}{12}, \frac{15}{18}$ **K** $\frac{14}{6}, \frac{21}{9}$ **L** $\frac{10}{24}, \frac{15}{36}$ **M** $\frac{20}{18}, \frac{30}{27}$ **N** $\frac{6}{40}, \frac{9}{60}$ **O** $\frac{8}{14}, \frac{12}{21}$ **P** $\frac{10}{18}, \frac{15}{27}$ **Q** $\frac{20}{22}, \frac{30}{33}$ **R** $\frac{3}{3}, \frac{6}{6}$ **S** $\frac{16}{10}, \frac{24}{15}$ **T** $\frac{2}{30}, \frac{3}{45}$ **U** $\frac{6}{32}, \frac{9}{48}$ **V** $\frac{2}{26}, \frac{3}{39}$ **W** $\frac{8}{42}, \frac{12}{63}$ **X** $\frac{8}{60}, \frac{12}{90}$ **2. A** $\frac{7}{8}$ **B** $\frac{5}{6}$ **C** 1 **D** $\frac{2}{7}$ **E** $\frac{3}{4}$ **F** $\frac{1}{6}$ **G** $\frac{9}{10}$ **H** $\frac{5}{8}$ **I** $\frac{7}{20}$ **J** $\frac{7}{9}$ **K** $\frac{12}{25}$ **L** $\frac{3}{7}$ **M** $\frac{2}{9}$ **N** $\frac{5}{7}$ **O** 3 **P** $\frac{4}{15}$ **Q** $\frac{2}{5}$ **R** $\frac{5}{6}$ **S** $\frac{12}{25}$ **T** $\frac{2}{15}$ **U** $\frac{4}{5}$ **V** $\frac{9}{11}$ **W** $\frac{3}{8}$ **X** $\frac{10}{30}$

S-13 SET 24

1. A no **B** yes **C** no **D** no **2. A** ≠ **B** ≠ **C** = **D** = **E** = **F** = **G** ≠ **H** ≠ **I** = **J** = **K** ≠ **L** ≠ **M** ≠ **N** ≠ **O** ≠ **3. A** 12 **B** 24 **C** 21 **D** 3 **E** 4 **F** 42 **G** 81 **H** 80 **I** 39 **J** 125

S-14 SET 25

1. A $\frac{8}{9}$ **B** $\frac{5}{8}$ **C** $\frac{7}{8}$ **D** $1\frac{2}{3}$ **E** $1\frac{1}{8}$ **F** $1\frac{1}{3}$ **G** $-1\frac{5}{18}$ **H** $1\frac{1}{6}$ **I** $\frac{3}{20}$ **J** $-\frac{1}{12}$ **K** $-\frac{13}{14}$ **L** $\frac{17}{18}$ **M** $-\frac{1}{10}$ **N** $1\frac{15}{56}, -\frac{9}{20}$ **2. A** $39\frac{7}{16}$ **B** $45\frac{3}{4}$ **C** $80\frac{1}{2}$ **D** $23\frac{1}{3}$ **E** $101\frac{1}{10}$ **F** $427\frac{9}{10}$ **G** $87\frac{1}{12}$ **H** $24\frac{44}{45}$ **I** $54\frac{13}{100}$ **J** $65\frac{49}{50}$ **K** $70\frac{25}{36}$ **L** $357\frac{11}{20}$

S-14 SET 26

1. A $\frac{3}{8}$ **B** $\frac{5}{12}$ **C** $\frac{1}{6}$ **D** $\frac{3}{10}$ **E** 0 **F** $\frac{13}{20}$ **G** $\frac{7}{20}$ **H** $\frac{8}{21}$ **I** $\frac{4}{5}$ **J** 0 **K** $\frac{1}{9}$ **L** $\frac{31}{36}$ **M** $\frac{1}{6}$ **N** $\frac{14}{45}$ **O** $\frac{16}{21}$ **2. A** $2\frac{1}{6}$ **B** $3\frac{1}{6}$ **C** $7\frac{1}{4}$ **D** $7\frac{7}{12}$ **E** $56\frac{17}{40}$ **F** $41\frac{19}{20}$ **G** $43\frac{14}{15}$ **H** $29\frac{5}{8}$ **I** $98\frac{91}{100}$ **J** $383\frac{4}{5}$ **K** $411\frac{1}{6}$ **L** $585\frac{91}{100}$

S-15 SET 27

1. A $\frac{1}{20}$ **B** $\frac{3}{8}$ **C** $\frac{1}{10}$ **D** 1 **E** $\frac{2}{3}$ **F** $\frac{1}{3}$ **G** $\frac{3}{4}$ **H** 1 **I** $\frac{1}{2}$ **J** 30 **K** $2\frac{8}{9}$ **L** 0 **M** $\frac{11}{12}$ **N** $\frac{7}{40}$ **O** $\frac{4}{15}$ **2. A** $-\frac{1}{6}$ **B** $-\frac{1}{15}$ **C** $\frac{1}{21}$ **D** $-\frac{1}{9}$ **E** $-\frac{1}{2}$ **F** $\frac{1}{3}$ **G** $-\frac{4}{15}$ **H** $\frac{3}{20}$ **I** $-\frac{1}{12}$ **J** −21 **K** $\frac{1}{8}$ **L** $-\frac{1}{2}$ **M** $-2\frac{7}{10}$ **N** 0 **O** $-\frac{1}{9}$ **3. A** $\frac{5}{12}$ **B** $17\frac{1}{2}$ **C** $-1\frac{1}{3}$ **D** $-1\frac{5}{6}$ **E** $1\frac{2}{3}$ **F** $-12\frac{1}{2}$ **G** $1\frac{13}{36}$ **H** $2\frac{1}{2}$ **I** 15 **J** $-38\frac{1}{3}$ **K** −78 **L** −6 **M** $-27\frac{5}{8}$ **N** $-121\frac{1}{2}$ **O** $\frac{81}{100}$

S-15 SET 28

1. A $1\frac{1}{2}$ **B** $3\frac{1}{2}$ **C** $\frac{2}{3}$ **D** $1\frac{1}{3}$ **E** $\frac{5}{12}$ **F** $-1\frac{2}{3}$ **G** $6\frac{2}{3}$ **N** $\frac{6}{7}$ **I** $\frac{17}{32}$ **J** $-\frac{1}{12}$ **K** $2\frac{2}{15}$ **L** $-1\frac{7}{8}$ **M** $\frac{1}{16}$ **N** $-2\frac{2}{5}$ **O** $2\frac{13}{16}$ **2. A** 14 **B** $10\frac{2}{3}$ **C** $\frac{3}{10}$ **D** $\frac{1}{14}$ **E** $\frac{2}{3}$ **F** $2\frac{4}{5}$ **G** $-5\frac{1}{2}$ **H** −3 **I** −6 **J** −3 **K** $2\frac{2}{3}$ **L** $-\frac{1}{5}$ **M** $-\frac{1}{4}$ **N** $-\frac{5}{10}$ **O** −6

S-16 SET 29

1. 18 **2.** $7.50 **3.** $70+ **4.** $1\frac{5}{12}$ hr **5.** 75 km/h− **6.** 8−

S-16 SET 30

1. A 7.93 **B** 12.791 **C** 246.85 **D** 9.007 **E** 6.746 **F** 251.248 **G** 39.375 **L** 70.7105 **I** 862.943 **J** 62.0379 **K** 2.7409 **L** 410.563 **2. A** 2.58 **B** 6.36 **C** 10.9 **D** 0.277 **E** 29.35 **F** 4.081 **G** 8.42 **H** 12.665 **I** 25.34 **J** 4.57 **K** 5.818 **L** 281.32 **M** 21.28 **N** 434.63 **O** 651.67 **P** 4506.029 **3. A** 1.26 **B** 3.6 **C** −13.43 **D** −32.08 **E** −1.55 **F** 0.166 **G** 4.83 **H** −0.0524 **I** 14.95 **J** 84.55 **K** 4.95 **L** 43.704

S-17 SET 31

1. A 28.32 **B** 12.8 **C** 3.01 **D** 156.91 **E** 220.92 **F** 1.544 **G** 25.976 **H** 5.369 **I** 13.086 **J** 0.90611 **K** 171.038 **L** 1.36279 **M** 13.3668 **N** 195.038 **O** 33.8355 **P** 414.15136 **Q** 0.32812 **R** 3688.626 **S** 12.04924 **T** 0.418573 **2. A** −0.27 **B** −12.6

C −2.8 **D** 4.25 **E** −3.569 **F** 6.585 **G** −10.132
H −848.16 **I** 29.96856 **J** −48824 **K** −1767.633
L 3523.455 **M** −53.049651 **N** −0.07425 **O** 202.4891

S-17 SET 32

1. A 4×10^2 **B** 3×10^3 **C** 6.3×10^3 **D** 2×10^4
E 5.4×10^4 **F** 3.5×10^5 **G** 9×10^6 **H** 4.56×10^{10}
I 8.03×10^9 **J** 7.905×10^{12} **K** 5×10^{-1}
L 8.49×10^0 **M** 1.4×10^{-1} **N** 7.0×10^{-2}
O 3.0×10^{-3} **P** 7.02×10^{-2} **Q** 9.0×10^{-5}
R 4.53×10^{-4} **S** 8.0×10^{-7} **T** 8.002×10^{-4}
2. A 42 000 **B** 561 **C** 3 700 000 **D** 92.2 **E** 405 000
F 3 200 000 000 **G** 9.55 **H** 107 000 000 **I** 90 050
J 35 400 000 **K** 0.6 **L** 0.008 **M** 0.15 **N** 0.034
O 0.000691 **P** 0.00000007 **Q** 0.00000000044
R 0.0000039 **S** 0.000000801 **T** 0.00000000001392

S-18 SET 33

1. A 2.4 **B** 63.4 **C** 9.3 **D** 3.8 **E** 3.51 **F** 0.32
G 0.2535384 **H** 0.93 **I** 63.4 **J** 3.61 **K** 6.5 **L** 0.37
M 23.5 **N** 0.36 **O** 0.99 **P** 0.114 **2. A** 0.82 **B** 5.15
C 0.345 **D** 1.05 **E** 0.062 **F** 115.5 **G** 40 **H** 4
I 32.28 **J** 460 **K** 0.15 **L** 135 **M** 45 **N** 0.55
O 6.5 **P** 4.1 **Q** 0.73 **R** 0.53 **S** 3.4 **T** 7.5110963

S-18 SET 34

1. A 3.2×10^7 **B** 2.4×10^5 **C** 1.2×10^4
D 2.8×10^{-1} **E** 4.9×10^3 **F** 1 **G** 4.8×10^4
H 1.5×10^5 **I** 3.5×10^2 **J** 2.7×10^2 **K** 2.4×10^3
L 3×10^7 **M** 2.8 **N** 35 **O** 4.5×10^{-5} **2. A** 5
B 200 **C** 20 **D** 15 000 **E** 0.3 **F** 200 **G** 50
H 1200 **I** 0.5 **J** 20 **K** 2 **L** 0.3

S-19 SET 35

1. A 0.5 **B** 0.53 **C** 0.533 **2. A** 0.88 **B** 0.38
C 0.11 **D** 0.36 **E** 0.39 **F** 0.87 **3. A** 0.688 **B** 0.889
C 0.938 **D** 0.711 **E** 0.577 **F** 0.974

S-19 SET 36

1. A 24 **B** 45 **C** 20 **D** 9 **E** 27 **F** 42 **G** 6
H 15 **2. A** $1.29 **B** 30 **C** 184 **D** 91 **E** 160 km

S-20 SET 37

1. A 12 cm **B** 24 cm **C** 1.5 cm **D** 22.5 cm
2. 279 km **3.** 18 cm **4.** 4 cm by 7.5 cm

S-20 SET 38

1. A 0.18 **B** 0.33 **C** 0.1 **D** 0.09 **E** 0.05 **F** 0.7
G 1.0 **H** 0.01 **I** 0.02 **J** 0.75 **K** 0.075 **L** 0.04
M 0.015 **N** 0.333 **2. A** 38% **B** 65% **C** 20% **D** 3%
E 150% **F** 15% **G** 130% **H** 205% **I** 9.3% **J** 550%
K 1630% **L** 17.8% **M** 1.7% **N** 30.9% **3. A** $\frac{37}{100}$ **B** $\frac{1}{2}$
C $\frac{2}{5}$ **D** $\frac{7}{10}$ **E** $\frac{3}{4}$ **F** $\frac{1}{20}$ **G** $\frac{1}{50}$ **H** $\frac{1}{100}$ **I** $\frac{3}{20}$ **J** $\frac{4}{25}$ **K** $\frac{7}{25}$
L $\frac{2}{25}$ **M** $5\frac{1}{5}$ **N** $\frac{14}{25}$ **4. A** 59% **B** 3% **C** 1% **D** 75%
E 34% **F** 28% **G** 36% **H** 56% **I** 39% **J** 42%
K 71% **L** 30% **M** 60% **N** 93%

S-21 SET 39

1. A $3.15 **B** $12.60 **2.** 80% **3. A** $169 **B** $481
4. A $9.30 **B** $164.30 **5.** $45 600 **6.** *A* by $1

S-21 SET 40

1. A 2.7 **B** 8.5 **C** 10 **D** 7.95 **E** 12.5 **F** 35
G 0.003 **H** 0.007 **I** 0.005 **J** 0.0035 **K** 0.0006
L 0.00125 **2. A** 1950 **B** 1700 **C** 2720 **D** 6300
E 2250 **F** 72 240 **G** 4.5 **H** 4.94 **I** 2.56 **J** 1.2
K 2.25 **L** 6.25 **3. A** 172 125 **B** 0.0075 m

S-22 SET 41

1. A 3 **B** 1.3 **C** 5000 **D** 15 000 **E** 4 **F** 0.74
G 4600 **H** 400 **I** 0.53 **J** 100 **K** 10 **L** 1 **M** 0.001
N 0.0001 **O** 3 **2. A** 9.8 **B** 15 **C** 8.2 **D** 10.8

S-22 SET 42

1. A 14.82 **B** 16.17 **C** $16\frac{3}{20}$ **D** 2086.5 **E** 27.0375
2. A 9.03 **B** 5.225 **C** 2161.08 **D** 2709.63 **E** 5.2768

S-23 SET 43

1. A 49 **B** 154 **C** $56\frac{7}{8}$ **D** 928.05 **2. A** 6975
B 20.8 **C** $18\frac{11}{40}$ **D** 43.26 **3. A** 10.66 **B** 75 **C** 175.5
D 1995

S-23 SET 44

1. A 65°48′56″ **B** 57°48′22″ **C** 99°11′49″ **D** 95°8″
E 130°42′1″ **F** 113°25′33″ **G** 90° **H** 183°51′10″
2. A 22°26′6″ **B** 27°23′47″ **C** 50°44′11″ **D** 27°20′47″
E 61°45′38″ **F** 24°54″ **G** 25°48′49″ **H** 46°7′13″

S-24 SET 45

1. A 10 **B** 25 **C** 17 **D** 13 **2. A** 24 **B** 20 **C** 12
D $7\frac{1}{2}$ **3. A** 25 **B** 26 **C** 30 **D** 35 **E** 12.7 **F** 20
G 30 **H** 3.3 **I** 3.2

S-24 SET 46

1. A 180 **B** 75.516 **C** 884.4 **D** 1 155 000 **E** 297.1
2. A 57.5 **B** 105 **C** 42.02 **D** 10.04

S-25 SET 47

1. A 11.66 **B** 441 **C** 3725 **D** 8450 **2. A** 28.7
B 700 **C** 1916.7 **D** 13

S-25 SET 48

1. A 3 **B** 8 **C** 32 **D** 7 **E** 39 **F** 2.1 **G** 3
H 0.45 **2. A** 4 **B** 12 **C** 9 **D** 5 **E** 48 **F** 30
G 3.9 **H** 4

S-26 SET 49

1. A E **B** e/d **C** EDF **2. A** $\overline{RS}/\overline{RQ} = \overline{TU}/\overline{UV}$
B $\angle P \cong \angle V$ **C** $\triangle SRQ \cong \triangle TUV$ **3. A** $\frac{2}{3}$ **B** $\frac{2}{3}$ **4. A** 3
B 4

S-26 SET 50

1. A 5.84 **B** 9.88 **2. A** 14.74 **B** 16.28 **3.** $t = 19.05$;
$x = 25$ **4. A** 0.53 **B** 0.47 **C** 0.88

S-27 SET 51

1. A 15.7 **B** 10.04 **C** 43.96 **D** 72.2 **E** 973.4
F 5.02 **G** 54.3 **H** 2.8 **2. A** 25.1 **B** 21.98 **C** 138.2
D 2763 **E** 329.7 **F** 52.1 **G** 2.20 **H** 38.62

S-27 SET 52

1. A 113.04 **B** 176.6 **C** 1256 **D** 452.2 **E** 7.07
F 30.2 **G** 32.2 **H** 0.50 **2. A** 379.94 **B** 632.06
3. A 520.96 **B** 298.98

S-28 SET 53

1. A 863.5 **B** 1161.8 **C** 368.95 **D** 6280 **E** 6782.4
F 156.9 **G** 23.25 **2. A** 150.7 **B** 942 **C** 211.95

D 196.25 **E** 100 480 **F** 4.57 **G** 1.21

S-28 SET 54

1. A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **2. A** $\frac{4}{6}$ **B** $\frac{5}{6}$ **C** $\frac{1}{3}$ **D** $\frac{2}{6}$ **3. A** $\frac{1}{6}$
B $\frac{5}{6}$ **C** $\frac{1}{2}$ **D** $\frac{1}{6}$ **E** $\frac{2}{3}$ **F** $\frac{2}{3}$ **G** $\frac{1}{3}$ **H** $\frac{1}{2}$ **I** 0

S-29 SET 55

1. A 24 **B** 720 **C** 5040 **D** 48 **E** 720 **F** 120
2. 24 **3.** 120 **4.** 40 320 **5.** 362 880 **6. A** 7! **B** 10
C 7·6 **D** 4! **E** 4! **F** 7! **G** 9! **H** 6·5·4 **I** 6

S-29 SET 56

1. A 6 **B** 6 **C** 36 **2.** 24 **3.** 15 **4.** 56 **5. A** 6
B 20 **C** 21 **D** 56 **E** 35 **F** 84

S-30 SET 57

1. A $\frac{1}{3}$ **B** $\frac{1}{2}$ **C** $\varnothing$, 0 **D** $\{1, 2, 3, 5, 6\}$, $\frac{5}{6}$ **2. A** $\frac{3}{8}$ **B** $\frac{1}{2}$
C $\frac{1}{4}$ **D** $\frac{5}{8}$ **3. A** $\frac{2}{3}$ **B** $\frac{3}{3}$ **C** 1

S-30 SET 58

1. A 27 **B** 67 **C** 241 **D** 120 **E** 218 **F** 4539
2. A 51 **B** 313 **C** 140 **D** 697 **E** 1291 **F** 88 **G** 55
H 53

S-31 SET 59

1. A 49 **B** 86 **C** 125 **D** 40 **E** 1350 **F** 10 **G** 19
H 313.5 **I** 1555 **J** 76 **2.** 75 **3.** 125 **4. A** 56.05
B 55.7 **C** mean

S-31 SET 60

1. **2.** 2–3 **3.** 16–20

S-32 SET 61

1. A 110 **B** 27.3% **C** 63.6% **2. A** 27°C **B** 22°C
C 4 **3. A** $150 **B** $90 **4. A** 30% **B** 15%

S-32 SET 62

1. A 13 **B** 23 **C** 21 **D** 16 **E** 34 **F** 71 **G** 54
H 47 **I** 56 **J** 86 **K** 83 **L** 94 **M** 9.9 **N** 8.8
O 8.4 **P** 6.1 **Q** 6.8 **R** 3.8 **S** 0.5708765 **T** 0.43
2. A 3.7 **B** 4.8 **C** 8.3 **D** 12.2 **E** 20.6 **F** 22.0
G 22.4 **H** 95.4 **I** 54.8 **J** 3.5 **K** 5.1 **L** 6.4 **M** 7.2
N 0.7 **O** 0.3

MAKING METRIC MEANINGFUL

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MAKING THE METRIC SYSTEM MEANINGFUL

To make the metric system meaningful it is useful to understand something of the origin of weights and measures, the emergence of early decimal measurement systems, and the development of the modern metric system. Although the metric system is simply the application of the decimal number system to measurement based on the metre and other standard units, its usefulness has not been readily accepted universally. The struggle to develop a universal language of measurement has been long and difficult.

It is a little known fact that even though our *old* measurement system is called the *British Imperial* or *English* measurement system, the British neither originated nor advocated this system in earlier days. The British used a system similar to metric nearly 1500 years ago, abandoning it about 700 years ago only to go metric again in the 1970's.

How did measurement begin? Probably back in pre-history, one cave man showed another just how big the fish was which he caught (or which got away) by using his out-stretched arms: since then, man has used parts of his body for measuring.

Ancient Egyptian history of measurement was recorded as a readable story with markings cut into rock used to keep track of the rise and fall of the Nile waters at floodtime. Every *tenth* line was emphasized, a measure roughly corresponding to the span of the hand. The units used in land surveying were as follows:

10 zebos (fingerwidths)	= 1 span
10 spans (100 zebos)	= 1 nent (armstretch)
10 nents	= 1 khet

The builders of pyramids and temples used the *cubit* (length of the forearm).

With the domestication of the Arabian camel, two Phoenician ports, Tyre and Sidon, became thriving commercial centres where trade flourished for centuries. Camels made possible caravan routes from Babylon to Egypt. Since Egypt was an important customer, the Phoenicians adopted the Egyptian *decimal* system. One small modification was made in order to measure cloth—half a nent (armstretch) was used, a unit close to a yard.

The Phoenicians had established trading centres along the Mediterranean shores including Greece. Around the 3rd to 4th century contact was made with the northern tribes, particularly with the Goths who had moved south along the Black Sea into Greece. Odin whose name survives in our word "Wednesday", was a mighty chieftain in Teutonic mythology, and guided his people from a semi-nomadic to a more settled way of life. It is said that he accepted the Phoenician way of measurement—only his "armstretch" unit was larger. And so the *decimal* measurement spread among the peoples known as Norse or German. Around this time portions of Britain were occupied by Roman legions who confined themselves mainly to the fortresses built along the defending walls. With the recall of the Roman soldiers Britan was left defenceless.

This situation allowed the migrations to Britain to take place from the 5th to the 8th century A.D. Across the North Sea came the Angles, Saxons, Jutes, and Danes. They brought with them the *decimal* system of measurement.

10 fingerwidths	= 1 span-of-length
10 spans	= 1 armstretch
10 armstretches	= 1 chain
10 chains	= 1 furlong
10 chains × 1 chain	= 1 acre

These measurements were suitable for surveying and building. The housewife used smaller units based on 10.

10 one-half fingerwidths	= 1 handbreadth (across knuckles)
10 handbreadths	= 1 wand (half an armstretch)

In most measurement systems, it is noteworthy that from the selected unit of length a cube is described which determines the unit of capacity or volume; the unit of mass is derived from the mass of cool water in this cube. For example, the capacity of a cubic handbreadth was called a *measure-full.* The mass of cool water in this cube gave the early Britons the basic mass unit called a *measure-weight;* the mass of a cubic wand of cool water was a *tun-weight.* Length received greater emphasis in this historical development because the derived units were determined by the unit of length and limited space.

However, about 700 years ago the foot was introduced to England from continental Europe. The foot did not originate with the English. The Romans used such a unit and they in turn had borrowed it from the Greeks who devised it in the first place.

The Greeks worked out a system for fighting men using paces (double steps) as units. Romans used *mille* for a thousand paces, giving us the word *mile.* Some areas affected by Roman units were Italy, Spain and France while those retaining the units of the Gothic tribe were England, Germany and the Scandinavian countries.

Around 1200 A.D., about 700 years after the fall of the Roman Empire, a new industrial era began in Europe. Trade increased, trade organizations were formed, fairs were established, and cathedrals were built. Unfortunately the measurement plan used was the "foot-length". With the introduction of the industrial era in England, there too the foot-length and other related units were adopted. Since it was non-decimal it did not fit the simple Gothic system. No doubt many in Britain at the time were dismayed by this decision to allow the Roman units to be used.

At the beginning of the 17th century, Simon Stevin, an inspector of dikes in the Low Countries, proposed a *decimal* measurement system, and shortly thereafter Father Gabriel Mouton suggested to the French Academy of Sciences a *decimal* system based on the circumference of the earth.

By the 18th century in Europe, there were *280* measures called "foot", all of different lengths. Imagine the quarrels and squabbles that this could cause. In the midst of this confusion people were clamouring for a simple system. One very important development in the U.S.A. captured the people's attention: namely, the *decimal* coinage plan approved by Congress in 1786.

10 mills = 1 cent
10 cents = 1 dime
10 dimes = 1 dollar

At this time Thomas Jefferson, as U.S. Secretary of State, prepared a unified system of weights and measures which was *decimal* in nature. He sub-divided the foot into 10 parts, larger units being formed by multiplying by 10.

Meanwhile the scientists at the Academy of Sciences of Paris began to devise a fully *decimalized* measurement system to dovetail with the decimal number system. At this time certain mathematicians were proposing a duodecimal number system and advocating a duodecimalized measurement system to match. This idea failed; instead the idea of linking the measurement system to the place-value number system based on 10, flourished.

In the 1790's the French instructed a small group of outstanding scientists to measure the distance from Dunkirk to Barcelona and this information was used to calculate the distance from the North Pole to the Equator. This land measurement was chosen because it was the longest segment of land along a meridian in Europe. Assuming the earth to be perfectly round, one ten-millionth of this arc was designated as the *metre* (greek metron—*a measure*).

Beginning with the metre, larger and smaller units were formed based on the decimal number system, i.e. multiples and sub-multiples of ten with the appropriate prefix.

PREFIX		SYMBOL	MULTIPLYING FACTOR
Greek	kilo	k	$1000 = 10^3$
	hecto	h	$100 = 10^2$
	deca	da	$10 = 10^1$
Latin	deci	d	$0.1 = 10^{-1}$
	centi	c	$0.01 = 10^{-2}$
	milli	m	$0.001 = 10^{-3}$

Some of the common metric units used in most daily situations are derived from the metre, gram and litre and these are:

QUANTITY	UNIT	SYMBOL
Length	millimetre	mm
	centimetre	cm
	metre	m
	kilometre	km
Capacity	millilitre	ml
	litre	l or ℓ
	kilolitre	kl
Mass	gram	g
	kilogram	kg
	tonne	t
Volume	cubic centimetre	cm^3
	cubic metre	m^3

It is remarkable how near the modern metric system is to the early decimal system. The wand of old was only 0.5 cm longer than the metre. Had the early system remained, little change would have been necessary in order to bring the *old* into the *new*. However, the metric system as proposed by France gained ready acceptance in certain Latin American and European countries. Germany adopted the metric system after defeating the French in the war of 1870-1871. The British however, in 1871, rejected the proposal to convert to the metric system by defeating the Bill in the House of Commons by a mere five votes.

Since Napoleonic times the original metric system has undergone changes and in 1960 the General Conference on Weights and Measures (Conférence Générale des Poids et Mesures) adopted the name *Systéme International d'Unités (International System of Units)* abbreviated **SI**. The SI system is an extension and refinement of the traditional systems (*CGS* and *MKS*) and superior because it is a rationalized, coherent, and simplified metric system.

The base units of SI are seven independent units: *metre* (length), *kilogram* (mass), *second* (time), *ampere* (electric current), *kelvin* (temperature), *candela* (luminous intensity), and *mole* (amount of substance). All other derived units are defined in terms of these units.

QUANTITY	UNIT	SYMBOL
Length	metre	m
Mass	kilogram*	kg
Time	second†	s
Electric current	ampere	A
Thermodynamic temperature	kelvin#	K
Luminous intensity	candela	cd
Amount of substance	mole[xx]	mol

In addition to the seven base units are the following SI units—supplementary units and derived units having special names.

	QUANTITY	UNIT	SYMBOL
Supplementary Units	Plane angle	radian	rad
	Solid angle	steradian	sr
Derived Units having Special Names	Frequency (cycle per second)	hertz	Hz
	Force	newton	N
	Energy	joule	J
	Power	watt	W
	Electric charge	coulomb	C
	Electric potential	volt	V
	Electric resistance	ohm	Ω
	Electric capacitance	farad	F
	Magnetic flux	weber	Wb
	Inductance	henry	H
	Magnetic induction	tesla	T
	Luminous flux	lumen	lm

*The *Kilogram* is the only base unit containing a prefix and is used because the gram is too small.

† As early as 1670, it was suggested by C. Wren that the length of the pendulum whose swing takes one second be the unit of length. By coincidence the length of this pendulum was very nearly 1 m long.

The *Celsius* temperature scale (previously called Centigrade) is common in everyday usage and even though it is a non-SI unit it is permissible to use it for temperature measurements. One degree Celsius equals one kelvin (1°C = 1 K).

xx The mole is defined in chemistry in terms of a number of elementary particles (Avogadro's number).

THE RULES GOVERNING THE USE OF SI UNITS

In order to preserve uniformity in the writing of the units, the symbols, the prefixes, and the numerals, the following list of rules should be followed. For a more complete treatment write to: *Canadian Standards Association,* 178 Rexdale Blvd., Rexdale, Ontario and request a copy of a CSA Standard Z234.1–1973, Metric Practice Guide.

A. SI Units when written out in full:

(1) Units are not capitalized except at the beginning of a sentence. Example: ampere, newton, kelvin. Only Celsius begins with a capital letter.
(2) The word *per* is used to denote division between the units. Example: kilometres per hour not kilometres/hour.

B. Symbols for SI Units:

(1) A symbol represents a unit and is the same in all languages.
(2) The symbols do not change in the plural.
Example: 10 metres 10 m
(3) Because the symbols are not abbreviations no periods are written after the symbol except at the end of a sentence: The symbols can be manipulated like algebraic variables i.e. multiplied, divided, squared, cubed, etc.
(4) Symbols are usually not capitalized except for a unit named after an individual. Example: A for ampere, N for newton, and K for kelvin. (Note: Hz for hertz, Pa for pascal, and Wb for weber).
(5) Symbols should not begin a sentence.
(6) The symbol of a derived unit formed by division is shown by using a solidus (/) or a negative exponent.
Example: km/h or $km \cdot h^{-1}$, not kmph or k.p.h. m/s^2 or $m \cdot s^{-2}$, not m/s/s/.
(7) The symbol must be separated from the numeral by a space. Example: 75 cm
(8) The symbol for degree ° is used to denote degree Celsius (°C) to differentiate it from a coulomb (C), a unit of electrical charge.

C. Prefixes for SI Units:

The prefixes larger than kilo and smaller than milli are listed as additional information to teachers and interested students.

PREFIX	SYMBOL	MULTIPLYING FACTOR
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

(1) The only prefix symbols capitalized are T for tera, G for giga, M for mega.
(2) The prefix symbol and the unit symbol are printed in roman (upright) type without spacing between them. Examples: kPa for kilopascal, mN for millinewton, MN for meganewton.
(3) Only one prefix is applied at one time to a given unit. Example: picometre (pm), not millinanometre (mnm).
(4) The prefix symbol combines with the unit symbol to form a new symbol which can be raised to a positive or negative power.
Example: $1\ cm^2 = 1\ (cm)^2 = (10^{-2}m)^2 = 10^{-4}m^2$
(5) The prefixes denoting multiples or sub-multiples of 1000 are preferred where it is practical. This implies that the prefixes h, da, d, c, are to be avoided; however, in measuring length the cm is acceptable.

D. Numerals used with SI Units:

(1) Instead of using commas to group blocks of three digits to the left and right of the decimal point, a space is used as a separator.
Examples: 654 321.234 56
(2) A prefix should be chosen so that the numerical values fall between 0.1 and 1000 if it is suitable.
Example: 23 000 J becomes 23 kJ
0.005 81g becomes 5.81 mg
(3) A zero is written in the units column for a number less than one.
Example: 0.3 in place of .3
(4) Decimal notation should be used where it is practical.
Example: 4/5 should be written 0.8

E. General comments:

(1) The following should *not* be used:
amp to mean ampere;
kilo to mean kilogram;
sq. cm to mean square centimetre (use cm^2);
cc to mean cubic centimetre (use cm^3).

(2) Metric (but non-SI) units are to be avoided.
Example: calorie, micron, dyne, erg.

(3) Some *NON-SI* units are permissible with the SI.
Example: minute, hour, day, litre, tonne, degree Celsius, nautical mile, knot, ängström, hectare.

THE ADVANTAGES OF SI

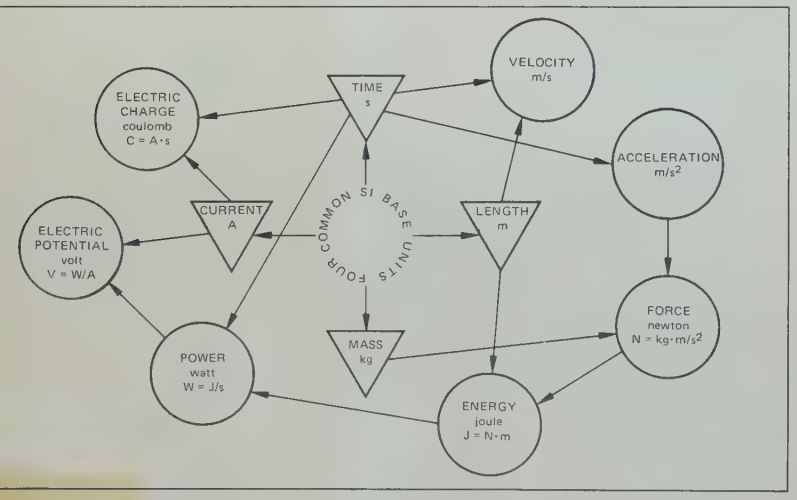
UNIQUE: The SI system is convenient in measuring in that there is only *one* unit for each physical quantity as illustrated by the seven base units such as the metre for length, second for time, etc. There is absolutely only *one* version of all the base units all over the world.

COHERENT: This can be defined
(a) as the act of sticking together or
(b) as a natural or logical connection.

Starting with a unique set seven base units all other derived units can be obtained by multiplying or dividing two units either basic or derived.

Example: velocity unit is the metre per second. i.e. m/s

The coherence can be better illustrated by the following schematic diagram.



Because of its coherence, complicated conversion factors are eliminated. When measuring physical quantities one unit applies to the mechanical, electrical and thermal fields.

Example: 1 watt = 1 newton x 1 metre per second
1 watt = 1 ampere x 1 volt
1 watt = 1 joule per second.

INTERRELATIONSHIP: The cubic model illustrated best describes the following interrelationships.

1 cubic centimetre (cm) container holds 1 millilitre of water of mass 1 gram (g) at 4° Celsius. Consequently a 1000 cubic centimetres (cm^3) or 1 cubic decimetre (dm^3) container holds 1000 millilitres (ml) or 1 litre (ℓ) of water of mass 1000 grams (g) of 1 kilogram (kg).

UNIVERSAL: The metric system (more specifically, SI units) has become the "lingua franca" of the world. The symbols are identical whenever and wherever SI units are used and thereby form part of a worldwide language of measurement.

THINK METRIC

The SI units are distinct and unique in that they are not derived by converting units from other systems. In order to preserve this "purity", conversions from one system of measurement to another should be avoided. Education research strongly supports the theory that the best way to learn the metric (SI) system is to *think metric.* Languages are best understood if the person thinks in the language spoken.

However, it is not always possible to strictly enforce this way of thinking. If conversion is necessary it is better to use the form 1 m = 1.094 yd. instead of 1 yd. = 0.9144 m.

It is easier to relate units of measurement to parts of the body.

1 fingerwidth–2 cm
1 small fingernail width–1 cm
1 handbreadth–10 cm
1 cubic handbreadth–1 dm^3 or 1 ℓ
(remember the mass of 1 ℓ water is 1 kg)

In this fashion the everyday units can be readily visualized. It is more difficult to visualize distances in kilometres. For this reason the more common relationships can be applied. For example:

60 miles is approximately 100 km
30 miles is approximately 50 km
20 miles is approximately 30 km
50 miles is approximately 80 km
100 miles is approximately 160 km.

It is convenient to think of a common paper clip to have a mass of 1 g. A litre is less than the Imperial or Canadian quart but greater than the U.S. quart.

In developing a *feel* for the temperatures expressed in degrees Celsius, it may be useful to relate it to some common experiences.

100°C boiling point of water
37°C normal body temperature
30°C hot summer day
20°C comfortable room temperature
10°C cool summer day
0°C freezing
-20°C cold winter day
-40°C extremely cold winter day (equivalent to -40°F)

CONCEPT OF MEASUREMENT

To measure is to count or mark off using a selected unit as the standard. In other words, to measure is to determine a number which indicates the number of times the standard unit has been applied.

Concepts of measurement are best learned if students are meaningfully involved in activity. The Chinese proverb "I do, I understand" applies very well to the development of measurement concepts. The U.S. Metric Study Interim Report states: "To begin with, measurement should not and really cannot be "taught" through a series of planned lessons. Learning to measure is a gradual process related to each child's experiences . . . children will learn best if it is not *taught* but *experienced* and used in some activity in the context of situations in which a child is actively involved. . . ."

The variety and extent of the hands-on approach is limited only to the imagination of the students and teacher. Although excellent commercial material is available, it is at times less expensive and more beneficial to the learner to use materials in the school or home. Cuisenaire rods and cubes are excellent to demonstrate the cm, cm^2, and cm^3. The container of a 2 quart milk carton is 10 cm × 10 cm. This would readily provide the student with a cubic decimetre, or 1 litre, if 10 cm high. This type of activity reinforces the *think metric* theme.

Teachers are strongly urged to allow students the opportunity to develop the skill of estimation. It is fun and very useful. Like any other skill, perfection is attained through meaningful practice.

When industry in Great Britain went metric the companies streamlined their operations. Forty different sizes of screws were reduced to seven. Going metric gives teachers an opportunity to streamline mathematical operations and methods of instruction of measurements concepts.

The adoption of the metric (SI) system will simplify and unify the teaching of mathematics and science, reduce calculation errors, save teaching time, and give a better understanding of basic physical principles.

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